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ON AN OBJECTIVE FUNCTION IN OPTIMIZATION PROBLEMS WITH A LOSS OF DYNAMIC STABILITY

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The paper concerns optimization problems at the loss of dynamic stability. The transverse vibrations of a few periodically loaded rods for different boundary conditions are considered. The rods have variable cross-sections and are made of viscoelastic material. Parameters determining the shape of the rods are defined. The aim of the paper is to introduce an objective function which is a measure of dynamic instability region. Hitherto for similar systems the amplitude of the exciting force oscillating term was the objective function. It appears that the optimality of system depends upon the objective function. The problem of system optimization at a parametric resonance is a complicated one – a few objective functions must be taken into account. The paper indicates some problems. The considerations are limited to a one-degree-of-freedom problem which needs further analysis.

1. Introduction

Lately the paper occur [1,2,3] concerning problems of optimization at the loss of dynamic stability. The papers [2,3] refer to internal resonance of parametric character in a system of rods. The amplitudes of vibration of parametrically excited elements were the objective functions. On the other hand in [1] the amplitude of the follower force oscillating term was the objective function.

A matter of this paper is to introduce, for problems connected with optimization at the loss of dynamic stability, such an objective function which would be a measure of instability region. The results of paper [1] indicate that the system is most stable if the rod is shaped like a cone. Then the amplitude of the follower force oscillating term is maximal but instability region is widest – the system is unstable in a large frequency range. Hence the idea of introducing a new objective function occurs.

We will consider a few kinds of rods with different boundary conditions. The problem of loss of stability for the prismatic rods is considered in detail in [4]. Methods of instability regions construction for such situations are known.

For the problems of optimization of rods with variable cross–sections we will look for such parameters of the rod shape for which some measures of dynamic instability regions (e.g. an area connected with the region or the width of it at a fixed place) will be minimal.

2. Differential equations of motion

The rods are made of Kelvin – Voigt viscoelastic material with the square cross–sections. Equation of motion of non–prismatic rod under the action of a longitudinal, periodically varying in time, force $P(t)$ has the form

$$\frac{\partial^2}{\partial x^2}\Big[EI(x)\frac{\partial^2 w}{\partial x^2} + \lambda I(x)\frac{\partial^3 w}{\partial x^2 \partial t}\Big] + \rho A(x)\frac{\partial^2 w}{\partial t^2} + P(t)\frac{\partial^2 w}{\partial x^2} = 0 \tag{2.1}$$

where $w(x,t)$ is a transverse displacement of the cross–section x in the time t, $I(x)$ is cross–sectional moment of inertia, $A(x)$ is cross–sectional area, E is Young's modulus, λ coefficient of internal damping and ρ mass density.

For example we will consider three different cases (Fig.1). For the situation presented in Fig.1a the boundary conditions are

$$\begin{aligned}
&\Big[EI(x)\frac{\partial^2 w}{\partial x^2} + \lambda I(x)\frac{\partial^3 w}{\partial x^2 \partial t}\Big](0,t) = 0, \qquad w(0,t) = 0\\
&\Big[EI(x)\frac{\partial^2 w}{\partial x^2} + \lambda I(x)\frac{\partial^3 w}{\partial x^2 \partial t}\Big](l,t) = 0, \qquad w(l,t) = 0.
\end{aligned} \tag{2.2}$$

However for the case like in Fig.1b the boundary conditions are following

$$\begin{aligned}
&w(0,t) = \frac{\partial w}{\partial x}(0,t) = 0\\
&w(l,t) = 0, \qquad \Big[EI(x)\frac{\partial^2 w}{\partial x^2} + \lambda I(x)\frac{\partial^3 w}{\partial x^2 \partial t}\Big](l,t) = 0.
\end{aligned} \tag{2.3}$$

We look for approximated solutions to above problems in the form of series of eigenfunctions of non–damped, natural vibrations

$$w(x,t) = \sum_{i=1}^{I} q_i(t) v_i^{(f)}(x) \tag{2.4}$$

where the form of functions $v_i^{(f)}(x)$ depend on the considered case; $f = 1,2$

$$v_i^{(1)}(x) \quad = \quad \sin\frac{i\pi x}{l} \tag{2.5}$$

$$
\begin{aligned}
v_i^{(2)}(x) &= \cos\lambda_i\Big(\operatorname{sh}\frac{\lambda_i x}{l} - \sin\frac{\lambda_i x}{l}\Big) - \\
&- \sin\lambda_i\Big(\operatorname{ch}\frac{\lambda_i x}{l} - \cos\frac{\lambda_i x}{l}\Big) \qquad (2.6)\\
\lambda_1 &= 3.9266.
\end{aligned}
$$

Inserting (2.4) to (2.1), multiplying it by $v_m^{(f)}(x)$ and integrating from 0 to l one gets

$$\sum_{i=1}^{l}\Big[\tilde{A}_{im}\ddot{q}_i + \lambda\tilde{B}_{im}\dot{q}_i + (E\tilde{B}_{im} + P(t)\tilde{C}_{im})q_i\Big] = 0 \qquad (2.7)$$

where

$$
\begin{aligned}
\tilde{A}_{im} &= \rho\int_0^l A(x)v_i^{(f)}(x)v_m^{(f)}(x)dx\\
\tilde{B}_{im} &= \int_0^l I(x)v_i^{(f)''}v_m^{(f)''}dx \qquad (2.8)\\
\tilde{C}_{im} &= \int_0^l v_i^{(f)''}v_m^{(f)}dx.
\end{aligned}
$$

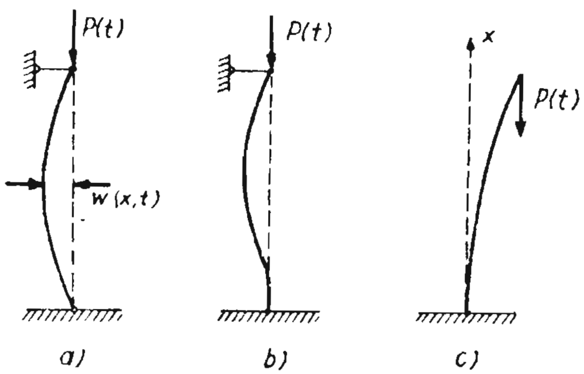

Fig. 1.

The similar set of equations we get for the third situation represented in Fig.1. This system of equations has been obtained on the base of the Lagrange's equations of the second kind. We introduce some parameters describing the rod shape. For the case like in Fig.1a we assume that the side of quadratic section $a^{(1)}(\alpha, \kappa^{(1)}, x)$

changes as quadratic function of x (Fig.2) and has the form

$$a^{(1)}(\alpha,\kappa^{(1)},x) = \alpha\Big\{4\kappa^{(1)}\Big(\frac{x^2}{l^2}-\frac{x}{l}\Big)+1\Big\} = \alpha\varphi^{(1)}(\kappa^{(1)},x) \tag{2.9}$$

where

$$\kappa^{(1)} = \frac{\alpha-\beta}{\alpha}, \qquad a(0)=a(l)=\alpha, \qquad a(\frac{l}{2})=\beta, \qquad \kappa^{(1)}\in(-\infty,1]. \tag{2.10}$$

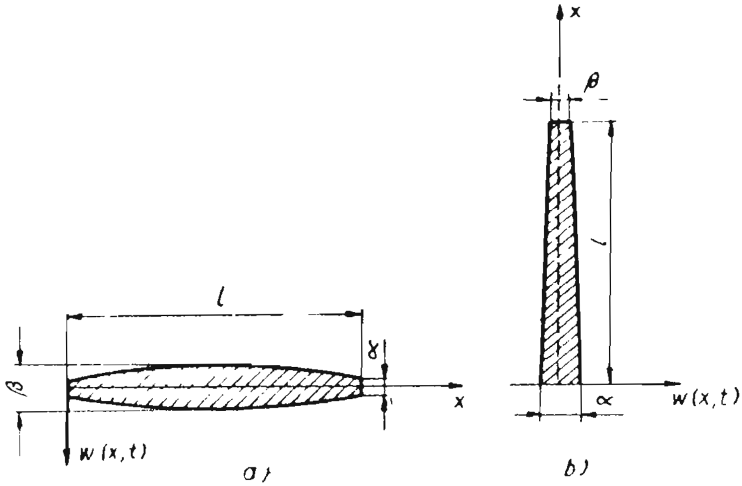

Fig. 2.

For the cases like in Fig.1b and Fig.1c we assume that the side of cross-section $a^{(2)}(\alpha,\kappa^{(2)},x)$ changes as a linear function of x

$$a^{(2)}(\alpha,\kappa^{(2)},x) = \alpha(1-\kappa^{(2)}\frac{x}{l}) \equiv \alpha\varphi^{(2)}(\kappa^{(2)},x) \tag{2.11}$$

where

$$\kappa^{(2)} = \frac{\alpha-\beta}{\alpha}, \qquad a(0)=\alpha, \qquad a(l)=\beta, \qquad \kappa^{(2)}\in(-\infty,1]. \tag{2.12}$$

So we obtain

$$A^{(k)} = [\alpha\varphi^{(k)}]^2, \qquad I^{(k)} = \frac{1}{12}[\alpha]^4[\varphi^{(k)}]^4 \tag{2.13}$$

where for $k=1$ one should take (2.9), (2.10), whereas for $k=2$ one should take (2.11), (2.12). Inserting these formulae to (2.7) and (2.8) we get

$$\sum_{i=1}^{l}\Big[\ddot{q}_i A_{im}^{(k,f)} + \tau\dot{q}_i B_{im}^{(k,f)} + \Big(B_{im}^{(k,f)} + P(t)C_{im}^{(f)}\Big)q_i\Big] = 0, \qquad \tau=\frac{\lambda}{E} \tag{2.14}$$

where

$$
\begin{aligned}
A_{im}^{(k,f)} &= \tilde{A}_{im} = \rho[\alpha]^2 \int_0^l [\varphi^{(k)}]^2 v_i^{(f)} v_m^{(f)} dx \\
B_{im}^{(k,f)} &= E\tilde{B}_{im} = E\frac{1}{12}[\alpha]^4 \int_0^l [\varphi^{(k)}]^4 v_i^{(f)''} v_m^{(f)''} dx \\
C_{im}^{(f)} &= \int_0^l v_i^{(f)''} v_m^{(f)} dx
\end{aligned}
\tag{2.15}
$$

(k) denotes the shape of the rod, $k = 1,2$; (f) denotes the applied mode of oscillation, $f = 1,2,3$.

3. Dynamic stability

To simplity the further considerations only the first term of series is taken into account. So the equation of motion is now

$$
\ddot{q}A^{(k,f)} + \tau\dot{q}B^{(k,f)} + q[B^{(k,f)} + P(t)C^{(f)}] = 0 \tag{3.1}
$$

where

$$
\begin{aligned}
A^{(k,f)} &= \rho[\alpha]^2 \int_0^l [\varphi^{(k)}]^2 [v^{(f)}]^2 dx \\
B^{(k,f)} &= E\frac{1}{12}[\alpha]^4 \int_0^l [\varphi^{(k)}]^4 [v^{(f)''}]^2 dx \\
C^{(f)} &= \int_0^l v^{(f)''} v^{(f)} dx.
\end{aligned}
\tag{3.2}
$$

Now we define natural frequency $\omega^{(k,f)}$, critical force $P^{(k,f)}$ and exciting parameter $\mu^{(k,f)}$ for non-prismatic rod, cf.[4]. If we consider the natural vibration of rod having a shape defined by (k), oscillation mode of which is (f), we have

$$
\ddot{q} + \frac{B^{(k,f)}}{A^{(k,f)}} q = 0. \tag{3.3}
$$

Hence the second power of natural angular frequency of vibration is

$$[\omega^{(k,f)}]^2 = \frac{B^{(k,f)}}{A^{(k,f)}} = \frac{\frac{1}{12}E[\alpha]^2 \int\limits_0^l [\varphi^{(k)}]^4 [v^{(f)''}]^2 dx}{\rho \int\limits_0^l [\varphi^{(k)}]^2 [v^{(f)}]^2 dx}. \tag{3.4}$$

Taking in (3.1) $P(t) = P_0 + P_t \cos\vartheta t$ we get

$$\ddot{q} + \frac{B^{(k,f)}}{A^{(k,f)}}\tau\dot{q} + [\omega^{(k,f)}]^2\Big[1 + \frac{P_0 C^{(f)}}{B^{(k,f)}}\Big]q + \Big[\frac{P_t C^{(f)}}{A^{(k,f)}}\cos\vartheta t\Big]q = 0. \tag{3.5}$$

We introduce, in a similar way as in [4], the following quantity

$$[\Omega^{(k,f)}]^2 = [\omega^{(k,f)}]^2\Big(1 + \frac{P_0 C^{(f)}}{B^{(k,f)}}\Big) = [\omega^{(k,f)}]^2\Big(1 - \frac{P_0}{P_{kr}^{(k,f)}}\Big). \tag{3.6}$$

Next by analogy with the equation of motion of prismatic rod, periodically loaded by longitudinal force, we obtain the formula for critical force for the rod of the variable cross-section of shape (k), with the mode of oscillation (f)

$$P_{kr}^{(k,f)} = -\frac{B^{(k,f)}}{C^{(f)}} = -\frac{\frac{1}{12}E[\alpha]^4 \int\limits_0^l [\varphi^{(k)}]^4 [v^{(f)''}]^2 dx}{\int\limits_0^l v^{(f)''} v^{(f)} dx}. \tag{3.7}$$

On the base of above formulae equation (3.5) takes the form

$$\ddot{q} + \frac{B^{(k,f)}}{A^{(k,f)}}\tau\dot{q} + [\Omega^{(k,f)}]^2 \cdot \\ \cdot\Big\{1 + \frac{P_t C^{(f)}}{A^{(k,f)}\Big(1 + \frac{P_0 C^{(f)}}{B^{(k,f)}}\Big)[\omega^{(k,f)}]^2}\cos\vartheta t\Big\}q = 0. \tag{3.8}$$

Denoting

$$\frac{P_t C^{(f)}}{A^{(k,f)}\Big(1 + \frac{P_0 C^{(f)}}{B^{(k,f)}}\Big)[\omega^{(k,f)}]^2} = -2\mu^{(k,f)} \qquad \text{or}$$

$$\mu^{(k,f)} = \frac{P_t}{2(P_{kr}^{(k,f)} - P_0)} \tag{3.9}$$

what is consistent with the parameter μ defined for prismatic beam, the equation (3.8) has the form

$$\ddot{q} + 2\varepsilon^{(k,f)}\dot{q} + [\Omega^{(k,f)}]^2(1 - 2\mu^{(k,f)}\cos\vartheta t)q = 0 \tag{3.10}$$

where

$$\frac{\tau B^{(k,f)}}{A^{(k,f)}} = 2\varepsilon^{(k,f)}.$$

We consider the possibility of parametric vibration occurrence, i.e. the possibility of existence of nonstable solutions to equation (3.10). We look for the solution with the vibration period $2T$ in the form

$$q(t) = \sum_{k=1,3}^{\infty}\Big(A_k \sin\frac{k\vartheta t}{2} + B_k \cos\frac{k\vartheta t}{2}\Big). \tag{3.11}$$

On the ground of (3.11) we will determine the boundaries of instability regions in coordinate system μ, $\vartheta/2\Omega$.

Confining ourselves to the first, most important instability region (the primary one) which occurs in the neighbourhood of the double value of the lowest natural frequency, the solution has the form

$$q(t) = A\sin\frac{\vartheta t}{2} + B\cos\frac{\vartheta t}{2}. \tag{3.12}$$

Inserting (3.12) to equation (3.10) and comparing the coefficients of $\sin(\vartheta t/2)$ and $\cos(\vartheta t/2)$ we get the system of algebraic equations in coefficients A and B. The nonzero solution to these equations exists if the following determinant equals to zero

$$\begin{vmatrix} 1+\mu^{(k,f)} - \dfrac{\vartheta^2}{4[\Omega^{(k,f)}]^2} & -\dfrac{\Delta^{(k,f)}}{2\pi}\dfrac{\vartheta}{\Omega^{(k,f)}} \\ \dfrac{\Delta^{(k,f)}}{2\pi}\dfrac{\vartheta}{\Omega^{(k,f)}} & 1-\mu^{(k,f)} - \dfrac{\vartheta^2}{4[\Omega^{(k,f)}]^2} \end{vmatrix} \tag{3.13}$$

where $\Delta^{(k,f)}$ denotes the damping decrement of vibrations of rod with the shape (k) and mode (f), under constant longitudinal force P_0

$$\Delta^{(k,f)} = \frac{2\pi\varepsilon^{(k,f)}}{\omega^{(k,f)}\sqrt{1-\dfrac{P_0}{P_{kr}^{(k,f)}}}} = \frac{2\pi\varepsilon^{(k,f)}}{\Omega^{(k,f)}}. \tag{3.14}$$

Solving the determinant (3.13) we have

$$\vartheta = 2\Omega^{(k,f)}\sqrt{1-\frac{1}{2}\Big(\frac{\Delta^{(k,f)}}{\pi}\Big)^2 \pm \sqrt{[\mu^{(k,f)}]^2 - \Big(\frac{\Delta^{(k,f)}}{\pi}\Big)^2 + \frac{1}{4}\Big(\frac{\Delta^{(k,f)}}{\pi}\Big)^4}}. \tag{3.15}$$

Since the damping decrement is usually small compared with a unit we reduce formula (3.15) neglecting the term with greatest power of $\Delta^{(k,f)}/\pi$. So we get

$$\vartheta = 2\Omega^{(k,f)}\sqrt{1 \pm \sqrt{[\mu^{(k,f)}]^2 - \Big(\frac{\Delta^{(k,f)}}{\pi}\Big)^2}}. \tag{3.16}$$

If the expression under the inner square root is positive the formula (3.16) gives two real values of critical frequency. These values give two boundaries of a primary instability region. The boundary condition $[\mu^{(k,f)}]^2 = [\Delta^{(k,f)}/\pi]^2$ describes the smallest value of exciting parameter, for which there is possible the existence of non–damped, parametric vibration. The critical value of exciting parameter is $\mu^* = \Delta^{(k,f)}/\pi$, where $\Delta^{(k,f)}$ is defined by (3.14). Comparing the formulae (3.15) which describe the boundaries of instability regions for rod with damping with the formulae describing instability regions without damping, cf.[4], we see that for $\mu > 2\mu^*$ the regions are practically the same. The advantage of this fact is taken in definition of a new objective function which is connected with the area of a part of instability region.

4. Objective function

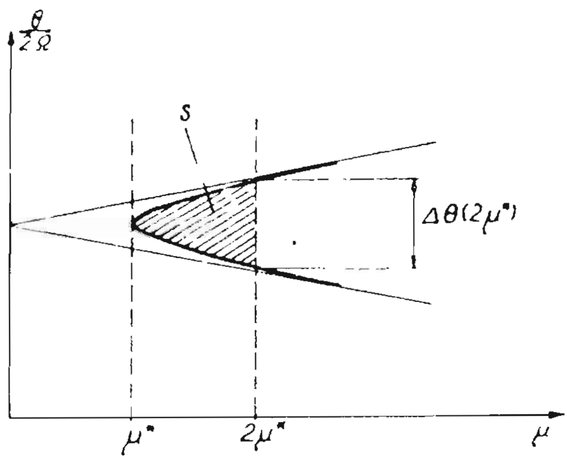

Fig. 3.

Let S denote the area of a part of instability region enclosed between μ^* and $2\mu^*$. For greater values of μ the influence of damping on boundaries of instability regions can be neglected (Fig.3). So we have

$$S = \int\limits_{\mu^*}^{2\mu^*} \sqrt{1 + \sqrt{\mu^2 - \Big(\frac{\Delta}{\pi}\Big)^2}}\, d\mu - \int\limits_{\mu^*}^{2\mu^*} \sqrt{1 - \sqrt{\mu^2 - \Big(\frac{\Delta}{\pi}\Big)^2}}\, d\mu. \tag{4.1}$$

Because μ^2 is near (Δ/π), the expression

$$\varepsilon = \sqrt{\mu^2 - \left(\frac{\Delta}{\pi}\right)^2}$$

is a small quantity in the region of μ changes: it is of order not higher then Δ/π. Hence transforming (4.1) we get

$$\begin{aligned} S &= \int_{\mu^*}^{2\mu^*} \sqrt{1+\varepsilon} d\mu - \int_{\mu^*}^{2\mu^*} \sqrt{1-\varepsilon} d\mu \cong \int_{\mu^*}^{2\mu^*} \sqrt{\mu^2 - \left(\frac{\Delta}{\pi}\right)^2} d\mu = \\ &= \mu^{*2}\left(\sqrt{3} + \frac{1}{2}\ln\frac{1}{2+\sqrt{3}}\right). \end{aligned} \tag{4.2}$$

In virtue of the formula for the critical value of exciting parameter we obtain the formula for the area S. The value of S depends on the parameters defining the shape of rod, the boundary conditions and material constants. The above considerations concern all the cases analysed in previous parts of this paper. One should only replace μ^* with $\mu^{*(k,f)}$. In virtue of (4.2) we get

$$S^{(k,f)} = [\mu^{*(k,f)}]^2 \cdot 1.074 = 1.074\frac{\tau^2[B^{(k,f)}]^2}{A^{(k,f)}(B^{(k,f)} + P_0C^{(f)})}. \tag{4.3}$$

Inserting $A^{(k,f)}$, $B^{(k,f)}$ and $C^{(f)}$ to this formula, in accordance with formulae (3.2), we get the final form of the objective function.

5. Parametric optimization

In the second part of the paper we have adopted some patterns of changing of the considered rods, cross–section cf.(2.9), (2.11), introducing the shape parameters κ. In optimization process we adopt the function S given by (4.3) as the objective function. The following constraints are assumed: the length and the volume of rod are constant. The shape parameters are the optimization parameters. The system is near to the main parametric resonance. The problem of optimization will be formulated as follows: we look for the values of parameters α and κ, which describe the shape of the rod, which satisfy the constraints and in addition minimize the function S

$$\min S^{(k,f)}[A^{(k,f)}(\alpha,\kappa), B^{(k,f)}(\alpha,\kappa), C^{(f)}], \tag{5.1}$$

where $A^{(k,f)}$, $B^{(k,f)}$, $C^{(f)}$ are defined by (3.2). The optimization problem may be formulated in a different way introducing instead of the area another objective function connected with it. We define the width of dynamic instability region for $\mu = 2\mu^*$

$$Z^{(k,f)} = \frac{\Delta\vartheta^{(k,f)}(2\mu^*)}{2\Omega}. \tag{5.2}$$

On the ground of the equation describing dynamic instability regions we have

$$Z(2\mu^*) = \sqrt{1+\sqrt{\mu^2-\left(\frac{\Delta}{\pi}\right)^2}} - \sqrt{1-\sqrt{\mu^2-\left(\frac{\Delta}{\pi}\right)^2}} = \sqrt{3}\mu^*. \tag{5.3}$$

It enables us to define the interval of exciting frequency for which the system is unstable

$$\Delta\vartheta(2\mu^*) = Z(2\mu^*)2\Omega = 2\sqrt{3}\frac{\tau B^{(k,f)}}{A^{(k,f)}}. \tag{5.4}$$

The quantities described by formulae (5.3), (5.4) can be optimized also. We look for the shape parameters which minimize the quantities (5.3), (5.4). The constraints and conditions of optimization are the same. Calculations and analysis will be presented.

Example 1. $k = 1$, $f = 1$, Fig.1a, Fig.2a.

Cross-section of the rod changes according to formula (2.9). The quantities A, B, C have the form

$$\begin{aligned} A^{(1,1)} &= \rho[\alpha]^2 l f_A(\kappa) \\ f_A(\kappa) &= 0.3910\kappa^2 - 0.8693\kappa + 0.500 \\ B^{(1,1)} &= \frac{E[\alpha]^4}{l^3} f_B(\kappa) \\ f_B(\kappa) &= 2.701\kappa^4 - 11.64\kappa^3 + 19.01\kappa^2 - 14.12\kappa + 4.058 \\ C^{(1)} &= -\frac{\pi^2}{2l}. \end{aligned} \tag{5.5}$$

The volume $V = l\alpha^2 f(\kappa)$, where $f(\kappa) = 1 - 4\frac{\kappa}{3} + \frac{8\kappa^2}{15}$. Therefore in virtue of (4.3)

$$S^{(1,1)} = 1.074 z_1 \frac{[f_B(\kappa)]^2}{f^3(\kappa) f_A(\kappa)\left[z_2 \frac{f_B(\kappa)}{f^2(\kappa)} - P_0 z_3\right]}. \tag{5.6}$$

The critical force for the rod of variable cross-section depends on shape parameters and it has the following form

$$P_{kr}^{(1,1)}(\kappa) = \frac{z_2}{z_3}\frac{f_B}{f^2}. \tag{5.7}$$

The critical value of exciting parameter is now

$$\mu^{*(1,1)} = z_5 \frac{f_B}{f^2\sqrt{z_4 \frac{f_A}{f}\left(z_2 \frac{f_B}{f^2} - P_0 z_3\right)}} \tag{5.8}$$

where

$$z_1 = \frac{\lambda^2 V^3}{\rho l^{10}}, \qquad z_2 = \frac{EV^2}{l^5}, \qquad z_3 = \frac{\pi^2}{2l}$$
$$z_4 = \rho V, \qquad z_5 = \tau z_2, \qquad z_6 = 3.464\frac{\lambda V}{\rho l^5}. \tag{5.9}$$

Using the formule (3.9) we have

$$P_t^* = 2\mu^*(P_{kr} - P_0). \tag{5.10}$$

Finally we calculate the range of frequency in which the dynamic instability occurs

$$\Delta\vartheta(2\mu^*) = z_6 \frac{f_B}{f f_A}. \tag{5.11}$$

It depends on the shape parameter.

Example 2. $k = 2$, $f = 2$, Fig.1b, Fig.2b.

Cross–section of the rod changes according to formula (2.11). The quantities A, B, C are the following

$$\begin{aligned} A^{(2,2)} &= \rho[\alpha]^2 l f_{\overline{A}} \\ f_{\overline{A}} &= 0.1747\kappa^2 - 0.5680\kappa + 0.500 \\ B^{(2,2)} &= \frac{E[\alpha]^4}{l^3} f_{\overline{B}} \\ f_{\overline{B}} &= 1.253\kappa^4 - 7.153\kappa^3 + 15.94\kappa^2 - 17.06\kappa + 9.89 \\ C^{(2)} &= -\frac{5.7518}{l} = -\overline{z}_3. \end{aligned} \tag{5.12}$$

The volume $V = l\alpha^2\overline{f}$, where $\overline{f} = 1 - \kappa + \frac{\kappa^2}{3}$.

Optimized functions $S^{(2,2)}$, $\Delta\vartheta^{(2,2)}$ and parameters $P_{kr}^{(2,2)}$, $\mu^{*(2,2)}$, $P_t^{*(2,2)}$ have now the forms

$$S^{(2,2)} = 1.074 z_1 \frac{f_{\overline{B}}^2}{\overline{f}^3 f_{\overline{A}}\left[z_2 \frac{f_{\overline{B}}}{\overline{f}^2} - \overline{z}_3 P_0\right]}$$

$$\Delta\vartheta^{(2,2)}(2\mu^*) = z_6 \frac{f_{\overline{B}}}{\overline{f} f_{\overline{A}}}$$

$$P_t = 2\mu^*(P_{kr} - P_0) \tag{5.13}$$
$$P_{kr}^{(2,2)} = \frac{z_2}{z_3}\frac{f_{\overline{B}}}{\overline{f}^2}$$
$$\mu^{*(2,2)} = z_5 \frac{f_{\overline{B}}}{\overline{f}^2 \sqrt{z_4 \frac{f_{\overline{A}}}{\overline{f}} (\frac{f_{\overline{B}}}{\overline{f}^2} - P_0 \overline{z}_3)}}.$$

Remaining quantities are the same as in example 1.

Example 3

Now we consider the cantilever beam of variable cross–section at loaded by periodic force $P(t)$ which is parallel to non–deformed rod axis (Fig.1c). The system of equations of motion is obtained from the Lagrange's equations in the form similar to (2.14). Assuming that the shape of rod is the same as in example 2 and confining to the first term of series we get

$$\frac{\rho V M}{<\varphi^2>}\ddot{q} + \frac{\mu^4 \lambda V^2 K}{12 l^5 <\varphi^2>}\dot{q} + \Big(\frac{\mu^4 E V^2 K}{12 l^5 <\varphi^2>^2} - \frac{\mu^2}{2l}P_0\Big)q -$$
$$-\frac{\mu^2}{2l}P_t q \cos\vartheta t = 0 \tag{5.14}$$

where

$$\varphi = 1 - \kappa\frac{x}{l},$$
$$<\varphi^2> = 1 - \kappa + \frac{1}{3}\kappa^2 = \overline{f}$$
$$M = \int_0^1 \varphi^2 (1 - \cos\frac{\pi\xi}{2})^2 d\xi = 0.2268 - 0.3720\kappa + 0.1575\kappa^2 \tag{5.15}$$
$$K = \int_0^1 \varphi^4 \cos^2\frac{\pi\xi}{2} d\xi = 0.500 - 0.5947\kappa + 0.3920\kappa^2 -$$
$$-0.1385\kappa^3 + 0.0205\kappa^4$$
$$\xi = \frac{x}{l}.$$

Adopting denotations, similar to (3.1), we have

$$A^{(2,3)} = \frac{\rho V M}{<\varphi^2>} = \frac{\rho V}{\overline{f}} f_{\overline{\overline{A}}}, \qquad f_{\overline{\overline{A}}} = M$$
$$B^{(2,3)} = \frac{\mu^4 V^2 E K}{12 l^5 \overline{f}^2} = \frac{E V^2}{l^5 \overline{f}^2} f_{\overline{\overline{B}}}, \qquad f_{\overline{\overline{B}}} = \frac{\pi^2}{48}K \tag{5.16}$$
$$C^{(3)} = -\frac{\mu^2}{2l} = -\frac{\pi^2}{8l} = -\overline{\overline{z}}_3.$$

Now one has assumed

$$w(x,t) = q(t)v^{(3)}(x), \qquad v^{(3)} = 1 - \cos\frac{\mu x}{l}, \qquad \mu = \frac{\pi}{2}. \tag{5.17}$$

Then the equation of motion has the following form

$$A^{(2,3)}\ddot{q} + B^{(2,3)}\tau\dot{q} + (B^{(2,3)} + C^{(3)}P_0)q + C^{(3)}P_t q\cos\vartheta t = 0. \tag{5.18}$$

Acting similarly as in examples 1 and 2 we get

$$\begin{aligned}
S^{(2,3)} &= z_1\frac{f^2_{\bar{\bar{B}}}}{\bar{f}^3 f_{\bar{\bar{A}}}\left(z_2\frac{f_{\bar{\bar{B}}}}{\bar{f}^2} - \bar{\bar{z}}_3P_0\right)}\\
P^{(2,3)}_{kr} &= \frac{z_2}{\bar{\bar{z}}_3}\frac{f_{\bar{\bar{B}}}}{\bar{f}^2}\\
P_t &= 2\mu^*(P_{kr} - P_0)\\
\mu^{*(2,3)} &= z_5\frac{f_{\bar{\bar{B}}}}{\bar{f}^2\sqrt{z_4\frac{f_{\bar{\bar{A}}}}{\bar{f}}\left(z_2\frac{f_{\bar{\bar{B}}}}{\bar{f}^2} - P_0\bar{\bar{z}}_3\right)}}\\
\Delta\vartheta^{(2,3)}(2\mu^*) &= z_6\frac{f_{\bar{\bar{B}}}}{\bar{f}f_{\bar{\bar{A}}}}
\end{aligned} \tag{5.19}$$

where

$$\begin{aligned}
f_{\bar{\bar{A}}} &= 0.2268 - 0.3720\kappa + 0.1575\kappa^2\\
f_{\bar{\bar{B}}} &= 0.1028 - 0.1223\kappa + 0.0859\kappa^2 - 0.0285\kappa^3 + 0.0042\kappa^4\\
\bar{f} &= 1 - \kappa + \frac{\kappa^2}{3}\\
\bar{\bar{z}}_3 &= \frac{\pi^2}{8l}.
\end{aligned} \tag{5.20}$$

6. Conclusions

Numerical calculations were carried out for example 2. The following values of parameters: $E = 2.2 \cdot 10^{11}$ Nm^{-2}, $\rho = 7.7 \cdot 10^3$ kg m^{-3}, $\eta = 1 \cdot 10^6$ N s m^{-2} or $\eta = 1 \cdot 10^8$ N s m^{-2}, $l = 8$ m, $V = 1$ m^3, $P_0 = 1 \cdot 10^7$ N has been adopted.

The results are presented in the figures. In Fig.4 one presents $S(\kappa)$ and $\Delta\vartheta(2\mu^*)$ versus shape parameter κ. Both these quantities are maximal for $\kappa = 1$, i.e. for

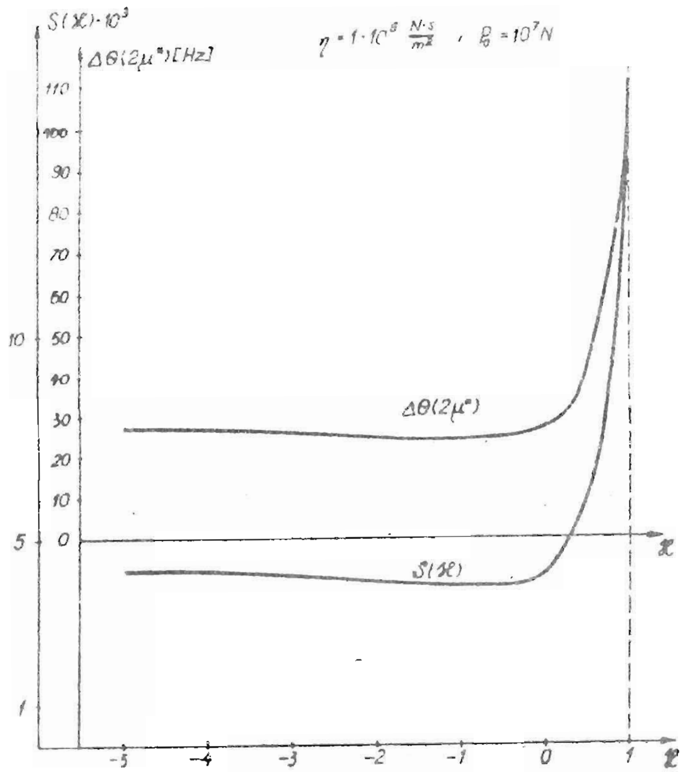
$S(\varkappa)\cdot 10^3$
$\Delta\Theta(2\mu^*)$ [Hz]
$\eta = 1\cdot 10^8 \frac{N\cdot s}{m^2}$, $P_0 = 10^7 N$
110
100
90
80
70
60
10
50
40
30
20
10
5
0
1
$\Delta\Theta(2\mu^*)$
$S(\varkappa)$
-5
-4
-3
-2
-1
0
1
$\varkappa$

Fig. 4.

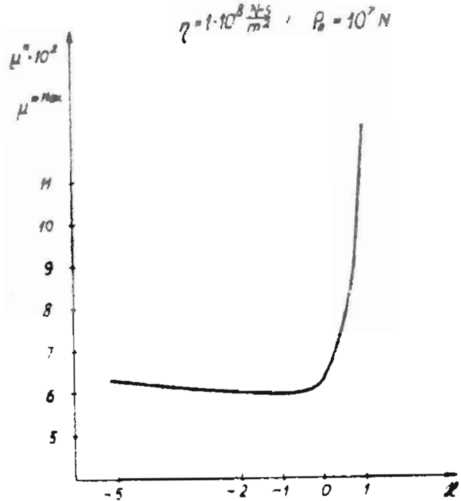
$\eta = 1\cdot 10^8 \frac{N\cdot s}{m^2}$, $P_0 = 10^7 N$
$\mu^*\cdot 10^2$
μ^{*max}
11
10
9
8
7
6
5
-5
-2
-1
0
1
$\varkappa$

Fig. 5.

the rod shaped like a cone. The loss of stability takes place for the geatest interval of frequency – this situation is the most unfavourable. Modifying the shape of rod one can change the width of this interval. The diagrams suggest that the minimum of $S(\kappa)$ and $\Delta\vartheta(2\mu^*)$ occurs near $\kappa = 0$, i.e. for almost prismatic rod.

Fig.5 shows the critical value of exciting parameter μ^* as a function of the shape parameter κ. Parameter μ^* attains its maximal value for $\kappa = 1$; the rod is then shaped like a cone. It is the most favourable situation considering the dynamic stability. In this situation the value of force which causes the unstable motion is maximal. From the above considerations arises that the optimal system on the ground of $S(\kappa)$ or $\Delta\vartheta(2\mu^*)$ is not optimal on the ground of μ^*. Optimization of the system at parametric resonance is a complicated problem. The optimization should be carried out taking into account a few aspects of the problem.

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Streszczenie

Praca jest poświęcona zagadnieniom optymalizacji przy utracie stateczności dynamicznej. Rozważono drgania poprzeczne kilku prętów, różniących się sposobem zamocowania, obciążonych siłami okresowo zmiennymi w czasie. Pręty wykonane z materiału lepkosprężystego, posiadają zmienne przekroje, przy czym przyjęto określony sposób zmiany tych przekrojów. Wprowdzono współczynniki, które określają kształt pręta. Istotą pracy jest wprowadzenie funkcji celu, która jest miarą obszaru niestateczności dynamicznej. Dotychczas optymalizowano podobne układy ze względu na amplitudę oscylującego składnika siły wymuszającej. Okazuje się, że układ optymalny ze względu na tę ostatnią funkcje celu nie jest optymalny ze względu na funkcje celu wprowadzone w pracy. Optymalizacja układu w warunkach rezonansu parametrycznego jest zagadnieniem złożonym, należy brać pod uwagę więcej niż jedną funkcję celu. Praca sygnalizuje istnienie pewnych problemów. Rozważania przeprowadzono ograniczając się tylko do jednego stopnia swobody. Zagadnienie wymaga dalszej, bardziej wnikliwej analizy.

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