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MACRO-DYNAMICS OF ELASTIC AND VISCOELASTIC MICRO-PERIODIC COMPOSITES[1]

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In this contribution we propose a new non-asymptotic method of macromodelling of periodic materials and structures. The obtained models can be applied to the analysis of vibration and wave propagation problems with the wavelength of an order of a periodicity cell dimension. The consideration are restricted to the linear elastic and viscoelastic materials and the small deformation gradient theory.

1. Statement of the problem

Asymptotic methods of macro-modelling for micro-periodic structures lead to macro-homogeneous media with the constant (averaged) mass density (cf Bakhvalov and Panasenko, 1984; Bensoussan et al., 1980). Hence, asymptotic equations are not able to describe dispersion effects due to the micro-heterogeneity of composites and can be applied solely to problems in which the time-dependent excitations of the structure produce wavelength much larger then the maximum length dimension of a periodicity cell (a long-wave approximation). The aim of this contribution is to propose a new non-asymptotic method of macro-modelling for micro-heterogeneous composite structures. The obtained equations of micro-macro dynamics can be applied to vibration and wave propagation problems with the wavelength of an order of a cell length dimension (a short-wave approximation).

2. Basic assumptions

The subject of the analysis is a linear elastic or viscoelastic micro-periodic composite body, which in its initial natural state occupies a region Ω in a 3−space

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parametrized by cartesian orthogonal coordinates x_i with a orthonormal basis $\boldsymbol{e}_i$ (i,j,k,l run over $1,2,3$; summation convention holds). The properties of these bodies are determined by a mass density $\rho(\cdot)$ and the tensor of elastic modulae $A_{ijkl}(\cdot)$ (for elastic materials) or by the constitutive function depending on a strain tensor and strain-rate tensor (for viscoelastic materials), which are V-periodic functions, where

$$V \equiv \Big(-\frac{l_1}{2},\frac{l_1}{2}\Big) \times \Big(-\frac{l_2}{2},\frac{l_2}{2}\Big) \times \Big(-\frac{l_3}{2},\frac{l_3}{2}\Big)$$

is a certain representative volume element (r.v.e.). Periods l_1, l_2, l_3 are sufficiently small related to the smallest characteristic length dimension of Ω.

In order to formulate basic hypotheses leading to macro-models of the micro-periodic body, we shall introduce two auxiliary concepts. The first of them is that of a V-macro function. It is a continuous function $F(\cdot)$ defined on Ω which for every $\boldsymbol{x},\boldsymbol{z} \in \Omega$ and $\boldsymbol{z}-\boldsymbol{x} \in V$, satisfies condition $F(\boldsymbol{x}) \cong F(\boldsymbol{z})$. Generally speaking V-macro functions describe the macroscopic behaviour of the body. The phenomena related to a heterogeneous micro-periodic material structure, from the qualitative point of view, will be described by means of independent functions $h_a(\cdot)$, $a = 1,...,n$, which are continuous in Ω and satisfy conditions: $h_a(\boldsymbol{x}) = h_a(\boldsymbol{x} + l_i\boldsymbol{e}_i)$, $<h_a> = 0$ and $<\rho h_a> = 0$ for every $\boldsymbol{x}$, $\boldsymbol{x} + l_i\boldsymbol{e}_i \in \Omega$, $i = 1,2,3$ (no summation over i), where $<\cdot>$ is the known averaging operator defined by

$$<f>(\boldsymbol{z}) = \frac{1}{|V|}\int\limits_V f(\boldsymbol{z}+\boldsymbol{y})dv(\boldsymbol{y})$$

$$dv(\boldsymbol{y}) \equiv dy_1dy_2dy_3$$

for an arbitrary integrable function $f(\cdot)$. Functions $h_a(\cdot)$ will be called micro-oscillatory shape functions. The choice of shape functions has to be postulated a priori in every special problem and depends on the character of micro oscillations which we are going to analyse.

The proposed method of macro-modelling is based on the assumption that displacement fields $u_i(\cdot,\tau)$ at an arbitrary time instant τ can be expected in the form (indices a,b run over $1,...,n$, summation convention holds)

$$u_i(\boldsymbol{x},\tau) = U_i(\boldsymbol{x},\tau) + h_a(\boldsymbol{x})Q_i^a(\boldsymbol{x},\tau) \qquad \boldsymbol{x} \in \Omega \tag{2.1}$$

where $U_i(\cdot,\tau)$, $Q_i^a(\cdot,\tau)$ are arbitrary V-macro functions together with their first and second order material and time derivatives. The V–macro fields $U_i(\cdot,\tau)$ are called macro-displacements and $h_aQ_i^a(\cdot,\tau)$ are oscillations due to the micro-periodic structure of the body. The V-macro functions $Q_i^a(\cdot,\tau)$ constitute the quantitive characteristics of micro-oscillations and will be called corrector fields.

The second assumption of the proposed method of macro-modelling takes into account the micro-oscillatory character of shape functions $h_a(\cdot)$; we shall assume that in formulas for material derivatives of $h_a Q_i^a(\cdot,\tau)$ terms involving $h_a(\cdot)$ can be neglected as small compared to terms involving derivatives $\nabla h_a(\cdot)$. Hence, the approximation

$$u_{i,j}(\boldsymbol{x},\tau) \cong U_{i,j}(\boldsymbol{x},\tau) + h_{a,j}(\boldsymbol{x}) Q_i^a(\boldsymbol{x},\tau) \tag{2.2}$$

will be used in the subsequent analysis.

3. Macro-modelling of elastic composites

The governing equations of the proposed micro-macro elastodynamics will be derived by applying assumptions of Section 2 to the well known action functional

$$\begin{aligned} \mathcal{A} &= \int\limits_{\Omega} \Big[\frac{1}{2}\rho(\boldsymbol{x})\dot{u}_i\dot{u}_i - \frac{1}{2}A_{ijkl}(\boldsymbol{x})u_{i,j}\,u_{k,l} + \rho(\boldsymbol{x})b_i u_i\Big] dv(\boldsymbol{x}) \\ dv(\boldsymbol{x}) &\equiv dx_1 dx_2 dx_3 \end{aligned} \tag{3.1}$$

where b_i are constant body forces. Substituting Eqs (2.1), (2.2) into (3.1), bearing in mind V-macro property of macro-displacements $U_i(\cdot,\tau)$, micro-correctors $Q_i^a(\cdot,\tau)$ and of their derivatives, and using condition $<\rho h_a> = 0$, after some manipulations we obtain the approximation $\mathcal{A} \cong \mathcal{A}_0$, where

$$\begin{aligned} \mathcal{A}_0 &= \int\limits_{\Omega} \Big[\frac{1}{2} <\rho> \dot{U}_i\dot{U}_i + \frac{1}{2} <\rho h_a h_b> \dot{Q}_i^a \dot{Q}_i^b - \frac{1}{2} <A_{ijkl}> U_{i,j}\,U_{k,l} - \\ &- <A_{ijkl}h_{a,j}> U_{k,l}\,Q_i^a - \frac{1}{2} <A_{ijkl}h_{a,j}\,h_{b,l}> Q_i^a Q_k^b + <\rho> b_i U_i\Big] dv(\boldsymbol{x}) \end{aligned} \tag{3.2}$$

Due to V-periodicity of $\rho(\cdot)$, $A_{ijkl}(\cdot)$, $h_a(\cdot)$ all averages in Eq (3.2) are constants which characterize the material and inertial properties of a dynamic system determined by action functional $\mathcal{A}_0$. This dynamic system will represent the macro-model of micro-heterogeneous periodic body. Lagrange equations for $\mathcal{A}_0$ read

$$\begin{aligned} &<A_{ijkl}> U_{k,jl} + <A_{ijkl}h_{b,l}> Q_{k,j}^b + <\rho> b_i = <\rho> \ddot{U}_i \\ &<\rho h_a h_b> \ddot{Q}_i^b + <A_{ijkl}h_{a,j}\,h_{b,l}> Q_k^b = - <A_{ijkl}h_{a,j}> U_{k,l} \end{aligned} \tag{3.3}$$

Eqs (3.3) constitute the governing equations of linear macro-elastodynamics for micro-heterogeneous periodic bodies. The above equations have to be considered together with the boundary and initial conditions for $U_i(\cdot)$ and initial conditions for $Q_i^a(\cdot)$ in the form consistent with Eqs (2.1) and (2.2). It has to be emphasized that the inertial properties of the obtained macro-model are described not only by an averaged mass density $<\rho>$ but also by modulae $<\rho h_a h_b>$ which depend on the length dimensions of the r.v.e. Due to this fact the scale and dispersion effects related to micro-heterogeneous material structure of the body can be investigated. Averages $<\rho h_a h_b>$ will be called micro-inertial modulae. Let us observe that for homogeneous materials

$$<A_{ijkl}h_{a,j}>= A_{ijkl} <h_{a,j}>= 0$$

and hence, under initial conditions

$$Q_i^a(x,0) = \dot{Q}_i^a(x,0) = 0 \qquad\qquad x \in \Omega$$

the second from Eqs (3.3) has only a trivial solution $Q_i^a \equiv 0$, and the first from Eqs (3.3) reduces to the well known equations of the linear elastodynamics. Thus we conclude that the correctors Q_i^a describe the effect of micro-heterogeneity on macro behaviour of the composite.

It has to be remembered that solutions to Eqs (3.3) have a physical sense only if $U_i(\cdot,\tau)$, $Q_i^a(\cdot,\tau)$ are V-macro functions for every τ.

In the asymptotic case term $<\rho h_a h_b> \ddot{Q}_i^b$ drops out from Eqs (3.3) and we arrive at the macro-model proposed by Woźniak (1987).

4. Macro-modelling of viscoelastic composites

Let us observe that Eqs (3.3) can be also written down in the form

$$\begin{aligned} &S_{ij,j} + <\rho> b_i =<\rho> \ddot{U}_i \\ &<\rho h_a h_b> \ddot{Q}_i^b + H_{ai} = 0 \end{aligned} \tag{4.1}$$

where

$$S_{ij} =< A_{ijkl} > U_{(k,l)} + < A_{ijkl}h_{a,(k}> Q^a_{l)}$$

$$H_{ai} =< A_{ijkl}h_{a,j}> U_{(k,l)} + < A_{ijkl}h_{a,j}\, h_{b,(l}> Q^b_{k)}$$

At the same time, setting $\Omega_0 \equiv \{x \in \Omega : V(x) \subset \Omega\}$, and denoting by σ_{ij} components of a stress tensor, by means of $\sigma_{ij} = A_{ijkl}(U_{k,l} + h_{a,k}Q_l^a)$, we obtain

$$\begin{aligned} &S_{ij}(x,\tau) =<\sigma_{ij}> (x,\tau) \\ &H_{ai}(x,\tau) =<\sigma_{ij}h_{a,j}> (x,\tau) \qquad\qquad x \in \Omega_0 \end{aligned} \tag{4.2}$$

The form of Eqs (4.1) ÷ (4.2) is independent of material (elastic) properties of a micro-periodic body under consideration.

Now assume that a composite is made of viscoelastic materials and its properties are described by the stress relation

$$\sigma_{ij} = g_{ij}(\boldsymbol{x}, \boldsymbol{e}, \dot{\boldsymbol{e}})$$

where $\boldsymbol{e}$ is a strain tensor with components $e_{kl} = U_{(k,l)} + h_{a,(k} Q^a_{l)}$ and $g_{ij}(\cdot, \boldsymbol{e}, \dot{\boldsymbol{e}})$ are the known V-periodic functions. Hence, after denotations $\mathbf{E} \equiv (U_{(k,l)})$, $\mathbf{Q} \equiv (Q^a_k)$, we also obtain

$$\sigma_{ij} = f_{ij}(\boldsymbol{x}, \mathbf{E}, \dot{\mathbf{E}}, \mathbf{Q}, \dot{\mathbf{Q}}) \tag{4.3}$$

where $f_{ij}(\cdot, \mathbf{E}, \dot{\mathbf{E}}, \mathbf{Q}, \dot{\mathbf{Q}})$ are also V-periodic functions. Substituting the right-hand sides of Eqs (4.3) into Eqs (4.2) and bearing in mind that $U_{(i,j)}$, Q^a_i are V-macro functions we obtain

$$\begin{aligned} S_{ij}(\boldsymbol{x}, \tau) &= < f_{ij} > (\mathbf{E}(\boldsymbol{x}, \tau), \mathbf{E}(\boldsymbol{x}, \tau), \mathbf{Q}(\boldsymbol{x}, \tau), \mathbf{Q}(\boldsymbol{x}, \tau)) \\ H_{ai}(\boldsymbol{x}, \tau) &= < f_{ij} h_{a,j} > (\mathbf{E}(\boldsymbol{x}, \tau), \dot{\mathbf{E}}(\boldsymbol{x}, \tau), \mathbf{Q}(\boldsymbol{x}, \tau), \dot{\mathbf{Q}}(\boldsymbol{x}, \tau)) \end{aligned} \tag{4.4}$$

It has to be emphasized that due to the V-periodicity of functions f_{ij}, $h_{a,j}$ (as functions of an argument $\boldsymbol{x}$), equations (4.4) are determined for every $\boldsymbol{x} \in \Omega$ (not only for $\boldsymbol{x} \in \Omega_0$!). Hence, Eqs (4.4) can be interpreted as macro-constitutive relations of a micro-periodic viscoelastic body. These equations together with Eqs (4.1) represent the system of governing equations of macro-dynamics for micro-periodic viscoelastic composite bodies.

5. Example

We shall apply Eqs (3.3) to the problem of a straight micro-periodic bar treated as an uniaxial structure. The r.v.e. V is now reduced to the straight-line segment $(-l/2, l/2)$ of the x-axis, $x \equiv x_1$.

We assume that the Young modulus $E(x)$ and mass density $\rho(x)$ are equal to E_1, ρ_1 on $(-a/2, a/2)$ and E_2, ρ_2 on $(-l/2, l/2) \setminus (-a/2, a/2)$, where $0 < a < l$. In this case we introduce only one (continuous and periodic) shape function $h(\cdot)$, $h(x) = h(x + l)$, $x \in \mathcal{R}$, which is piecewise linear and takes the values $h(-l/2) = h(0) = h(l/2) = 0$, $h(-a/2) = l$, $h(a/2) = -l$. After neglecting body forces and setting $(\cdot)' \equiv \partial(\cdot)/\partial x_1$, Eqs (3.3) in this special case yield

$$\begin{aligned} &< E > U''(x, \tau) + < E h' > Q'(x, \tau) = < \rho > \ddot{U}(x, \tau) \\ &< \rho h h > \ddot{Q}(x, \tau) + < E h' h' > Q(x, \tau) = - < E h' > U'(x, \tau) \end{aligned} \tag{5.1}$$

where

$$<\rho>= \frac{\rho_1 a + \rho_2(l-a)}{l} \qquad <E>= \frac{E_1 a + E_2(l-a)}{l}$$

$$<Eh'>= 2(E_1 - E_2) \qquad <Eh'h'>= 4l\Big(\frac{E_1}{a} + \frac{E_2}{l-a}\Big)$$

$$<\rho hh>= \frac{l^2}{3} <\rho>$$

Let us define

$$\mu^2 \equiv \frac{<Eh'h'>}{<\rho hh>} \qquad E^{\rm eff} \equiv <E> - \frac{<Eh'>^2}{<Eh'h'>}$$

and assume

$$\begin{aligned} U(x,\tau) &= U_0(x,\tau)\exp({\rm i}\omega\tau) \\ Q(x,\tau) &= Q_0(x,\tau)\exp({\rm i}\omega\tau) \end{aligned}$$

The analysis of free vibrations leads to the conclusion that:
(i) if

$$\Big(\frac{\omega}{\mu}\Big)^2 < \frac{E^{\rm eff}}{<E>} \qquad \text{or} \qquad \Big(\frac{\omega}{\mu}\Big)^2 > 1$$

then there exist sinusoidal vibrations

$$U_0(x) = A\cos kx \qquad Q_0(x) = B\sin kx$$

(ii) if

$$\frac{E^{\rm eff}}{<E>} < \Big(\frac{\omega}{\mu}\Big)^2 < 1$$

then there exist exponential vibrations

$$U_0(x) = A\cosh kx \qquad Q_0(x) = B\sinh kx$$

(iii) if

$$\Big(\frac{\omega}{\mu}\Big)^2 = 1 \qquad \text{or} \qquad \Big(\frac{\omega}{\mu}\Big)^2 = \frac{E^{\rm eff}}{<E>}$$

then we arrive at degenerated or trivial case, respectively. This classification holds for $<Eh'>\neq 0$; if $<Eh'> = 0$ (i.e. for a homogeneous bar) then only sinusoidal vibrations are possible.

In the case (i) (for sinusoidal vibrations), $U_0(\cdot)$, $Q_0(\cdot)$ are V-macro fields only if $lk \ll 1$. Treating lk as a small parameter it can be shown that

$$\omega^2 = \frac{E^{\rm eff}}{<\rho>}k^2\Big[1 - \frac{1}{3}(lk)^2\frac{<Eh'>^2}{<Eh'h'>^2}\Big] + o(l^2k^2) \tag{5.2}$$

The second term on the right hand side of Eq (4.2) describe the dispersion effect due to the micro-heterogeneous structure of the bar.

In the case (ii) (for exponential vibrations), we can prove that

$$\omega^2 = \frac{<E>}{<\rho>} k^2 \Big[\Big(\frac{\omega}{\mu}\Big)^2 - \frac{E^{\text{eff}}}{<E>}\Big] + \omega^2 \Big(\frac{\omega}{\mu}\Big)^2 \tag{5.3}$$

Eq (4.3) has a physical sense only for micro-heterogeneous bar because in the case of homogeneity $E^{\text{eff}}/<E> = 1$ and there are no exponential vibrations.

The detailed analysis of this problem will be given in a forthcoming paper.

6. Conclusions

The characteristic feature of the proposed approach is that the inertial properties of the obtained macro-model given by Eqs (3.3) and (4.1) are described not only by an averaged mass density $<\rho>$ but also by micro-modulae $<\rho h_a h_b>$ depending on length dimensions of r.v.e. Hence, the equations of micro-macro dynamics makes it possible to investigate the scale and dispersion effects due to the micro–heterogeneity of the body. The advantage of the proposed approach is a relatively simple form of Eqs (3.3) as well as Eqs (4.1), (4.4); from the illustrative example given above it follows that the general theory can be successfully applied to the analysis of engineering problems. The main drawback of the proposed method of macromodelling lies in an unprecise choice of shape functions which is often based on the intuition of the researcher.

Nonlinear treatment of micro-macro dynamics and its applications has been also analyzed and will be presented separately.

7. References

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Makrodynamika sprężystych i lepkosprężystych mikroperiodycznych kompozytów

Streszczenie

W pracy zaproponowano nową nieasymptotyczną metodę makromodelowania sprężystych i lepkosprężystych mikroperiodycznych kompozytów. Otrzymany model może być zastosowany do analizy mikrodrgań i propagacji fal o długościach rzędu wymiarów komórki periodyczności. Rozważania ograniczono do przypadku małych gradientów przemieszczenia (teoria geometrycznie liniowa).

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