SOME ASPECTS OF THE AXIAL EXTENSION MODE IN AN ELASTIC THIN-WALLED BEAM-COLUMN

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The present paper deals with the influence of the axial extension mode on static and dynamic interactive buckling of a thin-walled beam-column with imperfections subjected to uniform compression when the shear lag phenomenon and distortional deformations are taken into account. A plate model (2D) is adopted for the beam-column. One- and two-dimensional models of the elements are compared, too. The structure is assumed to be simply supported at the ends. Equations of motion of the component plates are obtained from Hamilton's Principle, taking into account all components of inertia forces. Within the frame of the first order nonlinear approximation, the dynamic problem of modal interactive buckling is solved by the transition matrix using the perturbation method and Godunov's orthogonalization.

Key words: axial extension mode, eigenvalue problem, dynamic interactive buckling, thin-walled structure

1. Introduction

Thin-walled structures composed of plate elements have many different buckling modes that vary in quantitative and qualitative aspects. The analysis of buckling of conservative systems belongs to the main problems in mechanical sciences. In the case of finite displacements, different buckling modes are interrelated even with the loads close to their critical values (eigenvalues of the respective boundary problem). The investigation of stability of equilibrium states requires application of a nonlinear theory that enables us to estimate the influence of different factors on the structure behavior. When the static post-critical behavior of each individual mode is stable, their interaction can lead to unstable behavior, and thus to an increase in the imperfection sensitivity (Ali and Sridharan, 1988; Benito and Sridharan, 1984-85; Byskov, 1987-88; Byskov

and Hutchinson, 1977; Kolakowski, 1993; Kolakowski and Kowal-Michalska, 1999; Kolakowski and Krolak, 2006; Kolakowski *et al.*, 1999; Kolakowski and Teter, 2000; Manevich, 1985, 1988; Moellmann and Goltermann, 1989; Pignataro and Luongo, 1987; Pignataro *et al.*, 1985; Sridharan and Ali, 1986; Sridharan and Benito, 1984; Sridharan and Peng, 1989; Teter and Kolakowski, 2004).

The concept of static interactive buckling (the so-called coupled buckling) involves the general asymptotic theory of stability. Among all versions of the general nonlinear theory, Koiter's theory (Koiter, 1976; Koiter and Pignataro, 1976; Koiter and van der Neut, 1980) of conservative systems is the most popular one, owing to its general character and development, even more so after Byskov and Hutchinson (1977) formulated it in a convenient way. The theory is based on asymptotic expansions of the postbuckling path and is capable of considering nearly simultaneous buckling modes.

The theory of interactive buckling of thin-walled structures subjected to static and dynamic loading has been already developed widely for over forty years. Although the problem of static coupled buckling can be treated as fairly well recognized, the analysis of dynamic interactive buckling is in practice limited to columns (adopting their beam model), single plates and shells (Ari-Gur and Simonetta, 1997; Huyan and Simitses, 1997; Kołakowski, 2010; Kowal-Michalska, 2007; Papazoglou and Tsouvalis, 1995; Petry and Fahlbusch, 2000; Schokker *et al.*, 1996; Volmir, 1972).

In the world literature, a substantial lack of the nonlinear analysis of dynamic stability of thin-walled structures with complex cross-sections can be felt.

In this study, special attention is focused on the influence of the axial extension mode on static and dynamic coupled buckling with local buckling modes of the thin-walled beam-column with closed cross-section. The axial extension mode should be included in dynamic analysis (Kołakowski, 2010).

1.1.   Static interactive buckling

When components of the displacement state for the first order approximation are taken into account, this can be followed by a decrease in values of global loads. The theoretical static load carrying capacity, obtained within the frame of the asymptotic theory of the nonlinear first order approximation, is always lower than the minimum value of critical load for the linear problem, and the imperfection sensitivity can only be obtained.

According to the assumptions made in Byskov and Hutchinson's theory (Byskov, 1987-88; Byskov and Hutchinson, 1977), local buckling modes do not

interact explicitly. However, an interaction occurs through an interaction of each of them with the global modes. It can be noticed that the global flexural (Euler) buckling can interact with an even number of modes that are symmetric or antisymmetric but the global flexural-torsional mode interacts only with pairs of symmetric and antisymmetric modes (Kolakowski, 1993; Kolakowski and Kowal-Michalska, 1999; Kolakowski *et al.*, 1999; Kolakowski and Teter, 2000; Sridharan and Ali, 1986; Sridharan and Benito, 1984; Sridharan and Peng, 1989; Teter and Kolakowski, 2004). However, the global axial extension mode can interact even with one symmetric mode or with a pair of antisymmetric modes with respect to the axis of overall bending.

Since the late 1980's, the Generalized Beam Theory (GBT) (Basaglia *et al.*, 2008; Camotim *et al.*, 2008; Camotin and Dinis, 2009; Davies, 2000; Silva *et al.*, 2008) has been developed extensively. Recently, a new approach has been proposed, i.e., the constrained Finite Strip Method (cFSM) (Adany and Schafer, 2006a,b; Dinis *et al.*, 2007; Schafer, 2006). These two alternative modal approaches to analyze the elastic buckling behavior were compared in Adany *et al.* (2006, 2007, 2008).

In the current decade, in more and more numerous publications by Adany and Schafer (2006a,b, 2008), Basaglia *et al.* (2008), Camotim *et al.* (2008), Dinis *et al.* (2007), Schafer (2006), Silva *et al.* (2008), attention has been paid to the global axial mode, which is considered only in the theoretical aspect in linear issues, that is to say, under critical loads. Adany and Schafer (2006a), have said that "it should be noted that this axial mode is a theoretically possible buckling mode, even though it has little practical importance". In the axial extension mode, longitudinal displacements of the cross-section dominate and this mode can be referred to as the shortening one (Fig. 1). The axial mode is symmetric with respect to the cross-section axis of symmetry and it is symmetric with respect to the axis of overall bending. The axial extension mode should be included in dynamic analysis (Kołakowski, 2010).
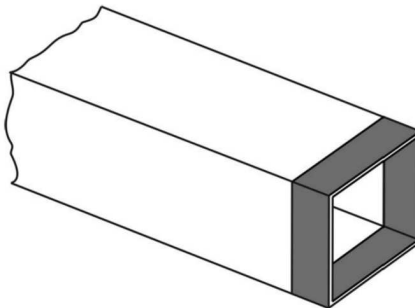

Fig. 1. Longitudinal displacement distributions of the axial mode of the beam-column

A trial to dispute with the statement included in the paper by Adany and Schafer (2006a), has been undertaken. Here special attention has been focused on the eigenvalue problem of the axial mode and on the interaction of the global axial extension mode with local buckling modes for a square box of the column in the first order nonlinear approximation of the perturbation method (Kołakowski and Kowal-Michalska, 2011).

In the present study, a plate model (2D) of the column has been adopted to describe all buckling modes. Instead of the finite strip method, the numerical method of the transition matrix using Godunov's orthogonalization is used in this case. In the solution obtained, the interaction between all the walls of structures being taken into account, the shear lag phenomenon and also the effect of cross-sectional distortions are included. The most important advantage of this method is such that it enables us to describe a complete range of behavior of thin-walled structures from all global to the local stability (Kolakowski and Kowal-Michalska, 1999; Kolakowski and Krolak, 2006; Kolakowski *et al.*, 1999; Kolakowski and Teter, 2000). The solution method has been partially presented in paper by Kolakowski and Krolak 2006.

In the present study, in order to simplify the analysis of the eigenvalue problem, the results obtained for the two-dimensional element have been compared to the results for the one-dimensional element.

1.2. Dynamic interactive buckling

The dynamic pulse load of thin-walled structures can be divided into three categories, namely: *impact* with accompanying perturbation propagation (a phenomenon that occurs with the sound wave propagation speed in the structure), *dynamic load* of a mean amplitude and a pulse duration comparable to the fundamental flexural vibration period, and *quasi-static load* of a low amplitude and a load pulse duration approximately twice as long as the period of fundamental natural vibrations.

The dynamic buckling of a beam-column can be treated as reinforcement of imperfections, initial displacements or stresses in the column through dynamic loading in such a manner that the level of the dynamic response becomes very high. When the load is low, the column vibrates around the static equilibrium position. On the other hand, when the load is sufficiently high, then the column can vibrate very strongly or can move divergently, which is caused by dynamic buckling (i.e., dynamic response). The effects of damping can be neglected in practice.

In the literature on this problem, various criteria concerning dynamic stability have been adopted. The most widely used is the Budiansky-Hutchinson criterion (Budiansky and Hutchinson, 1966; Hutchinson and Budiansky, 1966), in which it is assumed that a loss of dynamic stability occurs when the velocity with which displacements grow is the highest for a certain force amplitude. Other criteria were discussed in papers Ari-Gur and Simonetta (1997), Gantes *et al.* (2001), Huyan and Simitses (1997), Kleiber *et al.* (1987), Kowal-Michalska (2007), Papazoglou and Tsouvalis (1995), Petry and Fahlbusch (2000), Volmir (1972), for instance.

A dynamic response to the rectangular pulse load of the duration corresponding to the fundamental period of flexural free vibrations has been analyzed.

2.   Formulation of the problem

2.1.   Two-dimensional model – plate model

A long prismatic thin-walled beam-column built of panels connected along longitudinal edges has been considered. The beam-column is simply supported at its ends. In order to account for all modes and coupled buckling, a plate model of the thin-walled beam-column has been assumed. The material the column is made of is subject to Hooke's law.

For each plate component, precise geometrical relationships are assumed to enable the consideration of both out-of-plane and in-plane bending of the i-th plate (Kolakowski and Krolak, 2006; Kowal-Michalska, 2007)

$$\varepsilon_{xi} = u_{i,x} + \frac{1}{2}(w_{i,x}^2 + v_{i,x}^2 + u_{i,x}^2) \qquad \varepsilon_{yi} = v_{i,y} + \frac{1}{2}(w_{i,y}^2 + u_{i,y}^2 + v_{i,y}^2)$$

$$2\varepsilon_{xyi} = \gamma_{xyi} = u_{i,y} + v_{i,x} + w_{i,x}w_{i,y} + u_{i,x}u_{i,y} + v_{i,x}v_{i,y}$$

$$(2.1)$$

and

$$\kappa_{xi} = -w_{i,xx} \qquad \kappa_{yi} = -w_{i,yy} \qquad \kappa_{xyi} = -2w_{i,xy} \qquad (2.2)$$

where: u_i, v_i, w_i – components of the displacement vector of the i-th plate along the x_i, y_i, z_i axis direction, respectively, and the plane $x_i y_i$ overlaps the central plane before its buckling.

In the majority of publications devoted to stability of plate structures, the terms $(v_{i,x}^2 + u_{i,x}^2)$, $(u_{i,y}^2 + v_{i,y}^2)$ and $(u_{i,x}u_{i,y} + v_{i,x}v_{i,y})$ are neglected for $\varepsilon_{xi}, \varepsilon_{yi}, \gamma_{xyi} = 2\varepsilon_{xyi}$, correspondingly, in strain tensor components (2.1).

The main limitation of the assumed theory lies in the assumption of linear relationships between curvatures (2.2) and second derivatives of the displacement w (discussed in detail in, e.g., Opoka and Pietraszkiewicz, 2004; Pietraszkiewicz, 1989). This is the most often applied limitation in the theory of thin-walled structures.

From Hamilton's principle, having taken into account relationships (2.1) and (2.2), the following equations of motion are obtained

$$
\begin{aligned}
&N_{x,x} + N_{xy,y} + \{(N_x u_{,x})_{,x} + (N_y u_{,y})_{,y} + (N_{xy} u_{,x})_{,y} + (N_{xy} u_{,y})_{,x}\} \\
&\quad + [-h\rho u_{,tt}] = 0 \\
&N_{xy,x} + N_{y,y} + \{(N_x v_{,x})_{,x} + (N_y v_{,y})_{,y} + (N_{xy} v_{,x})_{,y} + (N_{xy} v_{,y})_{,x}\} \\
&\quad + [-h\rho v_{,tt}] = 0 \\
&M_{x,xx} + M_{y,yy} + 2M_{xy,xy} + (N_x w_{,x})_{,x} + (N_y w_{,y})_{,y} + (N_{xy} w_{,x})_{,y} \\
&\quad + (N_{xy} w_{,y})_{,x} + \left[-h\rho w_{,tt} + \frac{1}{12}h^3\rho(w_{,xxtt} + w_{,yytt})\right] = 0
\end{aligned} \tag{2.3}
$$

The attention has been drawn to the necessity of considering the full strain tensor and all the components of inertial forces in order to carry out a proper dynamic analysis in the whole range of length of the structures.

In the case static issues are analyzed, all dynamic components should be neglected, that is to say, all terms in square brackets should be omitted in (2.3).

After expanding the fields of displacements $\overline{U}$ and the fields of sectional forces $\overline{N}$ into a power series with respect to the mode amplitudes ζ_j (the dimensionless amplitude of the j-th mode), Koiter's asymptotic theory (Koiter, 1976; Koiter and Pignataro, 1976; Koiter and van der Neut, 1980; Kolakowski and Kowal-Michalska, 1999; Kolakowski *et al.*, 1999; Kowal-Michalska, 2007) has been employed

$$
\overline{U} = \lambda \overline{U}_i^{(0)} + \zeta_j \overline{U}_i^{(j)} + \ldots \qquad\qquad \overline{N} = \lambda \overline{N}_i^{(0)} + \zeta_j \overline{N}_i^{(j)} + \ldots \tag{2.4}
$$

where λ is the scalar load parameter, $\overline{U}_i^{(0)}$, $\overline{N}_i^{(0)}$ are the zero (e.g., prebuckling) state fields, and $\overline{U}_i^{(j)}$, $\overline{N}_i^{(j)}$ – j-th mode fields for the i-th plate. The range of indices is $[1, J]$, where J is the number of interacting modes.

For the case, the uncoupled mode for the axial extension mode are $j = J = 1$.

In all earlier publications by the author and his co-workers, it has been assumed that all the modes are normalized so that the maximum normal displacement of component plates of the j-th mode is equal to the first plate thickness h_1. Due to the fact that the axial mode is taken into consideration,

it is assumed in the present paper that the absolute maximum value of one of the components of the displacement field of the j-th mode is equal to the first plate thickness h_1.

The zero approximation describes the pre-buckling state, whereas the first order approximation allows for determination of eigenvalue problems (critical loads/natural frequencies and buckling modes/eigenvectors), being the linear problem of stability and initial post-buckling equilibrium paths.

The obtained static system of homogeneous ordinary differential equations, with the corresponding conditions of the interaction of walls, has been solved by the transition matrix method, having integrated numerically the equilibrium equations along the circumferential direction in order to obtain relationships between the state vectors on two longitudinal edges. During the integration of the equations, Godunov's orthogonalization method is employed. For a more detailed analysis of the solution in the first nonlinear approximation, see papers Kolakowski and Kowal-Michalska (1999), Kolakowski and Krolak (2006), Kolakowski *et al.* (1999), Kolakowski and Teter (2000), Teter and Kolakowski (2004). The solutions for components of the displacement fields for the pre-buckling (zero) and the first order approximation are presented in Kolakowski and Krolak (2006).

The natural frequencies of free vibrations have been determined analogously as in Teter and Kolakowski (2003).

If the components of membrane forces and displacements within the first order approximation are taken into account, a shear lag phenomenon and distortions of cross-sections can be considered then.

For thin-walled structures with initial deflections, Lagrange's equations of motion for the case of the interaction of J eigenmodes can be written as papers by Kowal-Michalska (2007), Schokker *et al.* (1996), Sridharan and Benito (1984), Teter and Kolakowski (2003)

$$\frac{1}{\omega_r^2}\zeta_{r,tt} + \left(1 - \frac{\sigma}{\sigma_r}\right)\zeta_r + a_{ijr}\zeta_i\zeta_j - \zeta_r^*\frac{\sigma}{\sigma_r} + \ldots = 0 \qquad \text{for} \quad r = 1, \ldots, J \quad (2.5)$$

where: ζ_r – dimensionless amplitude of the r-th buckling mode, $\sigma_r = \lambda_r\Delta E$ (at $r = 1, 2, \ldots$) – critical stress instead of the load parameter λ_r of the r-th buckling mode, ω_r, ζ_r^* – frequency of vibrations and the dimensionless amplitude of the initial deflection corresponding to the r-th mode, respectively.

The expressions for a_{ijr} are to be found in papers by Byskov (1987-88), Byskov and Hutchinson (1977), Kolakowski and Kowal-Michalska (1999), Kowal-Michalska (2007). In equations of motion (2.5), the inertia forces of the pre-buckling state have been neglected (Kowal-Michalska, 2007; Schokker *et al.*,

1996; Sridharan and Benito, 1984). The initial conditions have been assumed in the form

$$\zeta_r(t = 0) = 0 \qquad \zeta_{r,t}(t = 0) = 0 \qquad (2.6)$$

The static problem of interactive buckling of the thin-walled channel (i.e., for $\zeta_{r,tt} = 0$ in (2.5)) has been solved with the method presented in Kolakowski and Krolak (2006), the frequencies of vibrations have been determined analogously as in Teter and Kolakowski (2003), whereas the problem of interactive dynamic buckling (2.5) has been solved by means of the Runge-Kutta numerical method modified by Hairer and Wanner.

At the point where the load parameter for static problems σ reaches its maximum value σ_s (the so-called theoretical load carrying capacity) for the imperfect structure with regard to the imperfection of the mode with the amplitude ζ_r^*, the Jacobian of nonlinear system of equations (2.5) is equal to zero.

2.2.   One-dimensional model – beam-column

To simplify the analysis of the axial extension mode, a one-dimensional model of the beam-column has been assumed as well.

In the cases of one-dimensional structural elements, the relationships are reduced significantly. When this element is loaded by the axial compression load only, one speaks of a column but when it is under flexural loads, one calls this element a beam.

For one-dimensional structural elements, precise geometrical relationships (2.1) and equation (2.2) are reduced to the following form

$$\varepsilon_x = u_{,x} + \frac{1}{2}[w_{,x}^2 + u_{,x}^2] \qquad \kappa_x = -w_{,xx} \qquad (2.7)$$

whereas equilibrium equations (2.3) take the form

$$\hat{F}_{,x} + (\hat{F}u_{,x})_{,x} - \rho A u_{,tt} = 0 \qquad \hat{M}_{,xx} + (\hat{F}w_{,x})_{,x} - \rho A w_{,tt} = 0 \qquad (2.8)$$

where: $\hat{F}$ – longitudinal force, $\hat{M}$ – bending moment.

To consider the axial extension mode for the one-dimensional column only, the bending effect has been neglected, i.e., it has been assumed that equations (2.7) and (2.8) are reduced to the form

$$
\begin{aligned}
&\varepsilon_x = u_{,x} + \frac{1}{2}[u_{,x}^2] \\
&\hat{F}_{,x} + (\hat{F}u_{,x})_{,x} - \rho A u_{,tt} \equiv \hat{F}_{,x} + \hat{F}_{,x}u_{,x} + \hat{F}u_{,xx} - \rho A u_{,tt} = 0
\end{aligned}
\qquad (2.9)
$$

To compare the results for the one- and two-dimensional models of the column, identical boundary conditions have been assumed (i.e., the same as for the plate model)

$$u(x = \ell/2) = 0 \qquad u(x = 0) = -u(x = \ell) \qquad (2.10)$$

which corresponds to both ends moving and the bar being fixed in the middle of its length ℓ. Thus, the zero and the first order solutions have been assumed

$$u^{(0)} = (\ell/2 - x)\Delta \qquad u^{(1)} = \overline{U}^{(1)} \cos \frac{m\pi x}{\ell} \qquad (2.11)$$

where Δ is measure of applied pressure.

Having considered (2.4) and (2.11), the strain and the longitudinal force can be expressed as

$$\varepsilon_x = u_{,x} + \frac{1}{2}u_{,x}^2 \approx -\lambda\Delta + \zeta_1 u_{,x}^{(1)}(1 - \lambda\Delta) + 0(\lambda^2, \zeta_1^2, \lambda^2\zeta_1) = \lambda\varepsilon_x^{(0)} + \zeta_1\varepsilon_x^{(1)}$$
$$\hat{F} = EA\varepsilon_x \approx EA(\lambda\varepsilon_x^{(0)} + \zeta_1\varepsilon_x^{(1)}) = \lambda\hat{F}^{(0)} + \zeta_1\hat{F}^{(1)}$$

$$(2.12)$$

and equation of motion $(2.9)_2$ for the first-order approximation (i.e., with respect to ζ_1 – the dimensionless amplitude of the axial extension mode) takes the form

$$u_{,xx}^{(1)}(1 - 3\lambda\Delta) - \frac{1}{c^2}u_{,tt}^{(1)} + 0(\lambda^2, \zeta_1^2, \lambda^2\zeta_1) = 0 \qquad (2.13)$$

where: $c^2 = E/\rho$.

In the above-mentioned equation, the inertia forces of the zero state have been neglected, which is a commonly applied reduction.

For the static problem, the dynamic term, i.e., $u_{,tt}^{(1)} = 0$, should be neglected in relationship (2.13).

Considering (2.11) in the static version of equation (2.13), the critical eigeinvalue is the nontrivial solution

$$\lambda_{cr} = \frac{1}{3\Delta} \qquad (2.14)$$

and the respective critical compression force and the critical stress are then equal to

$$\hat{F}_{cr} = |(\hat{F})_{cr}| = EA\lambda_{cr}\Delta = \frac{EA}{3} \qquad \sigma_{cr} = \frac{\hat{F}_{cr}}{A} = \frac{E}{3} \qquad (2.15)$$

The above relationship defines the critical stress corresponding to the axial extension mode.

As can be easily seen, the static critical value corresponding to the axial mode neither depends on the bar length nor on the number of halfwaves that occur along the bar length ℓ.

In the case of the linear dynamic problem, the frequency of free vibrations can be determined from relation (2.13).

To achieve this, the forecast solution

$$u^{(1)} = \overline{U}^{(1)} e^{i\omega_m t} \cos \frac{m\pi x}{\ell} \tag{2.16}$$

has been substituted into (2.13) and the relationship that determines the free vibration frequency corresponding to the axial extension mode for the bar under compression is obtained

$$\omega_m = \frac{m\pi}{\ell} \sqrt{\frac{E}{\rho}(1 - 3\lambda\Delta)} \tag{2.17}$$

For the bar that is not subject to the longitudinal force (i.e., $\hat{F} = 0$ or $\lambda = 0$), we have

$$\omega_m = \frac{m\pi}{\ell} \sqrt{\frac{E}{\rho}} \tag{2.18}$$

As can be seen, the frequency of longitudinal vibrations of the bar corresponding to the axial extension mode is directly proportional to the number of halfwaves m and inversely proportional to the length ℓ.

In equation (2.13), only the terms standing at ζ_1 and $\lambda\zeta_1$ have been taken into account, whereas the terms of the type $\lambda^2\zeta_1$ have been omitted. If we additionally consider the terms $\lambda^2\zeta_1$ in (2.13), then the lowest critical value is expressed by the formula

$$(\lambda_1)_{cr} = \frac{3 - \sqrt{3}}{3\Delta} = \frac{0.423}{\Delta} \qquad \sigma_{cr} = E(\lambda_1)_{cr}\Delta = 0.423E \tag{2.19}$$

In the developed computer code, according to the characteristics of the solution method presented in Kolakowski and Krolak (2006), the critical value is equal to

$$(\lambda_1)_{cr} = \frac{3 - \sqrt{5}}{2\Delta} = \frac{0.382}{\Delta} \qquad \sigma_{cr} = E(\lambda_1)_{cr}\Delta = 0.382E \tag{2.20}$$

Flexural vibrations of the beam

To determine the frequency of free flexural vibrations of the beam, it has been assumed that $\hat{F} = 0$ and the longitudinal inertia force has been neglected (i.e., $\rho A u_{,tt} = 0$) in equations (2.8). In such a case, the system of equations (2.8), having taken into account that $\hat{M}_x = -EI_z w_{,xx}$, is reduced to the equation

$$EI_z w_{,xxxx} + \rho A w_{,tt} = 0 \qquad (2.21)$$

When the following boundary conditions are assumed for the beam

$$w(x = 0) = w(x = \ell) = 0 \qquad (2.22)$$

then, the frequency of flexural free vibrations of the beam equals

$$\varpi_m = \left(\frac{m\pi}{\ell}\right)^2 \sqrt{\frac{EI_z}{\rho A}} \qquad (2.23)$$

The frequency of beam flexural vibrations is inversely proportional to the squared length of the beam and directly proportional to the squared number of halfwaves that occur along its length.

3. Analysis of the calculation results

To verify the convergence of the calculation results obtained with two different methods, the results obtained by Camotim and Denis (2009) have been compared to the author's results. The results in Camotim and Denis (2009) were obtained for the elastic post-buckling behavior of the cold-formed steel lipped channel (Table 1 – case 3 in Camotim and Denis (2009)) and simply supported columns affected by coupled instabilities with local/distortional/global mode interaction effects through ABAQUS shell finite element analyses. A very good agreement of the results has been received when the axial extension mode is neglected in the considerations.

3.1. Eigenvalue problem

The most difficult for analysis within the first order approximation case of the column cross-section having at least two axes of symmetry, i.e., a thin-walled square cross-section, has been assumed. A detailed analysis of the

Table 1. Natural frequencies of free vibrations, critical stresses of the axial extension mode and ratios of maximum values of displacement components for the plate model corresponding to different lengths

ℓ [mm]	ω_{pl} [rad/s]	ω_{col} [rad/s]	σ_{pl} [MPa]	σ_{col} [MPa]	$\left\|\dfrac{v_{max}}{u_{max}}\right\|$	$\left\|\dfrac{w_{max}}{u_{max}}\right\|$
100000	158.5	158.5	76393		0.00026	0.00026
10000	1585	1585	76392	66666 (Eq. (2.15))	0.0026	0.0029
7500	2114	2114	76392		0.0035	0.0043
5000	3170	3171	76391		0.0052	0.0085
2500	6342	6342	76398		0.01	0.047
2000	7928	7928	76393		0.013	0.021
1000	15839	15857	76379		0.026	0.026
750	21139	21143	76362	76400 (Eq. (2.20))	0.035	0.041
500	31642	31714	76447		0.051	0.238
250	62851	63429	76141		0.11	0.15
200	77423	79286	76352		0.13	0.31
100	175675	158573	75217		0.29	0.32

calculations is conducted for the compressed beam-column of a square cross-section with the following dimensions (Fig. 2) (Kolakowski, 1993; Kolakowski *et al.*, 1999; Teter and Kolakowski, 2004)

$$b_1 = b_2 = b_3 = 100\,\text{mm} \qquad h_1 = h_2 = h_3 = 1\,\text{mm}$$

Each plate is made of steel characterized by the following mechanical properties

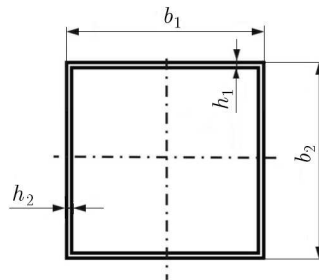$$E = 210\,\text{GPa} \qquad \nu = 0.3 \qquad \rho = 7850\,\text{kg/m}^3$$

Fig. 2. Geometry of the thin-walled beam-column

In Table 1, values of the natural frequencies of vibrations and the critical stresses for the axial extension mode corresponding to different lengths

$100 \leqslant \ell \leqslant 100000$ mm of the column under investigation are shown. Here ω_{pl}, ω_{col} denote the natural frequencies for the plate and the column model, respectively; σ_{pl}, σ_{col} are the critical stresses for the plate and the column model, respectively. The table also includes the ratios of maximum values of displacement components for each mode, i.e., $|v_{max}/u_{max}|$ and $|w_{max}/u_{max}|$ for the plate model of the column. The maximum values of displacement components u_{max}, v_{max}, w_{max} for the particular column length may appear in different points of the beam-column cross-section.

For the plate model of the column, vibration frequencies have been determined, taking into account all components of inertia forces (Kowal-Michalska, 2007; Teter and Kolakowski, 2003). Values of free vibrations determined for both the models under investigation are identical in practice, except for the length $\ell = 100$ mm, where the difference equals approx. 10%. Values of critical stresses, which are identical for the lengths considered for the assumed model, are collected. Values of critical stresses for both the models in the whole range of ℓ are virtually the same if σ_{col} is determined with relationship (2.18).

Values of critical stresses for the axial extension mode determined for the elastic range have very high stresses and one should totally agree with Adany and Schafer's statement quoted in Introduction.

For the plate model of the column whose length is $750 \leqslant \ell \leqslant 100000$ mm, maximum displacements in the cross-section plane (i.e., v, w) are equal to approx. 4% of the longitudinal displacements (i.e., u) at most. The displacements u are practically constant for the cross-section, for example $x = 0$. For the axial mode, the displacements are equal to the thickness h_1 (that is to say, $u_{max} \cong h_1$).

For the column length $\ell \leqslant 500$ mm (i.e., for the case when the dimensions of the cross-section are comparable to the length ℓ), the maximum values of displacement components $|v_{max}|$, $|w_{max}|$ are more than $10\%|u_{max}|$. Here, the assumption of a one-dimensional model to determine the eigenvector can prove insufficient.

A case of static and dynamic interactive buckling for which validity of this statement can be questioned is presented.

Next, detailed computations for the analyzed beam-column of the given length, i.e., $\ell = 2750$ mm, were conducted.

In Table 2, values of the critical stresses σ_r, values of the natural frequencies of free vibrations ω_r and their corresponding numbers of halfwaves m for the square beam-column are shown.

Now, the following index symbols have been introduced: 1 – flexural global mode for $m = 1$; 2 – primary local mode for $m = 28$; 3 – secondary local mode $m = 28$; 4 – axial extension mode for $m = 1$.

Table 2. Critical stresses σ_r and natural frequency ω_r of the beam-column

Index	m	σ_r [MPa]	ω_r [rad/s]
1	1	427.3	266.3
2	28	72.3	3069
3	28	102.7	3659
4	1	76415	5766

As obviously follows from Table 2, the vibration frequencies ω_2 and ω_4 satisfy the relation $\omega_4 = 1.88\omega_2 \approx 2\omega_2$, which in the case of system vibrations is very unfavorable taking into account the energy exchange between the global axial extension mode and the primary local buckling mode.

In Table 3, subsequent values of the natural frequencies of free vibrations ω for $m = 1$ and the ratios of maximum values of displacement components for each mode, i.e., $|v_{max}/u_{max}|$ and $|w_{max}/u_{max}|$, are displayed. The table includes also the condition on the symmetric axis of global bending corresponding to this mode. The following index symbols are introduced: A and S – antisymmetric and symmetric condition on the symmetric axis of global bending, respectively. The natural frequency of free vibrations corresponding to the axial extension mode is the fifth value of the free vibration frequency ($\omega = 5766$) for $m = 1$.

Table 3. Natural frequencies of vibrations ω for $m = 1$

ω $(m = 1)$ [rad/s]	$\left\lvert \dfrac{w_{max}}{u_{max}} \right\rvert$	$\left\lvert \dfrac{v_{max}}{u_{max}} \right\rvert$	Condition on the symmetric axis of global bending
266.3	17.9	17.7	A
1509	145967	18.9	S
2970	41.6	17.6	A
3416	205	0.129	S
5766	0.0186	0.0171	S (axial extension mode)
7646	325	16.2	A

3.2. Static interactive buckling

Detailed results of the static interactive buckling analysis of the first order nonlinear approximation are presented in Table 4 for the beam-column. The imperfections assumed are: $\zeta_1^* = |1.0|$, $\zeta_2^* = |0.2|$, $\zeta_3^* = \zeta_4^* = 0$.

In each case, the sign of the imperfections has been selected in the most unfavorable way, that is to say, as to obtain the lowest theoretical load carrying capability σ_S for the given level of imperfection when the interaction of buckling modes is accounted for.

Table 4 shows ratios of the theoretical load carrying capacity to the primary local critical stress σ_s/σ_2 for the first order nonlinear approximation.

It can be easily seen in Table 4 that when the axial mode is accounted for in the interaction, then the theoretical load carrying capacity σ_s is decreased considerably. A decrease in the load carrying capacity σ_S/σ_2 does not exceed 33%.

Table 4. Theoretical static load carrying capacity σ_s/σ_2 and critical dynamic load factors $DLF_{cr} = \sigma_D^{BH}/\sigma_2$

J-mode approach	Index of the coupled mode	σ_s/σ_2	DLF_{cr}
4	1; 2; 3; 4	0.6730	0.593
3	1; 2; 3	0.820	0.840
3	1; 2; 4	0.6733	0.602
2	2; 4	0.6733	0.610

In the case of the three-mode approach only (i.e., global flexural mode, primary and secondary local modes) for the square cross-section, the coefficients responsible for the interaction of buckling modes a_{122} and a_{133} are identical with zero due to vanishing integrals denoted for the opposed walls of the cross-section (accounting for the symmetry of the local buckling mode). This requires consideration of the interaction of the global flexural mode with the two local ones having the same number of buckling halfwaves m along the longitudinal direction. Apart from the fundamental local mode, the nontrivial local mode considered is referred to as the secondary one (Koiter and Pignataro, 1976; Koiter and van der Neut, 1980; Kolakowski and Kowal-Michalska, 1999; Kolakowski and Krolak, 2006; Kolakowski and Teter, 2000; Moellmann and Goltermann, 1989; Sridharan and Ali, 1986; Teter and Kolakowski, 2004).

The term a_{123}, i.e., the term $a_{123}\zeta_1\zeta_2\zeta_3$ in the potential energy expression, is responsible for the total effect of the interaction between the global flexural mode and local modes.

When additionally the axial extension mode is accounted for in the interaction, the two nonlinear coefficients of system of equations (2.5), namely: a_{123} and a_{422}, exert the main influence on the decrease in the load σ_s.

The key role in the interaction of the buckling mode is played by the coefficient a_{422}, i.e., the term $a_{422}\zeta_4\zeta_2^2$ of the third order in the expression for potential energy. It is related to the term $\sigma^{(4)}L_2(U^{(2)})$, following the notation of Byskov and Hutchinson (Byskov, 1987-88; Byskov and Hutchinson, 1977; Kowal-Michalska, 2007). This term arises from the product stress $\sigma^{(4)}$ associated with the axial mode and the term representing the midsurface strain $L_2(U^{(2)})$ associated with the primary local mode and integrating the same over the structure. The integral tends to vanish unless the longitudinal wave lengths (e.g., the number of halfwaves m) are the same. It makes the coefficient a_{422} play such a key role for the first order approximation (compare cases 1, 3, 4 in Table 4). In this case, the global axial extension mode can interact even with one symmetric local buckling mode.

This interaction of the axial extension mode with the primary local buckling mode can be better visualized by the fact that the additional axial compression $N_{ix}^{(4)}$ (i.e., $\sigma^{(4)}$) dramatically increases the deflections corresponding to the primary local buckling mode $w_i^{(2)}$ (i.e., $L_2(U^{(2)})$).

Thus, it can be stated that the most dangerous mode in the nonlinear problem may not comply with the linear analysis and the maximum values of the coefficients aijr should be defined on the basis of the nonlinear first order approximation.

It follows from this comparison that the consideration of the axial mode in the interaction is necessary as it results in a visible decrease in the theoretical load carrying capacity in the first order approximation.

3.3. Dynamic response

Further on, an analysis of dynamic interactive buckling (i.e., dynamic response) of the square box beam-column under consideration is conducted. Identically as in the static analysis, the interaction of the same modes is considered. A detailed analysis is carried out for a rectangular pulse load $\sigma(t) = \sigma_D$ for $0 \leqslant t \leqslant T_1$ and $\sigma(t) = 0$ for $T_1 < t$. This case corresponds to the pulse duration equal to the period of fundamental flexural free vibrations $T_1 = 2\pi/\omega_1$.

In Table 4, values of the critical dynamic load factors $DLF_{cr} = \sigma_D^{BH}/\sigma_2$ for the square box for the same imperfections under analysis are given, where σ_D^{BH} denotes the critical value of dynamic stress determined from the Budiansky-Hutchinson criterion (Budiansky and Hutchinson, 1966; Hutchinson and Budiansky, 1966).

For the case of the beam-column dimensions under analysis, the obtained values of DLF_{cr} are lower than the respective values of the theoretical static load carrying capacity σ_s/σ_2, except for case 2 in Table 4.

As can be easily seen in Table 4, the axial extension mode shall be included in dynamic stability analysis.

4.   Conclusion

In the present paper, a way of modeling and eigenvalues of the axial extension mode for the one- and the two-dimensional models of the structural elements have been presented. A very good agreement between the results of critical loads and free vibrations has been obtained.

When dimensions of the cross-section are comparable to the column length, the determined eigenvectors can differ considerably for both the models. In the study, special attention has been focused on static and dynamic coupled buckling of the local buckling mode with the global axial extension mode in the first nonlinear approximation of the perturbation method. In the world literature, it is probably the first study, to the author's knowledge, devoted to dynamic interactive buckling of the global axial extension mode and the primary local mode. This problem may be of great significance and it requires further investigations. According to the author's opinion, a further analysis of interactive buckling ought to include the interaction of the axial mode with global and local modes. Therefore, the interactive buckling should be further analyzed and comprehensively and thoroughly discussed.

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Pewne aspekty osiowej postaci wzdłużnej w sprężystej cienkościennej belce-słupie

Streszczenie

W prezentowanej pracy omówiono wpływ osiowej wzdłużnej postaci na statyczne i dynamiczne interakcyjne wyboczenie cienkościennej belki-słupa z niedokładnościami poddanej równomiernemu ściskaniu przy uwzględnieniu zjawiska *shear-lag* oraz dystorsyjnej deformacji. Przyjęto płytowy model (2D) belki-słupa. Porównano jedno- i dwuwymiarowe modele elementów. Konstrukcja jest przegubowo podparta na obu końcach. Równania ruchu płyt składowych otrzymano z zasady Hamiltona, biorąc

pod uwagę wszystkie składowe sił bezwładności. Dynamiczne zagadnienie modalne-
go interakcyjnego wyboczenia w ramach pierwszego rzędu nieliniowej aproksymacji
rozwiązano metodą macierzy przeniesienia, wykorzystując metodę perturbacyjną i or-
togonalizację Godunova.