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BIHARMONIC REPRESENTATION IN THE ANALYSIS OF PLATES MADE OF THE GRIOLI–TOUPIN MATERIAL

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To the memory of a great scientist and noble man, Professor Witold Nowacki (1911-1986) may favoured teacher.

In the present paper there is derived a general displacement vector representation, describing a generalized plane–stress state in the body with constrained rotations (Grioli–Toupin material), leading to the solution of a biharmonic equation.

1. Introduction

Under the notion of a generalized plane–stress state (GPSS) (cf Love [1], p.471) we understand a state in which at each point of the elastic layer the stress component σ_{33} vanishes and the layer faces $z = \pm h/2$ are traction–free. i.e. $\sigma_{3i}(x^\beta, \pm h/2) = 0$. In the bibliography of the classical symmetric elasticity theory one can find some particular representations of the displacement vector, relevant to the GPSS case. A certain representations of the displacement field, resulting in the biharmonic equation

$$\nabla^4 v(x^\alpha) = 0 \tag{1.1}$$

where v represents the deflection field, were for the first time given by Lévy, 1877 [2], cf also Michell [3] and Love [1]. The representation applies to the case of isotropic and homogeneous plates. The same equation (1.1) describes the case of transversely non–homogeneous plates provided the original representation is appropriately modified cf Sokołowski [4]. Further generalization to the case of plates made of transversely isotropic material distributed homogeneously or non-homogeneously, symmetrically or non-symmetrically in the transverse direction, is due to Lekhnitskii [5,6,7].

In the micropolar elasticity the state GPSS refers to the case when the components of stress and couple stress tensors are[1]

$$\boldsymbol{\sigma} = \begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & 0 \end{vmatrix}, \qquad \boldsymbol{\mu} = \begin{vmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{21} & \mu_{22} & \mu_{23} \\ \mu_{31} & \mu_{32} & 0 \end{vmatrix} \tag{1.2}$$

and on the faces we have

$$\sigma_{3i}\left(x^{\beta}, \pm\frac{h}{2}\right) = 0, \qquad \mu_{3i}\left(x^{\beta}, \pm\frac{h}{2}\right) = 0. \tag{1.3}$$

In the micropolar description the representations for the displacement and rotation vector fields, concerning the GPSS case, are unknown. However, one should emphasize here that even in the case of symmetric elasticity the representations adduced earlier (as we shall prove in the sequel) should not be treated as general representations.

In the present paper we shall deal with the elastic layer made of the material with constrained rotations. For this case we shall put forward a general representation for the displacement vector that leads to the equation (1.1) and that describes the GPSS state of stress.

The summation convention is adopted. Latin indices have the range 1,2,3, Greek indices – take values 1,2 only. A comma denotes partial differentiation.

2. Governing equations of the Grioli–Toupin material

The constitutive relations of the isotropic, homogeneous and centrosymmetric

[1]In Nowacki's monograph [8] (p.175) the GPSS is defined diferently. There this state is expressed in terms of averages

$$\sigma^{*}_{\alpha\beta} = \frac{1}{h}\int_{-h/2}^{h/2} \sigma_{\alpha\beta}dz, \qquad \mu^{*}_{\alpha 3} = \frac{1}{h}\int_{-h/2}^{h/2} \mu_{\alpha 3}dz$$

and the GPSS means that

$$\boldsymbol{\sigma}^{*} = \begin{vmatrix} \sigma^{*}_{11} & \sigma^{*}_{12} & 0 \\ \sigma^{*}_{21} & \sigma^{*}_{22} & 0 \\ 0 & 0 & 0 \end{vmatrix}, \qquad \boldsymbol{\mu}^{*} = \begin{vmatrix} 0 & 0 & \mu^{*}_{13} \\ 0 & 0 & \mu^{*}_{23} \\ \mu^{*}_{31} & \mu^{*}_{32} & 0 \end{vmatrix}.$$

media are assumed in the following form [9,10]

$$\begin{aligned}\sigma_{ji} &= 2\mu\Big[u_{j,i} + u_{i,j} + \frac{\nu}{1-2\nu}(u^{\gamma}{}_{,\gamma} + u^{3}{}_{,3})\delta_{ji}\Big] - \frac{1}{2}\epsilon_{kji}\mu^{lk}{}_{,l} \\ \mu_{ji} &= \gamma(\varphi_{i,j} + \varphi_{j,i}) + \varepsilon(\varphi_{i,j} - \varphi_{j,i}) = 4\mu l^2(\varphi_{i,j} + \eta\varphi_{j,i})\end{aligned} \tag{2.1}$$

where

$$\varphi_i = \frac{1}{2}\epsilon_i{}^{jk}u_{k,j} \tag{2.2}$$

$$l^2 = \frac{\gamma+\varepsilon}{4\mu}, \qquad \eta = \frac{\gamma-\varepsilon}{\gamma+\varepsilon} \tag{2.3}$$

σ_{ji} – represents the stress tensor, μ_{ji} – stands for the couple stress tensor, ϵ_{ijk} is a Levi–Civitá symbol, δ_{ij} is Kronecker delta, u_i are displacement and φ_i represent components of the vector of infinitesimal rotations. The Lamé constant is denoted by μ, ν is Poisson ratio. The moduli γ, ε, l and η describe micropolar properties.

Omitting the body forces one can write the equations of equilibrium as follows

$$\sigma^{ji}{}_{,j} = 0, \qquad \epsilon^{ijk}\sigma_{jk} + \mu^{ji}{}_{,j} = 0 \tag{2.4}$$

whereas the equilibrium equations expressed in terms of displacements can be cast in the form

$$\tilde{\nabla}^2 u_i + \frac{1}{1-2\nu}u^k{}_{,ki} + l^2\tilde{\nabla}^2\Big(u^k{}_{,ki} - \tilde{\nabla}^2 u_i\Big) = 0 \tag{2.5}$$

here $\tilde{\nabla}^2$ represents the Laplace operator in R^3.

3. Biharmonic representation

Consider an elastic layer of thickness h, freed from loads on the faces $x^3 = z = \pm h/2$. The solutions to the differential equations (2.5) with homogeneous boundary conditions (1.3) will be constructed by the semi–inverse technique.

Let us assume the following representation for the displacement vector

$$\begin{aligned}u_\alpha(x^\beta, z) &= t(z)v(x^\beta)_{,\alpha} + s(z)\nabla^2 v(x^\beta)_{,\alpha} \\ u_3(x^\beta, z) &= g(z)v(x^\beta) + f(z)\nabla^2 v(x^\beta).\end{aligned} \tag{3.1}$$

Here $t(z)$, $s(z)$, $g(z)$, $f(z)$ are unknown functions. They satisfy

$$\begin{aligned}t(z) &= -t(-z), \qquad & s(z) &= -s(-z) \\ g(z) &= g(-z), \qquad & f(z) &= f(-z).\end{aligned} \tag{3.2}$$

Applying (3.1) to (2.5) after substitution we conclude that there exist non–zero solutions of this system, if the function $v(x^\alpha)$ fulfils (1.1). The unknown functions in z should satisfy a certain complicated set of ordinary differential equations. Its solution satisfying the boundary conditions (1.3) has the form

$$
\begin{aligned}
s(z) &= -\frac{(2-\nu)h^2}{24(1-\nu)}z\Big(C_2 - C_1\frac{4z^2}{h^2}\Big) - l^2hC_1\frac{\mathrm{sh}\frac{z}{l}}{\mathrm{sh}\frac{h}{2l}}\\
t(z) &= -zC_1\\
g(z) &= C_1 \qquad\qquad (3.3)\\
f(z) &= -\frac{h^2}{24(1-\nu)}\Big[\Big(6 - \frac{12\nu z^2}{h^2}\Big)C_1 - (2-\nu)C_2\Big]
\end{aligned}
$$

where C_1, C_2 are arbitrary constans. These constans determine the physical meaning of the displacement field $v(x^\alpha)$.

After application of (3.3) to (3.1) and then into (2.1), the formulae for u_i and φ_i can be written as

$$
\begin{aligned}
u_\alpha(x^\beta,z) &= -\Big\{zC_1v(x^\beta)_{,\alpha} + \Big[\frac{(2-\nu)h^2}{24(1-\nu)}z\Big(C_2 - C_1\frac{4z^2}{h^2}\Big) +\\
&\quad + l^2hC_1\frac{\mathrm{sh}\frac{z}{l}}{\mathrm{sh}\frac{h}{2l}}\Big]\nabla^2v(x^\beta)_{,\alpha}\Big\} \qquad\qquad (3.4)\\
u_3(x^\beta,z) &= C_1v(x^\beta) - \frac{h^2}{24(1-\nu)}\Big[\Big(6 - \frac{12\nu z^2}{h^2}\Big)C_1 - (2-\nu)C_2\Big]\nabla^2v(x^\beta)
\end{aligned}
$$

$$
\begin{aligned}
\varphi_\alpha(x^\beta,z) &= \epsilon_\alpha{}^\beta\Big\{C_1v(x^\beta)_{,\alpha} + \Big[\frac{h^2}{24(1-\nu)}\Big((2-\nu)C_2 - 3C_1 -\\
&\quad -12(1-\nu)C_1\frac{z^2}{h^2}\Big) + \frac{1}{2}lhC_1\frac{\mathrm{ch}\frac{z}{l}}{\mathrm{sh}\frac{h}{2l}}\Big]\nabla^2v(x^\beta)_{,\alpha}\Big\} \qquad\qquad (3.5)\\
\varphi_3 &= 0.
\end{aligned}
$$

With the help of the eqs. (2.1), (3.4) and (3.5) we arrive at the formulae for stresses an couple stresses

$$
\begin{aligned}
\sigma_{\alpha\beta}(x^\gamma,z) &= -\frac{2\mu}{1-\nu}\Big\{C_1z\Big((1-\nu)v_{,\alpha\beta} + \nu\nabla^2v\delta_{\alpha\beta}\Big) +\\
&\quad + \Big[\frac{(2-\nu)h^2}{24}z\Big(C_2 - C_1\frac{4z^2}{h^2}\Big) + (1-\nu)l^2C_1h\frac{\mathrm{sh}\frac{z}{l}}{\mathrm{sh}\frac{h}{2l}}\Big]\nabla^2v(x^\gamma)_{,\alpha\beta}\Big\} \qquad (3.6)\\
\sigma_{\alpha3}(x^\beta,z) &= -\frac{\mu h^2}{4(1-\nu)}C_1\Big[\Big(1 - \frac{4z^2}{h^2}\Big) + 8(1-\nu)\frac{l}{h}\frac{\mathrm{ch}\frac{z}{l}}{\mathrm{sh}\frac{h}{2l}}\Big]\nabla^2v(x^\beta)_{,\alpha}
\end{aligned}
$$

$$\mu_{\alpha\beta}(x^\delta, z) = 4\mu l^2 \epsilon_\beta{}^\gamma \Big\{C_1 v_{,\gamma\alpha} + \Big[\frac{1}{2}C_1 hl \frac{\mathrm{ch}\frac{z}{l}}{\mathrm{sh}\frac{h}{2l}} -$$

$$-\frac{h^2}{24(1-\nu)}\Big(3C_1 + \frac{12z^2}{h^2}(1-\nu)C_1 - (2-\nu)C_2\Big)\Big]\nabla^2 v_{,\gamma\alpha}\Big\} \tag{3.7}$$

$$\sigma_{3\alpha}(x^\beta, z) = -\frac{\mu h^2}{4(1-\nu)}C_1\Big(1 - \frac{4z^2}{h^2}\Big)\nabla^2 v_{,\alpha} \tag{3.8}$$

$$\sigma_{33}(x^\alpha, z) = 0$$

$$\mu_{33}(x^\alpha, z) = 0 \tag{3.9}$$

$$\mu_{3\alpha} = -2\mu l^2 h C_1 \epsilon_\alpha{}^\beta \Big(2\frac{z}{h} - \frac{\mathrm{sh}\frac{z}{l}}{\mathrm{sh}\frac{h}{2l}}\Big)\nabla^2 v_{,\beta}$$

$$\mu_{\alpha 3} = \eta\mu_{3\alpha}. \tag{3.10}$$

One can easy check that if v satisfies (1.1) then all differential equations (2.5) and boundary conditions (1.3) are fulfilled. In its general form (3.4) $\div$ (3.10) these equations have not been reported in the hitherto existing literature.

Thus the constants C_1 and C_2 remain arbitrary. We shall choose them so as to assign a clear physical meaning to the function $v(x^\alpha)$. One can easy prove, that without any loss of generality, the constant C_1 can be assumed as equal to 1.

In the flexural plate theories the following functions describing a plate deflection are used

1. deflection of plate faces $\hat{w}(x^\alpha)$

$$\hat{w}(x^\alpha) \stackrel{\mathrm{df}}{=} u_3(x^\alpha, \pm\frac{h}{2}) \tag{3.11}$$

2. deflection of the mid-plane $w(x^\alpha)$

$$w(x^\alpha) \stackrel{\mathrm{df}}{=} u_3(x^\alpha, 0) \tag{3.12}$$

3. simple average $\overset{*}{w}(x^\alpha)$

$$\overset{*}{w}(x^\alpha) \stackrel{\mathrm{df}}{=} \frac{1}{h}\int\limits_{-h/2}^{h/2} u_3(x^\alpha, z)dz \tag{3.13}$$

4. weighted average $\tilde{w}(x^\alpha)$

$$\tilde{w}(x^\alpha) \stackrel{\mathrm{df}}{=} \frac{3}{2h} \int_{-h/2}^{h/2} (1 - 4\frac{z^2}{h^2}) u_3(x^\alpha, z) dz. \tag{3.14}$$

All these deflections have simple mathematical interpretations. The deflections w and $\hat{w}$ are first terms of the expansion of u_3 into appropriate power series. We shall prove that the averages $\overset{*}{w}$ and $\tilde{w}$ are first terms of the expansion of u_3 into Legendre polynomials. To this end let us expand u_3 (for the bending state) into Legendre polynomials

$$u_3 = \sum_{i=0,2,\dots}^{\infty} f_i(\zeta) v_i(x^\alpha) \tag{3.15}$$

$$u_3 = \sum_{i=1,3,\dots}^{\infty} \frac{df_i(\zeta)}{d\zeta} w_i(x^\alpha) \tag{3.16}$$

where

$$\zeta = \frac{2z}{h}, \qquad f_n = \frac{1}{2^n n!} \frac{d^n\left((\zeta^2-1)^n\right)}{d\zeta^n}, \qquad f_0 = 1, \qquad f_1 = \zeta$$
$$f_2 = \frac{1}{2}(3\zeta^2 - 1), \qquad f_3 = \frac{\zeta}{2}(5\zeta^2 - 3), \qquad f_4 = \frac{1}{8}(35\zeta^4 - 30\zeta^2 + 3)$$
$$f_5 = \frac{1}{8}\zeta(63\zeta^4 - 70\zeta^2 + 15). \tag{3.17}$$

Using the orthogonality property of the Legendre polynomials in the interval $-1 \le \zeta \le 1$ one finds

$$v_0 = \int_0^1 u_3 d\zeta = \overset{*}{w}, \qquad w_1 = \frac{3}{2}\int_0^1 (1 - \zeta^2) u_3 d\zeta = \tilde{w}. \tag{3.18}$$

Thus the averages $\overset{*}{w}$ and $\tilde{w}$ are first terms of the expansions of u_3 into Legendre polynomials (3.15), (3.16).

Using $(3.5)_2$ and the definitions (3.11) ÷ (3.14) we obtain subsequently the following values of the C_1, C_2 constants

1. for function $\hat{w}$ - $C_1 = 1, \quad C_2 = 3$,
2. for function w - $C_1 = 1, \quad C_2 = \frac{6}{2-\nu}$,
3. for function $\overset{*}{w}$ - $C_1 = 1, \quad C_2 = \frac{6-\nu}{2-\nu}$,
4. for function $\tilde{w}$ - $C_1 = 1, \quad C_2 = \frac{3(10-\nu)}{5(2-\nu)}$.

It is readily seen that depending on the definition of the deflection the formulae for displacements, stresses and stress resultants assume dissimilar forms.

4. Biharmonic representation of GPSS in the plate made of the Hookean material

To obtain the displacements and stresses in the plate made of the classical Hookean material one should pass to zero with l. On computing limits of the expressions (3.4) $\div$ (3.9) we arrive at

$$\begin{aligned} u_\alpha(x^\beta, z) &= -z\Big[C_1 v(x^\beta)_{,\alpha} + \frac{(2-\nu)h^2}{24(1-\nu)}\Big(C_2 - C_1\frac{4z^2}{h^2}\Big)\nabla^2 v(x^\beta)_{,\alpha}\Big] \\ u_3(x^\beta, z) &= C_1 v(x^\beta) - \frac{h^2}{24(1-\nu)}\Big[\Big(6 - \frac{12\nu z^2}{h^2}\Big)C_1 - (2-\nu)C_2\Big]\nabla^2 v(x^\beta) \end{aligned} \tag{4.1}$$

$$\begin{aligned} \sigma_{\alpha\beta}(x^\gamma, z) &= -\frac{2\mu}{1-\nu}z\Big\{C_1\Big((1-\nu)v_{,\alpha\beta} + \nu\nabla^2 v\delta_{\alpha\beta}\Big) + \\ &\quad + \Big[\frac{(2-\nu)h^2}{24}\Big(C_2 - C_1\frac{4z^2}{h^2}\Big)\Big]\nabla^2 v(x^\gamma)_{,\alpha\beta}\Big\} \\ \sigma_{3\alpha}(x^\beta, z) &= \sigma_{\alpha 3}(x^\beta, z) = -\frac{\mu h^2}{4(1-\nu)}C_1\Big(1 - \frac{4z^2}{h^2}\Big)\nabla^2 v_{,\alpha} \\ \sigma_{33} &= 0. \end{aligned} \tag{4.2}$$

The equations (4.1), (4.2) have been cast here in a general form which up till now have not been given in the literature.

It is Lévy who in 1877, as the first, found the particular form of the solution (4.1), (4.2). His solution refers to the case $C_1 = 1$, $C_2 = \frac{6}{2-\nu}$, cf [2] (pp 32–33). Under this choice of C_α the function $v(x^\alpha)$ stands for the deflection of the mid-plane. The method of Lévy was based on the power series expansion in $x^3 = z$.

The solution of Lévy, the whole one or a part (only formulae for stresses), were arrived at with the help of various methods in papers [3], [11$\div$24] subsequently. In 1900 Michell [3] obtained formulae for stresses only with the help of a different, complicated method. Love [1] (p.473), basing on the Michell's results, reported all formulae for displacements, stresses and stress resultants in terms of the mid-plane deflection. At different choices of C_1, C_2 one can obtain solutions of the following authors: Dougall [25] - for $C_1 = 4(1-\nu)$, $C_2 = \frac{12(1-\nu)}{2-\nu}$; Neuber [26] at $C_1 = \frac{1}{2\mu}$, $C_2 = \frac{3}{10\mu}$; Lur'e [27] at $C_1 = -\frac{2(1-\nu)}{1-2\nu}$, $C_2 = 0$; Donnell [28] at $C_1 = \frac{1}{D}$, $C_2 = \frac{3}{5}\frac{2+3\nu}{D(1-\nu)}$; Jemielita [21] at $C_1 = 1$, $C_2 = 3$.

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Streszczenie

W pracy wyznaczono ogólne przedstawienie wektora przemieszczenia, opisujące uogólniony płaski stan naprężenia w ciele ze związanymi obrotami (materiał Grioli–Toupina), prowadzące do rozwiązania równania biharmonicznego.

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