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INFLUENCE OF A MEASUREMENT TIME WINDOW ON THE FFT FREQUENCY SPECTRUM

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The rectangular measurement window effects loss of resolution and accuracy of investigated spectrum. In digital signal processing (in particular by FFT method) the problem is not simple and required detailed analysis. In this case the signal and obtained spectrum are given as numerical series of values. It is possible to prove that the discrete form of spectrum accomplishes function of filtr eliminating disturbed informations and distortions don't appear in the spectrum.

1. Introduction

Measurement of every diagnostic signal takes place in a certain limited period of time, it means that it has the beginning and the end. This signal exists independently of the measurement and we can suppose its existence before and after the measurement. This simple fact affects signal processing and final diagnosis resulting in important consequences. A mathematical model of this measurement can be represented by multiplication of the objective signal by zero before and after the measurement and by one during the measurement, respectively. Such an approach reminds observation of a way out of the window. We can see a small part of the way, limited by vertical edges of the window nevertheless we know that the way exists on the right and on the left beyond the field of view. Perhaps for this reason, the signal limiting phenomenon by the beginning and the end of measurement is named: a measurement window. It is well known that a measurement window provokes some disturbances during signal processing particularly in the case of spectral analysis.

The measurement window problem is known and widely presented in scientific bibliography acknowleged as classic now (cf Papoulis, 1962; Bath, 1974; Oran, 1978; Otnes and Enochson, 1978; Liferman, 1980; Bendat and Piersol, 1980). The window effect is treated as one of the reasons for spectrum disturbances (cf Papoulis, 1962; Bath, 1974; Kurowski, 1978; Liferman, 1980; Randall, 1987; Bendat and Piersol, 1980) or spectrum efluent (cf Enochson, 1977; Otnes and Enochson, 1978).

In all these papers the spectrum disturbances problem is treated generaly. Without the thorough exact analysis, one accepts that these disturbances are considerable and recommends time weighting functions as way of the window correction. In the case of investigation of a continuous signal spectrum the disturbances are evident and easy to estimate. However, in the case of digital signal processing the problem is more complex and requires a detailed analysis. Examination of the measurement effect on a spectral analysis of processing signal is the aim of this paper. It concerns in particular processing by FFT of a signal numerical representation obtained from analog-to-digital conversion.

2. Mathematical model of the time limited measurement

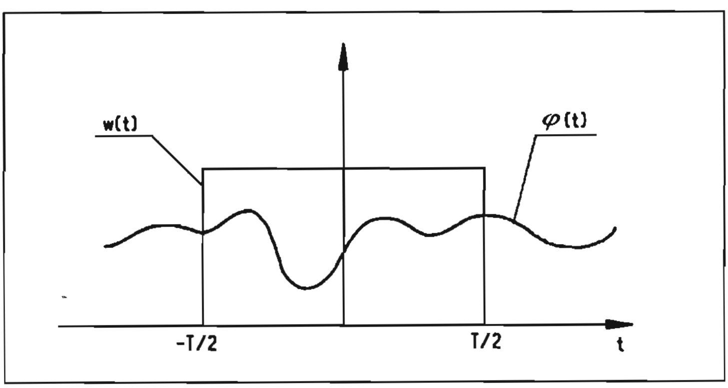

Fig. 1.

Let $\varphi(t)$ represent a signal generated by the investigated object. Let us assume that $\varphi(t)$ satisfies the Dirichlet conditions and is determined in time domain $t \in (-\infty, +\infty)$. This signal is shown in Fig.1; position of a zero point on the time axis results from an arbitrary choice and is not important. Let us suppose now that this signal is received for further processing. The receiving time $T > 0$, synonymous with the time of measurement, is determined by the measurement conditions, a kind of processing to be realised etc. It is well known that the longer measurement time assures receiving of more informations contained in the signal. On the other hand this time should not be too long because both the invesigated object and the measurement system exhibits tendency to lose stability, and the memory capacity of data preservation system is limited.

2.1. Measurement window function

Function $w(t)$ representing a measurement window can be written as the pulse function

$$w(t) = \begin{cases} 1 & \text{for} \quad -\frac{T}{2} \le t \le \frac{T}{2} \\ 0 & \text{for} \quad t < -\frac{T}{2} \text{ and } t > \frac{T}{2} \end{cases} \tag{2.1}$$

This function represents a rectangular time window of real measurement and only this type of the window will be considered in this paper.

Scientific sources publish variuos forms of the Fourier transform of the function $w(t)$. For this reason this transformation will be used below. Inserting $w(t)$ into the Fourier integral we have

$$W(\omega) = \int\limits_{-\infty}^{\infty} w(t)\exp(-j\omega t)\, dt$$

In result, the integral between the limits minus infinity and plus infinity can be written as a definite integral in the following form

$$W(\omega) = \int\limits_{-\frac{T}{2}}^{\frac{T}{2}} \exp(-j\omega t)\, dt$$

After integration we have

$$W(\omega) = \frac{\exp(-j\omega t)}{-j\omega}\Big|_{-\frac{T}{2}}^{\frac{T}{2}} = \frac{\cos\omega t - j\sin\omega t}{-j\omega}\Big|_{-\frac{T}{2}}^{\frac{T}{2}}$$

From this expression, for limits $-T/2$ and $T/2$ and for frequency $\omega = 2\pi f$ we obtain the Fourier transform of the rectangular window as

$$W(f) = \frac{\sin\pi f T}{\pi f} \tag{2.2}$$

The window function Fourier transform (2.2) is the same as the form presented by Bath (1974).

Because the form $\sin\alpha/\alpha$ is easier to be consider (properties of this function are well known) it is useful to write the transform (2.2) in the form (2.3). In the formula (2.2) a mid-band frequency has assumed a zero value. Generally this frequency can be of any value f_i. Therefore we can adopt as an argument of the function the difference representing frequency interval $f - f_i$, where f is running variable, and f_i the discrete value of a mid-band frequency. We have finally

$$W(f) = T\frac{\sin[\pi(f - f_i)T]}{\pi(f - f_i)T} \tag{2.3}$$

Functions (2.1) and (2.2) are shown in Fig.2a and 2b.

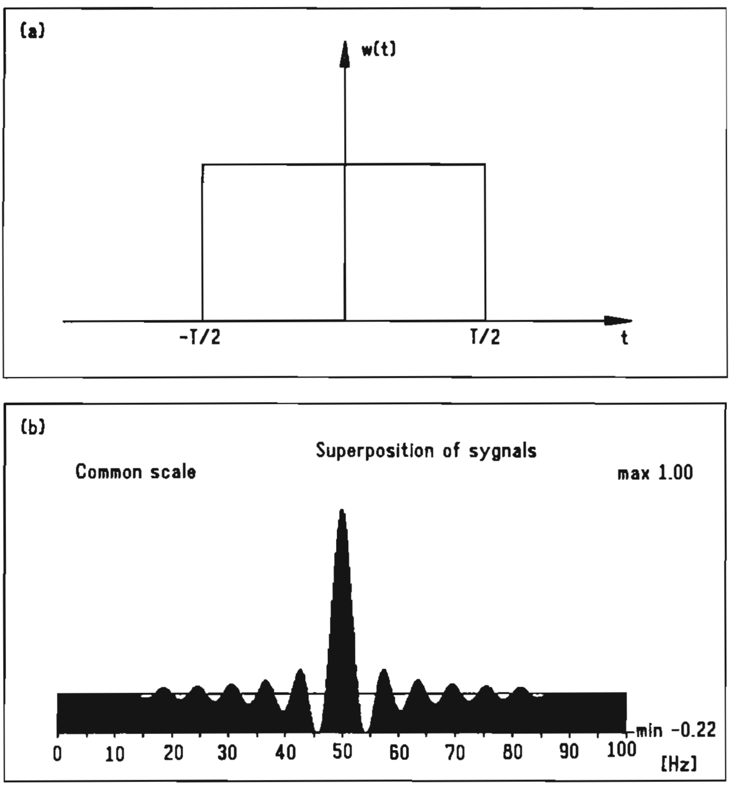

Fig. 2.

Let us consider the function $W(f)$. We may easily assert that this function takes certain characteristic values for the argument $\pi f T = \pm n\pi$, where $n = 0, 1, 2, ...$ For $n = 0$ the argument $\pi f T = 0$ (it means that $f = 0$ or $f - f_i = 0$), limit of this function is equal one. This case corresponds with the mid-band frequency or the $W(f)$ in Fig 2b. For all values of the integer n different from zero the frequency $f = \pm n/T$, and the transform $W(f)$ is equal to zero.

We can also notice that for the measurement time growing, the function $W(f)$ in Fig.2b becomes more and more narrow and in the limit it takes the form of the Delta function.

Assuming time $T = 1$, we can easily calculate that for $f = 1.5T$ it appears a side lobe local maximum of the absolute value equal to 0.21 of the peak value for $f = 0$. Next side lobe local maximum appears for $f = 2.5T$ and it is equal to 0.12 of the peak value for $f = 0$. In the same way we can calculate other side lobe maximums appearing for $f = 3.5T; 4.5T; \ldots$ We can remark that these maximums appear between zero values of $W(f)$ and they correspond to fractional values of n.

This considerations give an idea about a quality and quantity of errors which can appear in result of an unintended superposition of the window spectrum on the signal spectrum.

2.2. The measurement in the time limited

Let us assume that $f(t)$ represents the measured signal. Basing on the previous considerations we can write it in the form

$$f(t) = \varphi(t)w(t) \tag{2.4}$$

where the function $\varphi(t)$ represents the signal existing independently of the measurement.

The Fourier integral of $f(t)$ is

$$F(f) = \int_{-\infty}^{\infty} f(t)\exp(-j2\pi ft)\, dt \tag{2.5}$$

Substituting the expression (2.4) into (2.5) we obtain a transform of the signal with the measurement window

$$F(f) = \int_{-\infty}^{\infty} \varphi(t)w(t)\exp(-j2\pi ft)\, dt \tag{2.6}$$

The formula (2.6) is equivalent to Eq (2.5) with limits of integration: $-T/2, T/2$.

The Fourier transform of two functions product is equal to a convolution of their transforms

$$F(f) = \Phi(f) * W(f) \tag{2.7}$$

The convolution multiplication can be presented in an analytical form. Considering the expression (2.3) we obtain

$$F(f) = \frac{T}{\pi}\int_{-\infty}^{\infty} \Phi(f)\frac{\sin\pi(f-f_i)T}{(f-f_i)T}\, df \tag{2.8}$$

The expression (2.8) represents a function $\sin\alpha/\alpha$ "extended" on frequency peaks of the spectrum $\Phi(f)$. We can easily note that

$$F(f) \neq \Phi(f)$$

3. Transform of the signal with measurement window

In reality, after measurement we can obtain only signal $f(t)$ with a measurement window and only the transform of this signal can be determined in a signal processing. For investigator the objective source of informations about the object is the signal $\Phi(t)$ and its characteristics. But a measurement of the signal without a measurement window is impossible. For this reason a good knowkledge of the errors introduced by this rectangular measurement window is very usefull.

The signal generated by a real object can be treated as a superposition of many harmonic components of their mutual couplings and of stochastic noises. Informations about these components permit setting up the diagnosis of investigated object. It seems to be resonable to propose a simplified model of the signal in the form of sum of deterministic components. For these considerations we can neglect the noise, we can also assume that phase displacements of every component equel zero. Such a simplified signal can be presented in the form

$$\varphi(t) = \sum_{i=0}^{m} A_i \sin 2\pi f_i t \tag{3.1}$$

where: A_i – amplitude of ith component, f_i – frequency of ith component, $i = 1, 2, ..., m$, sequence of components of the signal. The Fourier transform of the signal (3.1) can be given as

$$\Phi(f) = \int\limits_{-\infty}^{\infty} \Big(\sum_{i=0}^{m} A_i \sin 2\pi f_i t\Big) \exp(-\mathrm{j}2\pi f t)\, dt \tag{3.2}$$

The Fourier transform of the same signal with measurement window determined by Eq (2.1) can be presented as

$$F(f) = \int\limits_{-\infty}^{\infty} \Big(\sum_{i=0}^{m} A_i \sin 2\pi f_i t\Big) w(t) \exp(-\mathrm{j}2\pi f t)\, dt \tag{3.3}$$

We can note that the window function is applied to each term of the sum.

Remembering Eqs (2.7) and (2.8) we shall determine the transform (3.2) of the signal (3.1) for frequencies $f_i \geq 0$. We have

$$\Phi(f) = \sum_{i=0}^{m} \frac{-\mathrm{j}A_i}{2}[I(f - f_i)] \tag{3.4}$$

where I – delta Dirac function for frequency $f > 0$. Eq (3.4) presents the series of peak pulses, each of modulus $A_i/2$, ovelaping the negative direction of the imaginary axis.

Substituing Eqs (2.3) and (3.4) into (2.8) and integrating this exspression we obtain the transform (3.5) of the signal $\varphi(t)$ with the rectangular time window

$$F(f) = \sum_{i=0}^{m} \frac{-\mathrm{j}A_iT}{2} \frac{\sin\pi(f - f_i)T}{\pi(f - f_i)T} \tag{3.5}$$

Expression (3.5) represents the sum of terms $\sin\alpha/\alpha$ of the maximum value, each of them equal $A_iT/2$, where $i = 0, 1, 2, ..., m$. Every ith maximum value corresponds to ith harmonic component.

4. Distortion of spectrum evoked by the measurement window

The previous considerations have proved that the transform of the signal $f(t)$ is evidently different from the transform of the signal $\varphi(t)$. It seems to be necessary to analyse the errors evoked by the measurement time window. These considerations will be carried out for two kinds of a signal processing: analog and numerical. The analog way concerns a general case of determination of the spectrum by a direct measurement or by an analytical computation. The numerical way concerns the computation of the spectrum by the FFT method from a digital representation of the signal.

4.1. General case

For analysis of this case, the signal (4.1) has been simulated on a microcomputer

$$\varphi(t) = A_i \sin 2\pi f_1 t + A_2 \sin 2\pi f_2 t \tag{4.1}$$

where

$$-\frac{A_1}{2} = -\frac{A_2}{2} = 1 \qquad f_1 = 300 \text{ Hz} \qquad f_2 = 700 \text{ Hz}$$

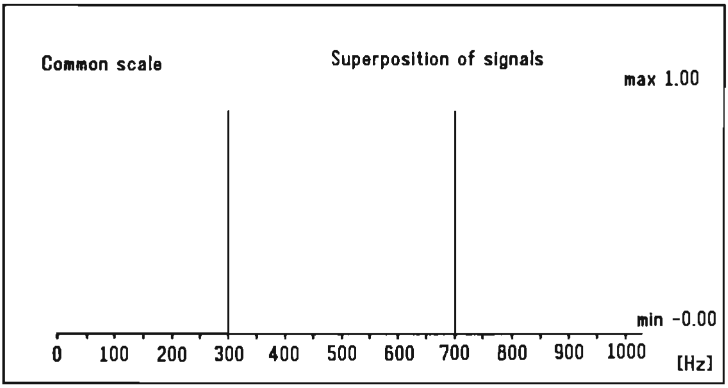

Fig. 3.

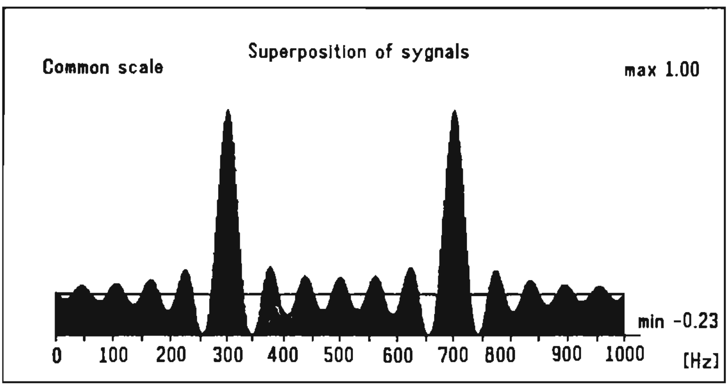

Fig. 4.

The spectrum of this signal, conformable to the expression (3.4), is shown in Fig.3. Then we have computed the spectrum of signal $f(t)$ according to Eq (4.1) with the rectangular time window (2.1). The time of measurement equal to 1/64 s has been assumed. The results of computation are presented in Fig.4; the amplitude values are divided by the time of measurement value. Fundamental informations about the signal $\varphi(t)$ presented in Fig.3 and in Fig.4 are similar. But we can easy forsee that the problem with accuracy appears when the frequencies f_1 and f_2 come near and courses of $\sin\alpha/\alpha$ "extended" on this frequencies are going to superpose. In Fig.5 and 6 are shown spectrums of signals $\varphi(t)$ and $f(t)$, respectively, for frequencies $f_1 = 490$ Hz and $f_2 = 510$ Hz. The spectrum of $\varphi(t)$ is agrees with our expectations but the spectrum of $f(t)$ differs from that one very much. We can notice a qualitative distorsion (one component instead of two) and a quantitative one (an amplitude almost two times greater). These two types of distortion result from a superposition of two adjacent courses and effect the loss of accuracy and resolution of two components.

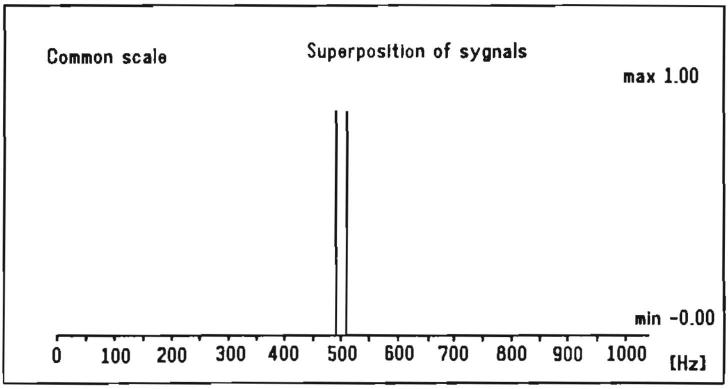

Fig. 5.

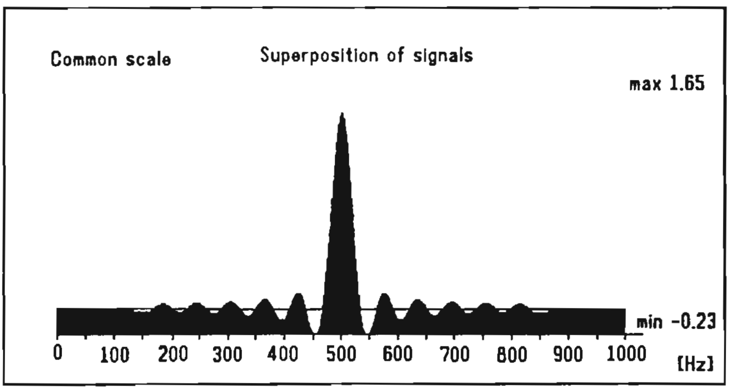

Fig. 6.

Analysing the values of function $\sin \pi(f - f_1)T/\pi(f - f_2)$, one can notice that the resolution between two adjacent components with the error less than 20% is possible for the interval $f_1 - f_2 = \Delta f \geq \Delta f_{\min}$. This value depends on the measurement time T value and can be estimated as $\Delta f_{\min} \geq 1.3/T$. For the example mentioned above the measurement time value has been assumed $T = 1/64$ s and it gives the interval frequency value $\Delta f_{\min} \sim 80$ Hz. So short a time has been taken for good illustration of the spectrum distortions. Under normal conditions this measurement time $T = 0.1 \div 1.0$ s and it results in the spectrum components discrimination $\Delta f = 1 \div 1.5$ Hz.

4.2. The spectrum computed by the FFT method

For computation of the transform with the aid of FFT, the signal has to be presented in the form of a numerical time series. The series is obtained by an analog-to-digital conversion with a certain sampling frequency. Total time of the conversion is equal to the signal measurement time.

Let us assume that the signal $\varphi(t)$ is sampled during the time T with sampling frequency f_e. Characteristics of the obtained train are following:

- number of samples $N = T/f_e$,
- the lowest frequency discriminated from the spectrum $f_1 = 1/T$ [Hz],
- Nyquist frequency $f_N = N/2T$ [Hz].

Computed by the FFT method the spectral lines will be arranged in sequence along the frequency axis, discretely for frequencies $f_n = n/T$, where $n = 0, 1, 2, ..., (N/2 - 1)$. The minimal interval between two succesive frequencies determining the frequency line discrimination can be preseted as

$$\Delta f_{\min} = \frac{n+1}{T} - \frac{n}{T} = \frac{1}{T} \tag{4.2}$$

The $\Delta f_{\min}$ value determines the minimal interval separating two succesive frequency informations of the descrete spectrum and it is equal to the lowest frequency discriminated from the spectrum. One can say that in the spectrum computed by FFT the frequency information appears only at points corresponding to the frequencies: $0/T, 1/T, 2/T, ...$ In intervals separating these points this information is equal to zero.

We may suppose that for some of these frequencies spectral peak lines will appear. Let us assume that these lines appear for frequencies $f_i = n_i/T$ (where $i = 0, 1, 2, ..., m$), corresponding to m the values of f_n (where $n = 0, 1, 2, ..., N/2 - 1$).

Substituing numerical values for frequencies f_n and f_i into Eq (3.5) we obtain the transform for the signal determined in the form of the of samples

$$F(n) = -\frac{\mathrm{j}T}{2} \sum_{i=0}^{m} A_i \frac{\sin \pi(n - n_i)}{\pi(n - n_i)} \tag{4.3}$$

Since n and n_i are always integers the difference $n - n_i$ will be also always an integer, it means $n - n_i = k$, where $k = 0, \pm 1, \pm 2, ...$ Numerical form of the expression (4.3) can be written as

$$W(n) = \frac{\sin \pi k}{\pi k} \tag{4.4}$$

The value $k = 0$ represents $n = n_i$ and we have the spectral peak line of ith harmonic component. The index n_i corresponds with mid-band frequency and then $W(n) = 1$. For the other integer values of $k \neq 0$, the transform $W(n) = 0$ because: $\sin \pi k = 0$ and $\pi k \neq 0$. Taking into account this reasoning in the expression (4.3) we can write

$$F(n) = \begin{cases} 1 & \text{for} \quad n = n_i \\ 0 & \text{for} \quad n \neq n_i \end{cases} \tag{4.5}$$

The same result can be obtained directly after considerations of the section 2.1. Frequency informations of the discrete spectrum are given on frequency axis only in points determined by Delta distribution function. One knows (Zemanian, 1969) that it is the Delta distribution carrier and that it is defined as the set of all points where this distribution is not equal zero. For the distribution defining the FFT spectrum, the carrier is the set, composed of points: $0/T, \pm 1/T, \pm 2/T, ...$ If, for the point n_i appears a spectral peak line corresponding to the frequency f_i, the nearest points where the information is not equal zero (conformable to the definition of the spectrum computation method) will be $n_i \pm 1, n_i \pm 2, ...$ These points correspond to the frequencies $f_i \pm 1/T, f_i \pm 2/T, ...$, distant always of the entire rate of the value $\Delta f_{\min} = 1/T$. For these points the function $\sin \alpha / \alpha$ given by expression (2.3) takes values equal to zero. In consequence in the transform $F(f)$ given by formulas (2.8) and (3.5) the term representing the rectangular window transform will be equal to zero at these points.

5. Conclusions

For an analog processing of the signal, the distortion resulting from the existance of the measurement window limits the resolution between the spectral lines. In the case of signal processing by FFT, both the signal and the obtained spectrum are given as a numerical series of values. The distortion values correspond to the frequencies at which the spectral amplitude informations are equal to zero. The discrete form of the spectrum accomplishes function of filter eliminating distorbed informations and they don't appear in the spectrum.

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Wpływ okna pomiarowego na widmo częstotliwości uzyskane metodą FFT

Streszczenie

Każdy rzeczywisty pomiar sygnalu odbywa się przez pewien czas (ma zatem początek i koniec) i tylko w tym czasie (w tzw. oknie pomiarowym) sygnał istnieje dla badacza. W konsekwencji dalsza obróbka musi dotyczyć iloczynu sygnalu istniejącego niezależnie od pomiaru oraz okna pomiarowego reprezentującego czas pomiaru. Przy wyznaczania np widma, transformacji podlega ten iloczyn co wprowadza poważne zakłócenia rezultatów.

W artykule omówiono zniekształcenia widm wynikające z istnienia naturalnego, prostokątnego okna pomiarowego. Uwzględniono analogowy i numeryczny sposób obróbki sygnalu. Dla obróbki analogowej (np analityczna transformacjia sygnalu), stwierdzono wyraźną utratę rozróżnialności i dokładności prążków w widmie.

W przypadku obróbki cyfrowej, widma częstotliwościowe oblicza się na podstawie ciągu wartości uzyskanych z próbkowania sygnalu przez czas T z wybraną częstotliwością próbkowania. Powszechnie stosowana do obliczania widma metoda FFT daje ciąg dyskretnych prążków widmowych, rozlożonych na osi częstotliwości w odstępach $1/T$. Tylko w tych punktach istnieje informacja częstotliwościowa, zaś między nimi jest ona równa zero.

W artykule udowodniono, że zniekształcenia widma wywołane istnieniem prostokątnego okna pomiarowego pojawiają się właśnie pomiędzy tymi punktami tzn. tam gdzie informacja częstotliwościowa jest rówa zero. Zaś w punktach gdzie pojawiają się prążki widmo nie jest obciążone błędem. W konsekwencji zniekształcenia wywolane istnieniem okna nie są widoczne w widmie uzyskanym metodą FFT.

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