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DUAL FINITE ELEMENT METHOD IN FRICTION PROBLEMS[1]

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The application of the equilibrium model in terms of the finite element method to plane stress and plane strain problems is considered in the paper. The influence of friction of Coulomb type is taken into account. The problem is set in the form of quasi-variational inequality. The Airy stress function is used for calculation of the statically admissible solution. The iterative method of solving of the problem and some numerical results are presented in the paper.

1. Setting of the problem

Let Ω be an open bounded domain in $\mathcal{R}^2$ with regular boundary $\partial\Omega$. The plane stress and plane strain problems for the isotropic elastic body are considered. It is assumed that the friction boundary conditions hold on the part Γ_μ of the boundary $\partial\Omega$. The solution of the problem satisfies the following relations

(i) the strain–displacement relations

$$\varepsilon_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) \qquad \text{on} \quad \Omega \tag{1.1}$$

where $\varepsilon_{\alpha\beta}$ is the tensor of small deformations

(ii) the equilibrium equations

$$\sigma_{\beta\alpha,\beta} + b_\alpha = 0 \qquad \text{on} \quad \Omega \tag{1.2}$$

(iii) the constitutive relations

$$\varepsilon_{\alpha\beta} = c_{\alpha\beta\gamma\delta}\, \sigma_{\gamma\delta} \tag{1.3}$$

[1]The paper rewarded with the first prize in The Polish Theoretical Papers on Mechanical Engineering Competition held by the Łódź Branch of Polish Society of Theoretical Applied Mechanics

where the flexibility tensor $\mathbf{c}$ has the form

$$c_{\alpha\beta\gamma\delta} = \frac{1}{E}\Big[(1+\nu)\delta_{\alpha\gamma}\,\delta_{\beta\delta} - \nu(1+\kappa\nu)\delta_{\alpha\beta}\,\delta_{\gamma\delta}\Big]$$

E is the Young modulus, ν is the Poisson ratio, and κ is equal to 0 or 1 for the plane stress problem or plane strain problem, respectively;

(iv) the boundary conditions

$$u_\alpha = U_\alpha \qquad \text{on} \quad \Gamma_u \tag{1.4}$$

$$\sigma_{\beta\alpha}\, n_\beta = T_\alpha \qquad \text{on} \quad \Gamma_\sigma \tag{1.5}$$

$$\left.\begin{array}{lll} u_N \leq 0 & \sigma_N \leq 0 & u_N\sigma_N = 0 \\ |\sigma_T| \leq \mu|\sigma_N| & & u_T\sigma_T \leq 0 \end{array}\right\} \qquad \text{on} \quad \Gamma_\mu \tag{1.6}$$

where

$$\bar{\Gamma}_u \cup \bar{\Gamma}_\sigma \cup \bar{\Gamma}_\mu = \partial\Omega$$

$$\Gamma_u \cap \Gamma_\sigma = \Gamma_u \cap \Gamma_\mu = \Gamma_\sigma \cap \Gamma_\mu = \emptyset$$

n is the unit vector outward normal to the boundary $\partial\Omega$

$$\begin{aligned} u_N &= u_\alpha\, n_\alpha \\ \sigma_N &= \sigma_{\alpha\beta}\, n_\alpha\, n_\beta \\ u_{T\alpha} &= u_\alpha - u_\beta\, n_\beta\, n_\alpha \\ \sigma_{T\alpha} &= (\sigma_{\alpha\beta}\, n_\beta - \sigma_{\beta\gamma}\, n_\beta\, n_\gamma\, n_\alpha) \end{aligned} \tag{1.7}$$

2. Variational formulation of the problem

We shall set our problem in the variational form. Using Eq (1.3), we can write the following equation

$$\int\limits_{\Omega} (c_{\alpha\beta\gamma\delta}\,\sigma_{\gamma\delta} - \varepsilon_{\alpha\beta})(\tau_{\alpha\beta} - \sigma_{\alpha\beta})\, dx = 0 \tag{2.1}$$

After using the relation (1.1) and the Green's formula, Eq (2.1) can be written as follows

$$\begin{aligned} &\int\limits_{\Omega} c_{\alpha\beta\gamma\delta}\,\sigma_{\gamma\delta}\,(\tau_{\alpha\beta} - \sigma_{\alpha\beta})\, dx - \int\limits_{\partial\Omega} u_\alpha\,(\tau_{\alpha\beta} - \sigma_{\alpha\beta})\, n_\beta\, ds + \\ &+ \int\limits_{\Omega} u_\alpha\,(\tau_{\alpha\beta} - \sigma_{\alpha\beta})_{,\beta}\, dx = 0 \end{aligned} \tag{2.2}$$

Let Y denote the set of statically admissible stress fields

$$Y = \Big\{\tau \in [L^2(\Omega)]^4 \; : \quad \tau_{\alpha\beta} = \tau_{\beta\alpha} \quad \tau_{\beta\alpha,\beta} + b_\alpha = 0 \quad \text{on} \quad \Omega; \\ \tau_{\alpha\beta}\, n_\beta = T_\alpha \quad \text{on} \quad \Gamma_\sigma \Big\} \tag{2.3}$$

We can see that Eq (2.2) induces the equation

$$\int_\Omega c_{\alpha\beta\gamma\delta}\, \sigma_{\gamma\delta}\, (\tau_{\alpha\beta} - \sigma_{\alpha\beta})\, dx - \int_{\Gamma_u} U_\alpha\, (\tau_{\alpha\beta} - \sigma_{\alpha\beta})\, n_\beta\, ds + \\ + \int_{\Gamma_\mu} u_\alpha\, (\tau_{\alpha\beta} - \sigma_{\alpha\beta})\, n_\beta\, ds = 0 \qquad \forall \tau \in Y \qquad \sigma \in Y \tag{2.4}$$

Let B denote the following set

$$B(\sigma_N) = \Big\{\tau_T \; : \quad |\tau_T| \leq \mu|\sigma_N|\Big\} \tag{2.5}$$

It follows from Eq (1.6) that the following inequality holds

$$u_T(\tau_T - \sigma_T) \geq 0 \qquad \forall \tau_T \in B \tag{2.6}$$

We see, from Eqs (1.6) and (2.6) that the last integral in Eq (2.4) is non-negative

$$\int_{\Gamma_\mu} u_\alpha\, (\tau_{\alpha\beta} - \sigma_{\alpha\beta})\, n_\beta\, ds = \\ = \int_{\Gamma_\mu} \Big[u_N(\tau_N - \sigma_N) + u_T(\tau_T - \sigma_T)\Big]\, ds \geq 0 \qquad \forall \tau \in K(\sigma)$$

where

$$K(\sigma_N) = \Big\{\tau \in Y \; : \quad \tau_N \leq 0 \quad |\tau_T| \leq \mu|\sigma_N| \quad \text{on} \quad \Gamma_\mu\Big\} \tag{2.7}$$

Then we can write our problem in the form of the quasi-variational inequality: find $\sigma \in K(\sigma_N)$ such that

$$\int_\Omega c_{\alpha\beta\gamma\delta}\, \sigma_{\gamma\delta}\, (\tau_{\alpha\beta} - \sigma_{\alpha\beta})\, dx \geq \int_{\Gamma_u} U_\alpha\, (\tau_{\alpha\beta} - \sigma_{\alpha\beta})\, n_\beta\, ds \\ \forall \tau \in K(\sigma_N) \tag{2.8}$$

The problem (2.8) has a unique solution. The theory of quasi-variational inequalities has already been described (cf [1,7]). Some applications of quasi-variational inequalities to contact problems in mechanics have been given by e.g. Bielski and Telega [2], Hlavaček et al. [6] and Telega [9].

3. Finite element discretization

The statically admissible stress fields, which belong to Y, are established in terms of the Airy stress function Φ

$$\tau_{\alpha\beta} = e_{\alpha\gamma}\, e_{\beta\delta}\, \Phi_{,\gamma\delta} + \delta_{\alpha\beta}\, \varphi \tag{3.1}$$

In Eq (3.1) φ denotes the potential of volume forces b, i.e. $b_\alpha = -\varphi_{,\alpha}$, $e_{\alpha\beta}$ is the permutation symbol. The functions mentioned above are to be of classes as follows

$$\Phi \in H^2(\Omega)/P_1 \equiv Z \tag{3.2}$$

$$\varphi \in H^1(\Omega) \tag{3.3}$$

where $H^2(\Omega)$ is the Sobolev space, and

$$P_1 = \left\{\Phi : \quad \Phi_{,\alpha\beta} = 0 \quad \forall \alpha,\, \beta = 1,2\right\}$$

Let Z_T denote the following subset of the space Z

$$Z_T = \left\{\Phi \in Z : \quad e_{\alpha\gamma}\, e_{\beta\delta}\, \Phi_{,\gamma\delta}\, n_\beta = T_\alpha \quad \text{on} \quad \Gamma_\sigma\right\} \tag{3.4}$$

It is easy to proof that the stress fields generated by the relation (3.1) belong to the set Y if the function Φ belongs to the set Z_T and $b_\alpha = -\varphi_{,\alpha}$.

Let $\mathcal{T}_h$ be a finite element mesh for Ω such that $\bar{\Omega} = \bigcup \bar{K}_i$, $K_i \in \mathcal{T}_h$, where K_i is the subdomain of Ω. Let $Z_h \subset Z$ denote the discrete space associated with $\mathcal{T}_h$. The relation (3.1) gives the stress fields fulfilling the equilibrium equations inside Ω only; the boundary equilibrium conditions (on Γ_σ) are fulfilled by the use of the Lagrange multipliers technique.

The discrete formulation suited to Eq (2.8) can be written in the form: find $\sigma_h \in K_h(\sigma_{Nh})$ such that

$$\int\limits_\Omega c_{\alpha\beta\gamma\delta}\, \sigma_{h\gamma\delta}\, (\tau_{\alpha\beta} - \sigma_{h\alpha\beta})\, dx \geq \int\limits_{\Gamma_u} U_\alpha\, (\tau_{\alpha\beta} - \sigma_{h\alpha\beta})\, n_\beta\, ds$$

$$\forall \tau \in K_h(\sigma_{Nh}) \tag{3.5}$$

where

$$K_h = K \cap Y_h \tag{3.6}$$

Y_h denotes the discrete subset of Y associated with Z_T and $\mathcal{T}_h$.

The rectangular element of Bogner, Fox and Schmit [3] is utilized in the present paper, so the space Z_h can be represent as follows

$$Z_h = \left\{\Phi \in Z : \quad \Phi \in C^1(\bar{\Omega})\ \&\ \Phi\Big|_{K_i} \in Q_3(K_i) \quad \forall K_i \in \mathcal{T}_h\right\} \tag{3.7}$$

where $Q_3(K)$ is the space of all polynomials of degree less than or equal to three with respect to both variables x_1, x_2. The friction conditions (1.6) on the boundary Γ_μ are checked at four points on the side of the rectangle $K_i \in \mathcal{T}_h$, coordinates of which are compatible with the Gauss quadrature rule.

4. The iterative solution

Let Λ be the following space

$$\Lambda = H^{\frac{1}{2}}(\Gamma_\mu)$$

and let Λ_h be a discrete subspace of Λ associated with the Gauss integration rule for the boundary Γ_μ.

The following iterative procedure is proposed for solution of the discrete problem

(i) initiation of $\lambda_N, \lambda_T \in \Lambda_h$: $\lambda_N^{(0)} = \lambda_T^{(0)} = 0$,

(ii) for $i = 1, 2, \ldots$, calculation of the successive estimation of $\sigma_h^{(i)}$, $\lambda_N^{(i)}$ and $\lambda_T^{(i)}$ from equations

$$\begin{aligned}\int_\Omega c_{\alpha\beta\gamma\delta}\, \sigma_{h\gamma\delta}^{(i)} \left(\tau_{\alpha\beta} - \sigma_{h\alpha\beta}^{(i)}\right) dx &= \int_{\Gamma_u} U_\alpha \left(\tau_{\alpha\beta} - \sigma_{h\alpha\beta}^{(i)}\right) n_\beta\, ds + \\ &+ \int_{\Gamma_\mu} \left[-\lambda_N^{(i-1)} \left(\tau_N - \sigma_{hN}^{(i)}\right) + \lambda_T^{(i-1)} \left(\tau_T - \sigma_{hT}^{(i)}\right)\right] ds \qquad \forall \tau \in Y_h\end{aligned} \tag{4.1}$$

$$\lambda_N^{(i)} = \begin{cases} \max(\lambda_N^{(i-1)} + \omega_N \sigma_N^{(i)}, 0) & \text{for} \quad \sigma_N > 0 \\ 0 & \text{for} \quad \sigma_N \le 0 \end{cases} \tag{4.2}$$

$$\lambda_T^{(i)} = \begin{cases} 0 & \text{for} \quad \sigma_N^{(i)} \ge 0 \quad \text{or} \quad \sigma_T^{(i)} = 0 \\ \max\left(\max(\lambda_T^{(i-1)}, 0) + \Delta, 0\right) & \text{for} \quad \sigma_N^{(i)} < 0 \ \ \& \ \ \sigma_T^{(i)} > 0 \\ \min\left(\min(\lambda_T^{(i-1)}, 0) - \Delta, 0\right) & \text{for} \quad \sigma_N^{(i)} < 0 \ \ \& \ \ \sigma_T^{(i)} < 0 \end{cases} \tag{4.3}$$

until the required accuracy of solution is achieved, where

$$\Delta = \omega_T \left(1 - \frac{\mu|\sigma_N^{(i)}|}{|\sigma_T^{(i)}|}\right) \tag{4.4}$$

$\omega_N, \omega_T > 0$.

5. Numerical results

The plane stress problem for the rectangular elastic plate compressed by two rigid plates is considered (see Fig.1).

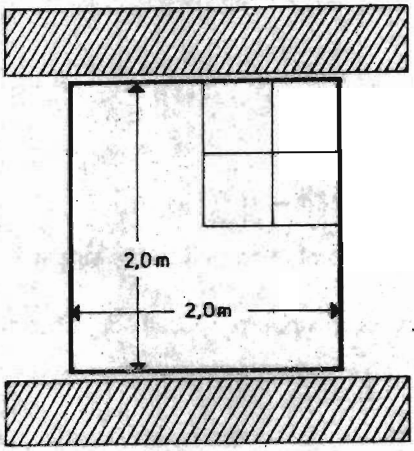

Fig. 1.

The relative vertical translation U between the rigid plates is given. The contact between the elastic body and rigid bodies is assumed to be of Coulomb type. Calculations have been made with the following data: $E = 2.09 \cdot 10^5$ MPa, $\nu = 0.3$, $U = 0.001$ for several values of the friction coefficient. The quarter of the plate has been divided into 4 square elements. The obtained maps of principal stresses are shown in Figs. 2 $\div$ 5 for the following values of friction coefficient μ: 0, 0.1, 0.2, and 0.3.

For each value of μ, 30 iterations have been performed. The value of ω_T has been experimentally defined as equal to $1 \cdot 10^{-5}$. Tab.1. consists the magnitudes of vertical force P which cause the contraction U.

Tab.1

μ	$P/2$
0.00	209.00 (exact)
0.00	209.15
0.05	210.67
0.10	212.05
0.15	212.86
0.20	213.44
0.25	213.81
0.30	213.89

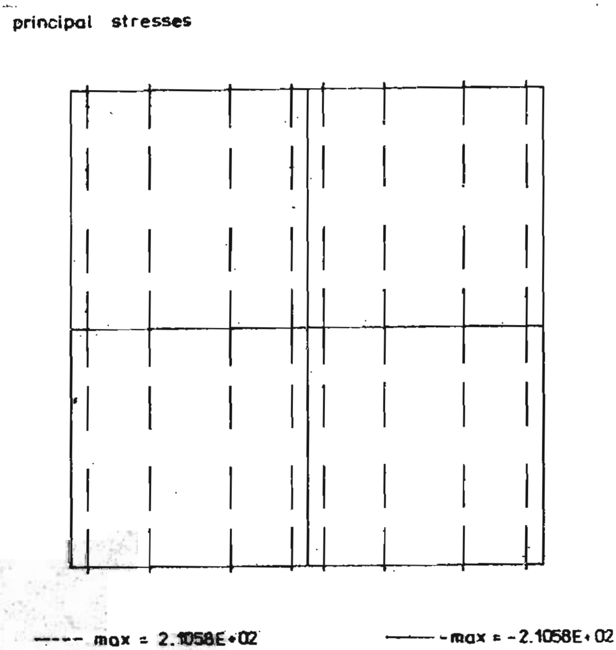

Fig. 2. $\mu = 0$

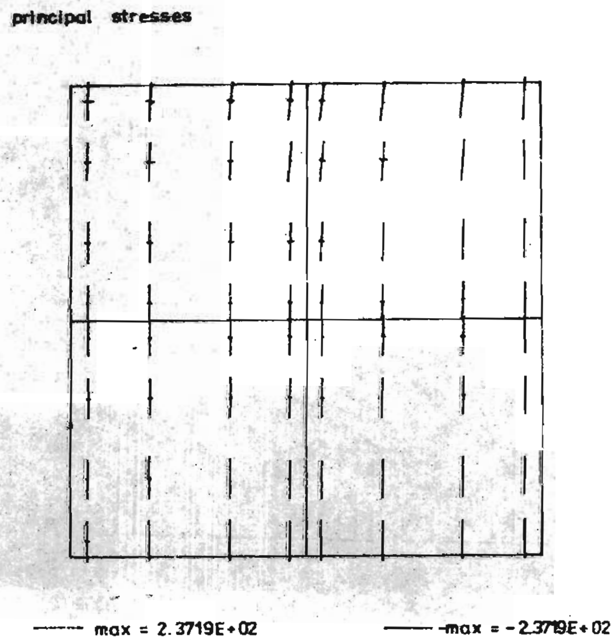

Fig. 3. $\mu = 0.1$

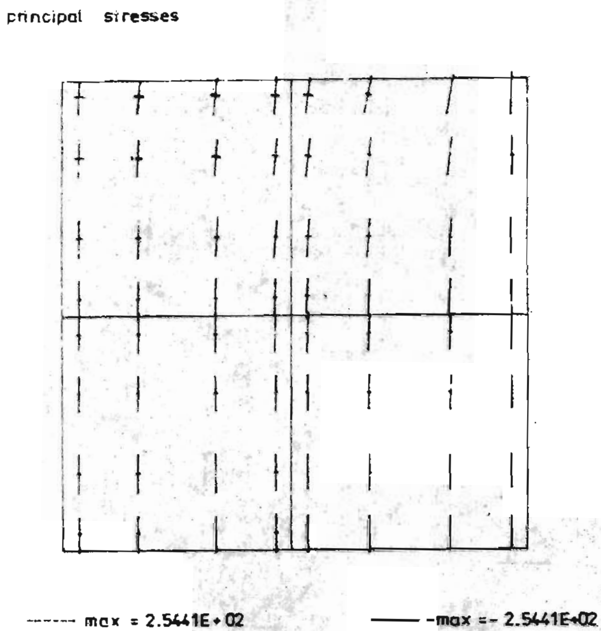

Fig. 4. $\mu = 0.2$

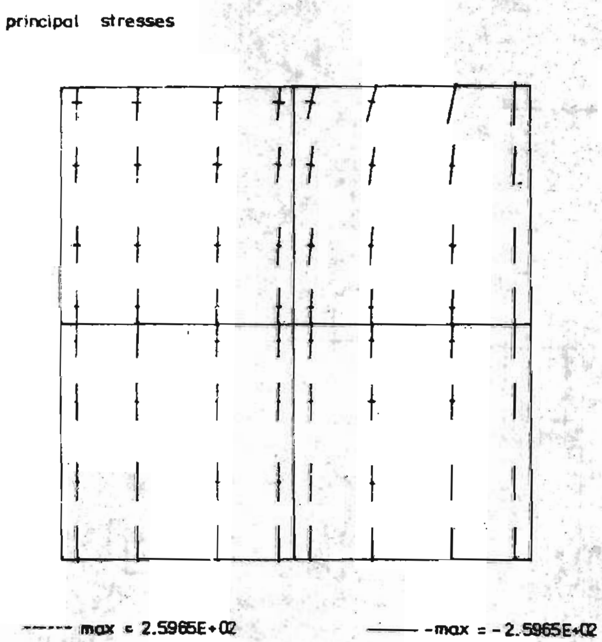

Fig. 5. $\mu = 0.3$

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Naprężeniowa metoda elementów skończonych w zagadnieniach tarcia

Streszczenie

W pracy opisano zastosowanie naprężeniowej wersji metody elementów skończonych w zagadnieniu tarcia Coulomba. Rozważono płaskie zagadnienia teorii sprężystości. Zagadnienie sformułowano w postaci nierówności quasi-wariacyjnej. Statycznie dopuszczalne pola naprężeń skonstruowano przy pomocy funkcji naprężeń Airy'ego. Podano iteracyjny algorytm rozwiązania zagadnienia oraz przykład obliczeń.

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