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ON THE DETERMINATION OF ELASTIC MODULI OF THE GRIOLI–TOUPIN MATERIAL

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The paper deals with statics of plates made of the Cosserat material with constrained rotations. Several relations are derived which can serve as a basis for experiments aimed at determining micropolar moduli of the material.

1. Introduction

An experimental verification of validity of the micropolar theory of elasticity meets with obstacles of technical nature [1,2]. However, one can indicate under which circumstances the influence of the extra micropolar constans comes up, thus changing the material response qualitatively. These extra effects can be observed in the following tests: determination of the stress concentration factors in a plate (under in-plane loading) with an opening or with a rigid inclusion [1,3,4]; wave propagation test [5–7]; determination of the torsional stiffness of a bar and bending rigidity of a plate [8,9], etc.

Therefore, the experimental tests for determining the micropolar constants should be performed with the help of such samples in which these extra effects could be observable and measurable.

In the case of discrete systems for which a micropolar description is adopted a method of determination of effective elastic constants is described in monograph [10]. Several examples are given in papers [1,10–14]. For continuous media the micropolar constants have been found by scrutiny of: stress concentration along the holes and inclusions [15–17], bending of beams and plates [17–20], torsion and bending of bars [19–24], vibrations and the velocity of wave propagation [22, 25–27].

In most experiments the samples were made of aluminium alloys, low–carbon steel, dense polyurethane foam and also human compact bones. Unfortunately, in majority of cases, the experimental results were vague. Only in few cases some micropolar constants were satisfactorily evaluated [18,19,23–26]. In my opinion, the most interesting results were found in papers [15], [24] and [26].

With the help of photoelasticity method, Shirayev [15] examined the stress concentration along an opening and a rigid inclusion. His results turn out to compare favourably with theoretical predictions of Mindlin [3] and Banks and Sokołowski [4]. Basing on the work of W.Nowacki and W.K.Nowacki [7], Gauthier and Jahsman [26] examined the velocity of propagation of waves in a plate and thus determined some of the micropolar moduli. Lakes [24] investigating specimens made of dense polyurethane foam and syntactic foam, determined the six Cosserat elastic constants.

Nonetheless there is a need of further searching for the solutions of the micropolar elasticity equations which are characterized by the following features: the unknown quantities should be essentially affected by the micropolar constants, these unknown quantities should be measurable.

In the present paper, using the solutions obtained in paper [28], we derive some interrelations that can be helpful in setting an experiment for finding the extra constants of the medium with constrained rotations.

2. Bending moments in a micropolar plate made of the Grioli–Toupin material

The components of the vector field of the plate in bending with homogeneous boundary conditions on the faces

$$\begin{aligned} \sigma_{3j}\Big(x^\alpha, \pm\frac{h}{2}\Big) &= 0 \\ \mu_{3j}\Big(x^\alpha, \pm\frac{h}{2}\Big) &= 0 \end{aligned} \tag{2.1}$$

can be represented in the form, cf [28]

$$\begin{aligned} u_\alpha(x^\beta, z) &= -\Big\{zv(x^\beta)_{,\alpha} + \Big[\frac{(2-\nu)h^2}{24(1-\nu)} z\Big(C_2 - \frac{4z^2}{h^2}\Big) + \\ &\quad + l^2 h \frac{\operatorname{sh}\frac{z}{l}}{\operatorname{sh}\frac{h}{2l}}\Big] \nabla^2 v(x^\beta)_{,\alpha}\Big\} \\ u_3(x^\beta, z) &= v(x^\beta) - \frac{h^2}{24(1-\nu)}\Big[6\Big(1 - \frac{2\nu z^2}{h^2}\Big) - (2-\nu)C_2\Big] \nabla^2 v(x^\beta) \end{aligned} \tag{2.2}$$

$$\varphi_\alpha = \frac{1}{2}\epsilon_\alpha{}^\beta(u_{3,\beta} - u_{\beta,3}) = \epsilon_\alpha{}^\beta\Big\{v(x^\beta)_{,\alpha} + \Big[\frac{h^2}{24(1-\nu)}\Big((2-\nu)C_2 - 3 - -12(1-\nu)\frac{z^2}{h^2}\Big) + lh\frac{\operatorname{ch}\frac{z}{l}}{\operatorname{sh}\frac{h}{2l}}\Big]\nabla^2 v(x^\beta)_{,\alpha}\Big\} \tag{2.3}$$

$$\varphi_3 = 0.$$

Here C_2 is an arbitrary constant, h represents plate thickness, l is a material constant of length dimension and ν is Poisson ratio. The function $v(x^\alpha)$ stands for the plate deflection. This function satisfies the biharmonic equation

$$\nabla^4 v(x^\alpha) = 0. \tag{2.4}$$

The constant C_2 is chosen in such a way that the $v(x^\alpha)$ field gets a simple physical meaning. For instance, for $C_2 = 3$ we have

$$v(x^\alpha) = \hat{w}(x^\alpha) \overset{\text{df}}{=} u_3\Big(x^\alpha, \pm\frac{h}{2}\Big)$$

and v is a deflection of the faces. If $C_2 = 6/(2-\nu)$ we obtain

$$v(x^\alpha) = w(x^\alpha) \overset{\text{df}}{=} u_3(x^\alpha, 0)$$

hence v is a mid–plane deflection.

Let us define the integral quantities

$$M_{\alpha\beta}(x^\gamma) = \int\limits_{-h/2}^{h/2} (z\sigma_{\alpha\beta} + \epsilon_\beta{}^\gamma\mu_{\alpha\gamma})dz$$
$$Q_\alpha(x^\gamma) = \int\limits_{-h/2}^{h/2} \sigma_{\alpha 3}dz \tag{2.5}$$

where $\sigma_{\alpha\beta}$ and $\mu_{\alpha\beta}$ are components of the stress tensor and couple stress tensor. The formulae for stresses and couple stresses are reported in [28]. $M_{\alpha\beta}$ are bending moments and Q_α represent transverse forces of the micropolar plate made of the material with constrained rotations. Performing integration in eq (2.5) (cf eqs (3.6)–(3.10) of [28]) one finds

$$M_{\beta\alpha} = -D\{(1-\nu)\Big[\Big(1 + 24\frac{l^2}{h^2}\Big)v_{,\beta\alpha} + \frac{\nu}{1-\nu}\nabla^2 v\delta_{\beta\alpha} - 24\eta\frac{l^2}{h^2}\epsilon_\alpha{}^\gamma\epsilon_\beta{}^\delta v_{,\delta\gamma}\Big] +$$

$$
\begin{aligned}
&+ \quad h^2\Big[\frac{(2-\nu)h^2}{120}(5C_2-3)+12(1-\nu)\frac{l^3}{h^3}\mathrm{cth}\frac{h}{2l}\Big]\nabla^2 v_{,\beta\alpha} - \\
&- \quad l^2\Big[\Big(4-\nu-(2-\nu)C_2\Big)(\nabla^2 v_{,\beta\alpha}-\eta\epsilon_\alpha{}^\gamma\epsilon_\beta{}^\delta\nabla^2 v_{,\delta\gamma}) + \\
&+ \quad 24(1-\nu)\eta\frac{l^2}{h^2}\epsilon_\alpha{}^\gamma\epsilon_\beta{}^\delta\nabla^2 v_{,\delta\gamma}\Big]\Big\}
\end{aligned}
\tag{2.6}
$$

$$
Q_\alpha(x^\beta) = -\overset{*}{D}\,\nabla^2 v_{,\alpha} \tag{2.7}
$$

where $\overset{*}{D}$ represents a bending stiffness of the micropolar plate. This stiffness is given by the following formula

$$
\overset{*}{D} = \frac{\mu h^3}{6(1-\nu)} + h(\gamma+\varepsilon) = \frac{\mu h^3}{6(1-\nu)}\Big(1+\frac{24(1-\nu)}{h^2}l^2\Big). \tag{2.8}
$$

The Lamé constant is denoted by μ, the constants γ, ε, η being new elastic constants of the material with constrained rotations. The interrelations between the constants l, η and γ, ε are

$$
\begin{aligned}
l^2 &= \frac{\gamma+\varepsilon}{4\mu} &\qquad \eta &= \frac{\gamma-\varepsilon}{\gamma+\varepsilon} \\
\gamma &= 2\mu l^2(1+\eta) &\qquad \varepsilon &= 2\mu l^2(1-\eta).
\end{aligned}
\tag{2.9}
$$

Computing the fields $w(x^\alpha)$, $\hat{w}(x^\alpha)$ from eq $(2.2)_2$ one finds a relation

$$
w(x^\alpha) - \hat{w}(x^\alpha) = -\frac{\nu h^2}{8(1-\nu)}\nabla^2 v. \tag{2.10}
$$

Using eq (2.6) and considering (2.4) one can note that the invariant $\mathcal{M} = M^{\alpha\beta}\delta_{\alpha\beta}$ is defined by the following formula

$$
\mathcal{M} = M^{\alpha\beta}\delta_{\alpha\beta} = -\frac{\mu h^3(1+\nu)}{6(1-\nu)}\Big(1+\varepsilon\frac{12(1-\nu)}{(1+\nu)\mu h^2}\Big)\nabla^2 v. \tag{2.11}
$$

On eliminating $\nabla^2 v$ from eqs (2.10), (2.11) we arrive at the equality

$$
\mathcal{M}(x^\alpha) = M_{11} + M_{22} = \frac{2Eh}{3\nu}\Big(1+\varepsilon\frac{24(1-\nu)}{Eh^2}\Big)\Big(w(x^\alpha)-\hat{w}(x^\alpha)\Big) \tag{2.12}
$$

where $E = 2(1+\nu)\mu$ is a Young modulus. Similarly, the transverse forces can be written as follows

$$
Q_\alpha(x^\beta) = \frac{4\mu h}{3\nu}\Big(1+\frac{24(1-\nu)}{h^2}l^2\Big)\Big(w(x^\alpha)-\hat{w}(x^\alpha)\Big)_{,\alpha}. \tag{2.13}
$$

It is readily seen that the sum of principal moments M_{11} and M_{22} in each point of the plate mid–plane is proportional to the difference of these deflections[1]. The above relation is valid for all plates loaded along the lateral edge surface only (the body forces are omitted) and such that the through thickness distribution of tractions on this surface coincides with the formulae (3.6)–(3.10) of paper [28].

Assumling that in each point we know the values of $\mathcal{M}$ and Q_α and we can measure the quantites w, $\hat{w}$, $w_{,\alpha}$, $\hat{w}_{,\alpha}$. Then, with the help of eqs (2.12) and (2.13) we can determine both micropolar constants γ and ε or l and η (see eq (2.9)). For instance, the value of $\mathcal{M}$ is known for a circular plate of radius R and thickness h, immovably supported along

$$\Big(r = R,\ \ z = \pm\frac{h}{2}\Big), \qquad \Big(u_3(R, \pm\frac{h}{2}) = 0,\ \ u_r(R, \pm\frac{h}{2}) = 0\Big)$$

loaded on the edge with moments of intensity M = const. In such a plate

$$M_r = M_\varphi = M, \qquad \mathcal{M} = 2M, \qquad Q_r = 0$$

and from eq (2.12) one can obtain

$$M = \frac{Eh}{3\nu}\Big(1 + \varepsilon\frac{24(1-\nu)}{Eh^2}\Big)\Big(w(r) - \hat{w}(r)\Big).$$

Similarly, in a clamped, cantilever strip loaded by a transverse force P we can easily compute the $\mathcal{M}$ invariant and we know the transverse force $Q_1 = P$.

However, an essential problem remains: how to measure the difference between deflections $\Big(w(x^\alpha) - \hat{w}(x^\alpha)\Big)$ as well as its derivatives in the course of real experiments.

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Streszczenie

W pracy wyprowadzono pewne zależności występujące w płytach wykonanych z materiału ze związanymi obrotami. Zależności te mogą być podstawą do przeprowadzenia doświadczeń w celu wyznaczenia stałych sprężystych ciała mikropolarnego.

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