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CONSTITUTIVE RELATIONSHIPS FOR ELASTIC AND PLASTIC BEHAVIOUR OF ISOTROPIC MATRIX REINFORCED WITH THREE FAMILIES OF FIBRES

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The nonpolynomial anisotropic tensor function theory is employed together with the theorems of their representations. The tensor function approach combines clarity with the necessary generality of the constitutive equations to be formulated with the help of this type of description. As known, constitutive relationships can be presented for virtually any deformable continuum provided the necessary tensor generators are established together with the minimal set of fundamental invariants. In this paper the constitutive equations are formulated for nonlinear elasticity and perfect plasticity of innately anisotropic media. The isotropic matrix is reinforced with three families of straight fibres. Within each family the fibres are evenly distributed and made of the same material. Three cases of reinforcement are considered: a) nonorthogonal fibres, b) orthogonal fibres, c) orthogonal fibres, each family being made of the same material.The introduction of reinforcement causes the material symmetry group to be a finite one, which can be suitably characterized by parametric tensors. General constitutive relations are formulated to describe nonlinearly elastic behaviour. The flow rules and yield criteria are derived from the condition that the constitutive equations must be homogeneous zero order functions of strain rates. Derivation of those equations is based on the concept developed by Sawczuk, Stutz and Boehler. Their linearization leads to the relatively simple expressions. The simplified perfect plasticity models are proposed. The simplification consists in application of an equivalent stress tensor which is a tensor transformed by using a fourth-order symmetric tensor dependent of parametric tensors describing the material symmetry group.

1. Introduction

The theory of tensor functions has turned out to be very useful in those situations in which no beforehand knowledge is available as to the type of functional dependence between the mechanical variables to enter the constitutive relationships. It is, in particular, the theorems of representations that can reveal all the necessary arguments and their mutual interdependence.

In the paper the nonpolynomial representations of tensor-valued functions are assumed as firstly employed by Pipkin and Wineman [11,21]. Construction of a representation of a scalar-valued function consists in the determination of a functional basis for a given set of arguments and symmetry groups. If, for instance, a second-order tensor-valued function is considered, an irreducible set of generators must be found in addition. It was Wang [18 ÷ 20], Smith [13] and Boehler [2], who tackled the problem of determination of irreducible nonpolynomial representation of isotropic scalar- and vector-valued functions as well as symmetric and antisymmetric second-order tensor-valued functions depending on the finite number of vectors and second-order tensors, both symmetric and antisymmetric. Nonpolynomial representations of anisotropic scalar-valued functions and symmetric second-order tensor-valued functions depending on symmetric second-order tensors were dealt with by Boehler and Raclin [3], Boehler [4,5] and Basista [1].

It has been shown that nonpolynomial representations, as compared with polynomial representations of the same tensor-valued function, contain fewer tensor generators and invariants. Thus the former ones have two advantages: first- are more general as to the type of functions admitted and second- enable the obtained constitutive equations to have more concise forms.

By imposing certain additional constraints on the considered representations of tensor-valued functions, various types of constitutive relationships can be arrived at. For example, it is an inherent feature of plastic behaviour that the stresses are time-independent, i.e. the material stays insensitive to the rate at which strains develop in the mechanical process. In mathematical terms this means that the corresponding constitutive relations have to remain zero degree homogeneous with respect to the strain rates, see Sawczuk and Stutz [12]. This property was later used for transversally isotropic material by Boehler [4], Boehler and Sawczuk [6,7], and Murakami and Sawczuk [10] and for reinforced concrete by Jemioło et al. [8,9].

2. Formulation of the problem

An isotropic matrix, in which a system of reinforcing fibres is embedded becomes an innately anisotropic body, similarly as it is in the case of crystalline materials. On the contrary, a deformation induced anisotropy is caused by irreversible strains or structural microdefects developing as the deformation process proceeds. This is shown in Fig.1, in which three families of straight bars enforce a defined macroscopic structure of a composite body. The symmetry group of it

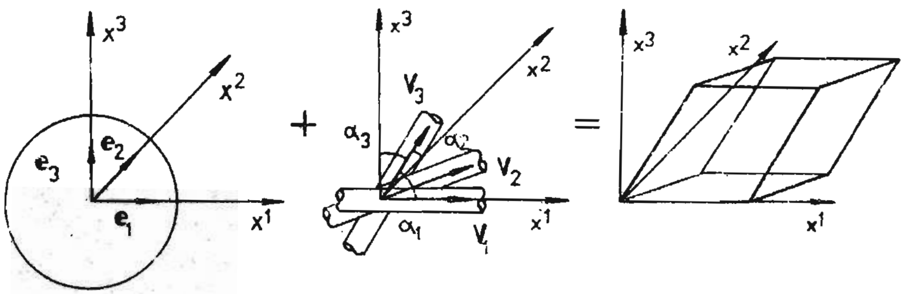

Fig. 1.

strictly corresponds to a certain group of material symmetry of an equivalent monocrystal. Unit vectors e_i, $i = 1, 2, 3$ are the versors of the Cartesian coordinate system x^i fixed to the considered body. Vectors v_i are directed along the fibres and can be described as follows

$$v_1 = k_1[1,0,0] \qquad v_2 = k_2[c,s,0] \qquad v_3 = k_3[c_1,c_2,c_3] \tag{2.1}$$

where $c = \cos\beta$, $s = \sin\beta$, $c_i = \cos\alpha_i$, $k_1 = 1$ (direction 1 is taken as reference), $k_2, k_3 > 0$ are coefficients of relative intensity of reinforcement in the directions 2 and 3, respectively. Product of symmetry groups of the versors v_i does not, in general, coincide with the material symmetry group S. The latter is clearly dependent on both the locations of these vectors, i.e. angles α_i, β and their lengths i.e. k_2, k_3. Our aim is to formulate for the described composite body the nonlinear and linear constitutive relationships for elastic and perfectly plastic behaviour in generally anisotropic and orthotropic situations.

Composites with one and two families of fibres were considered by Spencer [14,15] by means of polynomial representations. Nonpolynomial ones were employed by Basista [1] to describe nonlinear elastic behaviour of skew anisotropic medium.

Constitutive relationships for a matrix with three families of straight fibres can be expressed as an isotropic second-order tensor-valued function

$$\mathbf{T} = \mathbf{F}\big(\mathbf{A}, \mathbf{P}_m\big) \qquad m = 1, ..., M \tag{2.2}$$

which, according to the principle of physical (Euclidian) space isotropy, has to satisfy the following relations

$$\forall\, \mathbf{Q} \in \mathcal{O} \qquad \mathbf{QTQ}^{\mathsf{T}} = \mathbf{F}\big(\mathbf{QAQ}^{\mathsf{T}}, \bar{\mathbf{P}}_m\big) \tag{2.3}$$

where $\mathbf{T}$ is a symmetric stress tensor, $\mathbf{A}$ stands for a symmetric strain tensor $\mathbf{E}$ for elasticity and denotes a symmetric strain rate tensor $\mathbf{D}$ for plasticity, $\mathbf{Q}$ is an orthogonal tensor $\mathbf{Q}\mathbf{Q}^{\mathsf{T}} = \mathbf{Q}^{\mathsf{T}}\mathbf{Q} = \mathbf{I}$ belonging to the full group of orthogonal transformations $\mathcal{O}$, $\mathbf{I}$ denotes a unit tensor, $\mathbf{P}_m$ are parametric tensors to describe reinforcement (in other words, a group of material symmetry $\mathcal{S}$ induced by the fibres), $\bar{\mathbf{P}}_m$ stands for $\mathbf{P}_m$ after the transformation $\mathbf{Q}$. The structural tensors satisfy the condition

$$\forall\, \mathbf{Q} \in \mathcal{S} \subset \mathcal{O} \qquad \bar{\mathbf{P}}_m = \mathbf{P}_m \tag{2.4}$$

which yields

$$\forall\, \mathbf{Q} \in \mathcal{S} \qquad \mathbf{F}\left(\mathbf{Q}\mathbf{A}\mathbf{Q}^{\mathsf{T}}, \mathbf{P}_m\right) = \mathbf{Q}\mathbf{F}\left(\mathbf{A}, \mathbf{P}_m\right)\mathbf{Q}^{\mathsf{T}} \tag{2.5}$$

Function $\mathbf{F}$ is thus invariant with respect to the transformation group $\mathcal{S}$; it is an anisotropic function with respect to the argument $\mathbf{A}$, (Eq (2.5)), and, at the same time, remains isotropic according to the condition (2.3).

The relationships (2.1) will describe plastic behaviour once $\mathbf{A}$ is replaced by $\mathbf{D}$ and the condition of zero degree homogeneity with respect to the strain rates (cf [13]) is insisted upon

$$\frac{\partial \mathbf{T}}{\partial \mathbf{D}}\mathbf{D} = \mathbf{0} \qquad \text{if} \qquad \frac{\partial \mathbf{T}}{\partial \mathbf{D}} \neq \tilde{\mathbf{0}} \tag{2.6}$$

where $\mathbf{0}$ and $\tilde{\mathbf{0}}$ are second-order and fourth-order zero tensors, respectively.

3. Three families of nonorthogonal fibres

When an isotropic matrix is supplied with three families of nonorthogonal fibres, each having different mechanical properties, the product of symmetry groups of vectors $\boldsymbol{v}_i$ Eq (2.1) corresponds to the material symmetry group and the resulting composite body behaves anisotropically.

3.1. Elasticity

Similarly as Eq (2.2), the constitutive relations for elastic behaviour of a composite take the form of an isotropic second-order tensor-valued function

$$\mathbf{T} = \mathbf{F}\left(\mathbf{E}, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\right) \tag{3.1}$$

having in mind the condition

$$\forall\, \mathbf{Q} \in \mathcal{O} \qquad \mathbf{Q}\mathbf{T}\mathbf{Q}^{\mathsf{T}} = \mathbf{F}\left(\mathbf{Q}\mathbf{E}\mathbf{Q}^{\mathsf{T}}, \mathbf{Q}\boldsymbol{v}_1, \mathbf{Q}\boldsymbol{v}_2, \mathbf{Q}\boldsymbol{v}_3\right) \tag{3.2}$$

Simultaneously, Eq (3.1) is anisotropic with respect to E since

$$\forall\, \mathsf{Q} \in \mathcal{S} = \{\mathsf{I}, -\mathsf{I}\} \qquad \mathsf{Q}\mathsf{T}\mathsf{Q}^{\mathsf{T}} = \mathsf{F}\Big(\mathsf{Q}\mathsf{E}\mathsf{Q}^{\mathsf{T}}, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\Big) \tag{3.3}$$

According to the theorems of Wang [18 ÷ 20], Smith [13] and Boehler [2], the representation of Eq (3.1) is as follows

$$\begin{aligned}
\mathsf{T} = {} & \alpha_1\mathsf{I} + \alpha_2\mathsf{E} + \alpha_3\bar{\mathsf{M}}_{11} + \alpha_4\bar{\mathsf{M}}_{22} + \alpha_5\bar{\mathsf{M}}_{33} + \alpha_6\Big(\bar{\mathsf{M}}_{12} + \bar{\mathsf{M}}_{21}\Big) + \\
+ {} & \alpha_7\Big(\bar{\mathsf{M}}_{13} + \bar{\mathsf{M}}_{31}\Big) + \alpha_8\Big(\bar{\mathsf{M}}_{23} + \bar{\mathsf{M}}_{32}\Big) + \alpha_9\Big(\bar{\mathsf{M}}_{11}\mathsf{E} + \mathsf{E}\bar{\mathsf{M}}_{11}\Big) + \\
+ {} & \alpha_{10}\Big(\bar{\mathsf{M}}_{22}\mathsf{E} + \mathsf{E}\bar{\mathsf{M}}_{22}\Big) + \alpha_{11}\Big(\bar{\mathsf{M}}_{33}\mathsf{E} + \mathsf{E}\bar{\mathsf{M}}_{33}\Big) + \alpha_{12}\mathsf{E}^2 + \\
+ {} & \alpha_{13}\Big(\bar{\mathsf{M}}_{11}\mathsf{E}^2 + \mathsf{E}^2\bar{\mathsf{M}}_{11}\Big) + \alpha_{14}\Big(\bar{\mathsf{M}}_{22}\mathsf{E}^2 + \mathsf{E}^2\bar{\mathsf{M}}_{22}\Big) + \alpha_{15}\Big(\bar{\mathsf{M}}_{33}\mathsf{E}^2 + \mathsf{E}^2\bar{\mathsf{M}}_{33}\Big) + \\
+ {} & \alpha_{16}\Big[\Big(\bar{\mathsf{M}}_{12}\mathsf{E} + \mathsf{E}\bar{\mathsf{M}}_{21}\Big) - \Big(\bar{\mathsf{M}}_{21}\mathsf{E} + \mathsf{E}\bar{\mathsf{M}}_{12}\Big)\Big] + \alpha_{17}\Big[\Big(\bar{\mathsf{M}}_{13}\mathsf{E} + \mathsf{E}\bar{\mathsf{M}}_{31}\Big) - \\
- {} & \Big(\bar{\mathsf{M}}_{31}\mathsf{E} + \mathsf{E}\bar{\mathsf{M}}_{13}\Big)\Big] + \alpha_{18}\Big[\Big(\bar{\mathsf{M}}_{23}\mathsf{E} + \mathsf{E}\bar{\mathsf{M}}_{32}\Big) - \Big(\bar{\mathsf{M}}_{32}\mathsf{E} + \mathsf{E}\bar{\mathsf{M}}_{23}\Big)\Big]
\end{aligned} \tag{3.4}$$

where $\bar{\mathsf{M}}_{ij} = \boldsymbol{v}_i \otimes \boldsymbol{v}_j$, $i,j = 1,2,3$, (no summation is perfomed when $i = j$), α_k are scalar-valued functions of invariants

$$\begin{aligned}
\alpha_k = {} & \alpha_k\Big(\mathrm{tr}\mathsf{E}, \mathrm{tr}\mathsf{E}^2, \mathrm{tr}\mathsf{E}^3, \mathrm{tr}\bar{\mathsf{M}}_{11}\mathsf{E}, \mathrm{tr}\bar{\mathsf{M}}_{22}\mathsf{E}, \mathrm{tr}\bar{\mathsf{M}}_{33}\mathsf{E}, \mathrm{tr}\bar{\mathsf{M}}_{12}\mathsf{E}, \\
& \mathrm{tr}\bar{\mathsf{M}}_{13}\mathsf{E}, \mathrm{tr}\bar{\mathsf{M}}_{23}\mathsf{E}, \mathrm{tr}\bar{\mathsf{M}}_{11}\mathsf{E}^2, \mathrm{tr}\bar{\mathsf{M}}_{22}\mathsf{E}^2, \mathrm{tr}\bar{\mathsf{M}}_{33}\mathsf{E}^2, \mathrm{tr}\bar{\mathsf{M}}_{12}\mathsf{E}^2, \\
& \mathrm{tr}\bar{\mathsf{M}}_{13}\mathsf{E}^2, \mathrm{tr}\bar{\mathsf{M}}_{23}\mathsf{E}^2\Big) \qquad k = 1,...,18
\end{aligned} \tag{3.5}$$

The obtained representation appears to be reducible. To make the process easier, parametric tensors $\bar{\mathsf{M}}_{ij}$ can be rewritten in the form dependent on tensors $\mathsf{M}_{ij} = \boldsymbol{e}_i \otimes \boldsymbol{e}_j$. Detailed analysis of interrelations between generators and invariants in the Cartesian coordinates yields the irreducible representation of Eq (3.4) in the form

$$\begin{aligned}
\mathsf{T} = {} & \alpha'_1\bar{\mathsf{M}}_{11} + \alpha'_2\bar{\mathsf{M}}_{22} + \alpha'_3\bar{\mathsf{M}}_{33} + \alpha'_4\Big(\bar{\mathsf{M}}_{12} + \bar{\mathsf{M}}_{21}\Big) + \\
+ {} & \alpha'_5\Big(\bar{\mathsf{M}}_{13} + \bar{\mathsf{M}}_{31}\Big) + \alpha'_6\Big(\bar{\mathsf{M}}_{23} + \bar{\mathsf{M}}_{32}\Big)
\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}
\alpha'_l = {} & \alpha'_l\Big(\mathrm{tr}\bar{\mathsf{M}}_{11}\mathsf{E}, \mathrm{tr}\bar{\mathsf{M}}_{22}\mathsf{E}, \mathrm{tr}\bar{\mathsf{M}}_{33}\mathsf{E}, \mathrm{tr}\bar{\mathsf{M}}_{12}\mathsf{E}, \mathrm{tr}\bar{\mathsf{M}}_{13}\mathsf{E}, \mathrm{tr}\bar{\mathsf{M}}_{23}\mathsf{E}\Big) \\
& l = 1,...,6
\end{aligned} \tag{3.7}$$

The form of Eq (3.6) and (3.7) coincide with those arrived at by Boehler (cf [3 ÷ 5]) since the product of symmetry groups for vectors $\boldsymbol{e}_i$ corresponds to the symmetry group of vectors $\boldsymbol{v}_i$.

Eqs (3.6) can be linearized by adopting functions α_l' as linear combinations of invariants in Eq (3.7) and assuming the existence of neutral state $\mathbf{E} = \mathbf{0} \Rightarrow \mathbf{T} = \mathbf{0}$. Then

$$
\begin{aligned}
\alpha_l' &= a_l \operatorname{tr}\bar{\mathbf{M}}_{11}\mathbf{E} + b_l \operatorname{tr}\bar{\mathbf{M}}_{22}\mathbf{E} + c_l \operatorname{tr}\bar{\mathbf{M}}_{33}\mathbf{E} + d_l \operatorname{tr}\bar{\mathbf{M}}_{12}\mathbf{E} + \\
&+ e_l \operatorname{tr}\bar{\mathbf{M}}_{13}\mathbf{E} + f_l \operatorname{tr}\bar{\mathbf{M}}_{23}\mathbf{E}
\end{aligned}
\tag{3.8}
$$

Eq (3.6) with scalar-valued function (3.8) contains as many as 36 independent elasticity constants. It describes elastic material in the sense of Cauchy, i.e. material capable of dissipating energy on a closed loading cycle. When elastic potential is assumed to exist, the number of elastic constants reduces to 21.

Jemioło et al. [8] took the relationships (3.4), (3.5) as a starting point to formulate equations simpler than those in Eqs (3.6) and (3.7). The following simplifying assumptions were made: mutual influence of families of fibres were disregarded by putting $\alpha_k = 0$ for $k = 6,7,8,16,17,18$; invariants containing $\bar{\mathbf{M}}_{ij}$, $i \neq j$, were neglected; strain in the reinforcement of direction $\boldsymbol{v}_i$ was described by tensor depending solely on $\boldsymbol{v}_i$. Therefore

$$
\mathbf{E}(\boldsymbol{v}_i) = \varepsilon_{si}(\boldsymbol{v}_i \otimes \boldsymbol{v}_i) = \varepsilon_{si}\bar{\mathbf{M}}_{ii} \qquad \text{(no summation)} \tag{3.9}
$$

and

$$
\begin{array}{lll}
\varepsilon_{si} = \operatorname{tr}\bar{\mathbf{M}}_{ii}\mathbf{E} & & \alpha_k = \bar{\alpha}_k\varepsilon_{si} = \bar{\alpha}_k \operatorname{tr}\bar{\mathbf{M}}_{ii}\mathbf{E} \\
i = 1 & k = 3,9,13 & \\
i = 2 & k = 4,10,14 & \\
i = 3 & k = 5,11,15 &
\end{array}
\tag{3.10}
$$

The above set of assumptions appears to be equivalent to the application of the function (2.2) in the form

$$
\mathbf{T} = \mathbf{F}\Big(\mathbf{E}, \bar{\mathbf{M}}_{11}, \bar{\mathbf{M}}_{22}, \bar{\mathbf{M}}_{33}\Big) \tag{3.11}
$$

with the simultaneous condition

$$
\forall\, \mathbf{Q} \in \mathcal{O} \qquad \mathbf{Q}\mathbf{T}\mathbf{Q}^{\top} = \mathbf{F}\Big(\mathbf{Q}\mathbf{E}\mathbf{Q}^{\top}, \mathbf{Q}\bar{\mathbf{M}}_{11}\mathbf{Q}^{\top}, \mathbf{Q}\bar{\mathbf{M}}_{22}\mathbf{Q}^{\top}, \mathbf{Q}\bar{\mathbf{M}}_{33}\mathbf{Q}^{\top}\Big) \tag{3.12}
$$

On using the theorems of representations [2,13,18 ÷ 20], the relations (3.11), (3.12) lead to the relations (3.4) with $\alpha_k = 0$, $k = 6,7,8,16,17,18$; the remaining α_k's depend on the invariants $\operatorname{tr}\mathbf{E}^i$, $\operatorname{tr}\bar{\mathbf{M}}_{ii}\mathbf{E}^m$, $i = 1,2,3$, $m = 1,2$.

In the case of orthotropy (when parametric tensors in Eq (3.11) are $\mathbf{M}_{ii}$ instead of $\bar{\mathbf{M}}_{ii}$) Eq (3.4) is simplified since, in addition, $\alpha_k = 0$ for $k = 1,2,13,14,15$; invariants $\operatorname{tr}\mathbf{E}^m$, $m = 1,2$ are reducible.

The simplest linear equations for independent families of fibres and the assumptions (3.9), (3.10) were given by Jemiolo et al. [8]. For different mechanical properties of fibres it becomes

$$\mathbf{T} = \alpha_1 \mathbf{I} + \alpha_2 \mathbf{E} + \bar{\alpha}_3 \Big(\mathrm{tr}\bar{\mathbf{M}}_{11}\mathbf{E}\Big)\mathbf{M}_{11} + \bar{\alpha}_4 \Big(\mathrm{tr}\bar{\mathbf{M}}_{22}\mathbf{E}\Big)\bar{\mathbf{M}}_{22} + \bar{\alpha}_5 \Big(\mathrm{tr}\bar{\mathbf{M}}_{33}\mathbf{E}\Big)\bar{\mathbf{M}}_{33} \tag{3.13}$$

where

$$\begin{aligned} \alpha_1 &= \frac{\nu_c E_c}{(1+\nu_c)(1-2\nu_c)}\mathrm{tr}\mathbf{E} \qquad & \alpha_2 &= \frac{E_c}{(1+\nu_c)} \\ \bar{\alpha}_3 &= E_{s1}\mu_1 \qquad \bar{\alpha}_4 = E_{s2}\mu_2 \qquad & \bar{\alpha}_5 &= E_{s3}\mu_3 \end{aligned} \tag{3.14}$$

ν_c and E_c denote Poisson's ratio and Young's modulus, both for matrix (subscript c stands for concrete), E_{si} stand for Young moduli for reinforcing fibres (subscript s denotes steel), μ_i are the reinforcement intensities in the directions of $\mathbf{v}_i$.

Eqs (3.13), (3.14) were used to formulate linear constitutive relationships for plane stress and to describe, after making elastic moduli dependent upon the current strain state, the behaviour of reinforced concrete slab according to the deformation theory of plasticity (cf [8]).

3.2. Perfect plasticity

Description adopted in the paper is based on the concept of Sawczuk and Stutz [12] as used for transversal isotropy (cf [4,6,7]). The expression (2.2) together with the homogeneity requirement (2.6) for parametric tensors $\bar{\mathbf{M}}_{ij}$ was discussed by Jemiolo et al. ([8]).

The equation

$$\begin{aligned} \mathbf{T} &= \beta_1\bar{\mathbf{M}}_{11} + \beta_2\bar{\mathbf{M}}_{22} + \beta_3\bar{\mathbf{M}}_{33} + \beta_4\Big(\bar{\mathbf{M}}_{12} + \bar{\mathbf{M}}_{21}\Big) + \beta_5\Big(\bar{\mathbf{M}}_{13} + \bar{\mathbf{M}}_{31}\Big) + \\ &+ \beta_6\Big(\bar{\mathbf{M}}_{23} + \bar{\mathbf{M}}_{32}\Big) \\ \beta_l &= \beta_l\Big(\mathrm{tr}\bar{\mathbf{M}}_{11}\mathbf{D}, \mathrm{tr}\bar{\mathbf{M}}_{22}\mathbf{D}, \mathrm{tr}\bar{\mathbf{M}}_{33}\mathbf{D}, \mathrm{tr}\bar{\mathbf{M}}_{12}\mathbf{D}, \mathrm{tr}\bar{\mathbf{M}}_{13}\mathbf{D}, \mathrm{tr}\bar{\mathbf{M}}_{23}\mathbf{D}\Big) \\ & \qquad l = 1, ..., 6 \end{aligned} \tag{3.15}$$

with the condition (2.6) describes plastic behaviour of the considered anisotropic composite.

Let us follow the consequencies of zero degree homogeneity requirement (2.6) as applied to the trace of equation (3.15). To this end, let the following new kinematic variables be introduced

$$
\begin{array}{ll}
E_1 = \mathrm{tr}\bar{\mathbf{M}}_{11}\mathbf{D} & E_2 = \mathrm{tr}\bar{\mathbf{M}}_{22}\mathbf{D} \\
E_3 = \mathrm{tr}\bar{\mathbf{M}}_{33}\mathbf{D} & E_4 = \mathrm{tr}\bar{\mathbf{M}}_{12}\mathbf{D} \\
E_5 = \mathrm{tr}\bar{\mathbf{M}}_{13}\mathbf{D} & E_6 = \mathrm{tr}\bar{\mathbf{M}}_{23}\mathbf{D}
\end{array}
\tag{3.16}
$$

What results is the set of Euler's equations

$$
\frac{\partial \beta_k}{\partial E_l}E_l = 0 \qquad k,l = 1,...,6
\tag{3.17}
$$

which, after adopting the subsitution for x and y_m

$$
\begin{array}{ll}
x = \ln|E_6| & y_m = \ln\Big|\dfrac{E_m}{E_6}\Big| \\
E_6 \neq 0 & m = 1,...,5
\end{array}
\tag{3.18}
$$

can be solved to yield the functions

$$
\beta_k = A_k\Big(\frac{E_m}{E_6}\Big) = A_k(p_m)
\tag{3.19}
$$

Allowing for (3.19) in (3.15) some auxiliary magnitudes such as $\bar{\mathbf{M}}_{11}\mathbf{T}$, $\bar{\mathbf{M}}_{22}\mathbf{T}$, $\bar{\mathbf{M}}_{33}\mathbf{T}$, $\bar{\mathbf{M}}_{12}\mathbf{T}$, $\bar{\mathbf{M}}_{13}\mathbf{T}$, $\bar{\mathbf{M}}_{23}\mathbf{T}$ are calculated together with their traces. Six relations are thus obtained dependent on 5 kinematic variables which characterize plastic flow. It follows that an additional scalar relation must exist among the stress invariants, termed "a plasticity criterion"

$$
f\Big(\mathrm{tr}\bar{\mathbf{M}}_{11}\mathbf{T}, \mathrm{tr}\bar{\mathbf{M}}_{22}\mathbf{T}, \mathrm{tr}\bar{\mathbf{M}}_{33}\mathbf{T}, \mathrm{tr}\bar{\mathbf{M}}_{12}\mathbf{T}, \mathrm{tr}\bar{\mathbf{M}}_{13}\mathbf{T}, \mathrm{tr}\bar{\mathbf{M}}_{23}\mathbf{T}\Big) = 0
\tag{3.20}
$$

It can be readily seen that six independent functions enter the rule (3.15) and Eq (3.19). Let us prove that additional five relations among A_k take place once the principle of maximum dissipation power (cf [12]) is employed. The dissipation power d is calculated as

$$
\begin{aligned}
d &= \mathrm{tr}\mathbf{T}\mathbf{D} = A_1\mathrm{tr}\bar{\mathbf{M}}_{11}\mathbf{D} + A_2\mathrm{tr}\bar{\mathbf{M}}_{22}\mathbf{D} + A_3\mathrm{tr}\bar{\mathbf{M}}_{33}\mathbf{D} + 2A_4\mathrm{tr}\bar{\mathbf{M}}_{12}\mathbf{D} + \\
&+ 2A_5\mathrm{tr}\bar{\mathbf{M}}_{13}\mathbf{D} + 2A_6\mathrm{tr}\bar{\mathbf{M}}_{23}\mathbf{D} = A_1E_1 + A_2E_2 + A_3E_3 + \\
&+ 2A_4E_4 + 2A_5E_5 + 2A_6E_6
\end{aligned}
\tag{3.21}
$$

Simple geometrical interpretation can be given to the foregoing expression. Namely, consider the functions A_1, A_2, A_3, $2A_4$, $2A_5$, $2A_6$ to represent generalized stresses Z_k and E_k to stand for generalized strain rates z_k, all visualized in the six–dimensional space of invariants. Thus Eq (3.21) simply becomes the scalar product $Z_k z_k > 0$, $k = 1, ..., 6$. Its maximum clearly takes place when the vectors Z and z are coaxial and their senses are the same. The lengths of vectors satisfy

$$\frac{A_1}{E_1} = \frac{A_2}{E_2} = \frac{A_3}{E_3} = \frac{2A_4}{E_4} = \frac{2A_5}{E_5} = \frac{2A_6}{E_6} \qquad E_l \neq 0 \tag{3.22}$$

Thus the number of independent functions in Eqs (3.15) and (3.21) is reduced to one, for example, A_6. Inserting Eq (3.22) into Eq (3.21), we get

$$d = 2A_6 E_6 (p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2 + 1) = \max \tag{3.23}$$

The hypersurface $d =$ const in the space of generalized strain rates is clearly a sphere and serves as a potential to calculate generalized stresses.

Applying Eqs (3.22) and (3.15), the following relationship arrives

$$\begin{aligned} \mathbf{T} &= \frac{A_6}{\mathrm{tr}\mathbf{M}_{23}\mathbf{D}}\Big[2\Big(\mathrm{tr}\bar{\mathbf{M}}_{11}\mathbf{D}\Big)\bar{\mathbf{M}}_{11} + 2\Big(\mathrm{tr}\bar{\mathbf{M}}_{22}\mathbf{D}\Big)\bar{\mathbf{M}}_{22} + 2\Big(\mathrm{tr}\bar{\mathbf{M}}_{33}\mathbf{D}\Big)\bar{\mathbf{M}}_{33} + \\ &+ \mathrm{tr}\bar{\mathbf{M}}_{12}\mathbf{D}\Big(\bar{\mathbf{M}}_{12} + \bar{\mathbf{M}}_{21}\Big) + \mathrm{tr}\bar{\mathbf{M}}_{13}\mathbf{D}\Big(\bar{\mathbf{M}}_{13} + \bar{\mathbf{M}}_{31}\Big) + \mathrm{tr}\bar{\mathbf{M}}_{23}\mathbf{D}\Big(\bar{\mathbf{M}}_{23} + \bar{\mathbf{M}}_{32}\Big)\Big] \end{aligned} \tag{3.24}$$

from which no unique strain rate tensor $\mathbf{D}$ can be obtained. Inverse of Eq (3.24) is recognized as a flow law. The form of Eq (3.24) is clearly dependent on the way in which Eq (3.18) is assumed.

4. Three families of orthogonal fibres

Let the mechanical properties of particular families of fibres differ but the families be mutually perpendicular. The product of symmetry groups of tensors $\mathbf{M}_{ii}$ corresponds to the symmetry group of the considered composite which can be now called orthotropic.

4.1. Elasticity

General expression for nonlinearly elastic behaviour is an orthotropic second–order tensor–valued function (cf [3 ÷ 5])

$$\mathbf{T} = \mathbf{F}\Big(\mathbf{E}, \mathbf{M}_{11}, \mathbf{M}_{22}, \mathbf{M}_{33}\Big) \tag{4.1}$$

with the requirements

$$\forall\, \mathsf{Q} \in \mathcal{O} \qquad \mathsf{QTQ}^T = \mathsf{F}\Big(\mathsf{QEQ}^T, \mathsf{QM}_{11}\mathsf{Q}^T, \mathsf{QM}_{22}\mathsf{Q}^T, \mathsf{QM}_{33}\mathsf{Q}^T\Big) \tag{4.2}$$

$$\forall\, \mathsf{Q} \in \mathcal{S} = \{\mathsf{I}, -\mathsf{I}, \mathsf{S}_1, \mathsf{S}_2, \mathsf{S}_3\} \qquad \mathsf{QTQ}^T = \mathsf{F}\Big(\mathsf{QEQ}^T, \mathsf{M}_{11}, \mathsf{M}_{22}, \mathsf{M}_{33}\Big) \tag{4.3}$$

where S_1 is a reflection with respect to the e_2, e_3 plane.

Representation of Eq (4.1) with the conditions (4.2), (4.3) can be shown to be

$$\begin{aligned} \mathsf{T} &= \alpha_1 \mathsf{M}_{11} + \alpha_2 \mathsf{M}_{22} + \alpha_3 \mathsf{M}_{33} + \alpha_4\Big(\mathsf{M}_{11}\mathsf{E} + \mathsf{E}\mathsf{M}_{11}\Big) + \\ &+ \alpha_5\Big(\mathsf{M}_{22}\mathsf{E} + \mathsf{E}\mathsf{M}_{22}\Big) + \alpha_6\Big(\mathsf{M}_{33}\mathsf{E} + \mathsf{E}\mathsf{M}_{33}\Big) + \alpha_7 \mathsf{E}^2 \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} \alpha_i &= \alpha_i\Big(\mathrm{tr}\mathsf{M}_{11}\mathsf{E}, \mathrm{tr}\mathsf{M}_{22}\mathsf{E}, \mathrm{tr}\mathsf{M}_{33}\mathsf{E}, \mathrm{tr}\mathsf{M}_{11}\mathsf{E}^2, \mathrm{tr}\mathsf{M}_{22}\mathsf{E}^2, \mathrm{tr}\mathsf{M}_{33}\mathsf{E}^2, \mathrm{tr}\mathsf{E}^3\Big) = \\ &= \alpha_i(I_k) \qquad\qquad k = 1, ..., 7 \end{aligned} \tag{4.5}$$

The above description corresponds to the Cauchy material. For the Green material, in which the existence of elastic potential is postulated, the constitutive law (Eqs (4.4), (4.5)) must be supplemented with the condition

$$\frac{\partial \alpha_k}{\partial I_l} = \frac{\partial \alpha_l}{\partial I_k} \tag{4.6}$$

$$\alpha_k = \frac{\partial W(\mathsf{E}, \mathsf{M}_{11}, \mathsf{M}_{22}, \mathsf{M}_{33})}{\partial I_k} \tag{4.7}$$

where W is an orthotropic scalar-valued function of the invariants I_k (cf [3 ÷ 5]). Linearization of Eqs (4.4) ÷ (4.7) leads to the following formulae for α_k

$$\begin{aligned} &\alpha_1 = b_1 \mathrm{tr}\mathsf{M}_{11}\mathsf{E} + c_1 \mathrm{tr}\mathsf{M}_{22}\mathsf{E} + d_1 \mathrm{tr}\mathsf{M}_{33}\mathsf{E} \\ &\alpha_2 = c_1 \mathrm{tr}\mathsf{M}_{11}\mathsf{E} + c_2 \mathrm{tr}\mathsf{M}_{22}\mathsf{E} + d_2 \mathrm{tr}\mathsf{M}_{33}\mathsf{E} \\ &\alpha_3 = d_1 \mathrm{tr}\mathsf{M}_{11}\mathsf{E} + d_2 \mathrm{tr}\mathsf{M}_{22}\mathsf{E} + d_3 \mathrm{tr}\mathsf{M}_{33}\mathsf{E} \\ &\alpha_4 = a_1 \qquad\qquad \alpha_5 = a_2 \\ &\alpha_6 = a_3 \qquad\qquad \alpha_7 = 0 \end{aligned} \tag{4.8}$$

Nine elastic constants are here involved.

4.2. Perfect plasticity

Using procedure similar to that of section 3.2, the relationship between the stress tensor and the strain rate tensor is found to be

$$
\begin{aligned}
\mathbf{T} = {} & A_1\mathbf{M}_{11} + A_2\mathbf{M}_{22} + A_3\mathbf{M}_{33} + \frac{A_4}{E_7}\Big(\mathbf{M}_{11}\mathbf{D} + \mathbf{D}\mathbf{M}_{11}\Big) + \\
& + \frac{A_5}{E_7}\Big(\mathbf{M}_{22}\mathbf{D} + \mathbf{D}\mathbf{M}_{22}\Big) + \frac{A_6}{E_7}\Big(\mathbf{M}_{33}\mathbf{D} + \mathbf{D}\mathbf{M}_{33}\Big) + \frac{A_7}{E_7^2}\mathbf{D}^2
\end{aligned}
\tag{4.9}
$$

where

$$
\begin{array}{llll}
A_k = A_k(p_l) & p_l = \dfrac{E_l}{E_7} & (E_7 \neq 0 \quad k = 1,...,7 & l = 1,...,6) \\
E_1 = \mathrm{tr}\mathbf{M}_{11}\mathbf{D} & E_2 = \mathrm{tr}\mathbf{M}_{22}\mathbf{D} & E_3 = \mathrm{tr}\mathbf{M}_{33}\mathbf{D} & \\
E_4 = \sqrt{\mathrm{tr}\mathbf{M}_{11}\mathbf{D}^2} & E_5 = \sqrt{\mathrm{tr}\mathbf{M}_{22}\mathbf{D}^2} & E_6 = \sqrt{\mathrm{tr}\mathbf{M}_{33}\mathbf{D}^2} & \\
E_7 = \sqrt[3]{\mathrm{tr}\mathbf{D}^3} & & &
\end{array}
\tag{4.10}
$$

After calculating the magnitudes $\mathbf{M}_{ii}\mathbf{T}^m + \mathbf{T}^m\mathbf{M}_{ii}$, $\mathbf{T}^3$, $m = 1,2$, $i = 1,2,3$, finding their traces and eliminating parameters p_l, the yield criterion is obtained in the form

$$
f\Big(\mathrm{tr}\mathbf{M}_{11}\mathbf{T}, \mathrm{tr}\mathbf{M}_{22}\mathbf{T}, \mathrm{tr}\mathbf{M}_{33}\mathbf{T}, \mathrm{tr}\mathbf{M}_{11}\mathbf{T}^2, \mathrm{tr}\mathbf{M}_{22}\mathbf{T}^2, \mathrm{tr}\mathbf{M}_{33}\mathbf{T}^2, \mathrm{tr}\mathbf{T}^3\Big) = 0 \tag{4.11}
$$

It can be proved, as before, that a single independent function enters both the relation (4.9) and the flow law. The latter can be obtained by inversion of Eq (4.9) and has the form

$$
\begin{aligned}
\frac{\mathbf{D}}{\sqrt[3]{\mathrm{tr}\mathbf{D}^3}} = {} & \beta_1\mathbf{M}_{11} + \beta_2\mathbf{M}_{22} + \beta_3\mathbf{M}_{33} + \beta_4\Big(\mathbf{M}_{11}\mathbf{T} + \mathbf{T}\mathbf{M}_{11}\Big) + \\
& + \beta_5\Big(\mathbf{M}_{22}\mathbf{T} + \mathbf{T}\mathbf{M}_{22}) + \beta_6(\mathbf{M}_{33}\mathbf{T} + \mathbf{T}\mathbf{M}_{33}\Big) + \beta_7\mathbf{T}^2
\end{aligned}
\tag{4.12}
$$

$$
\beta_k = \beta_k\Big(\mathrm{tr}\mathbf{M}_{11}\mathbf{T}, \mathrm{tr}\mathbf{M}_{22}\mathbf{T}, \mathrm{tr}\mathbf{M}_{33}\mathbf{T}, \mathrm{tr}\mathbf{M}_{11}\mathbf{T}^2, \mathrm{tr}\mathbf{M}_{22}\mathbf{T}^2, \mathrm{tr}\mathbf{M}_{33}\mathbf{T}^2, \mathrm{tr}\mathbf{T}^3\Big)
$$

Conditions (4.11) must be borne in mind.

5. Three families of orthogonal fibres showing the same mechanical properties

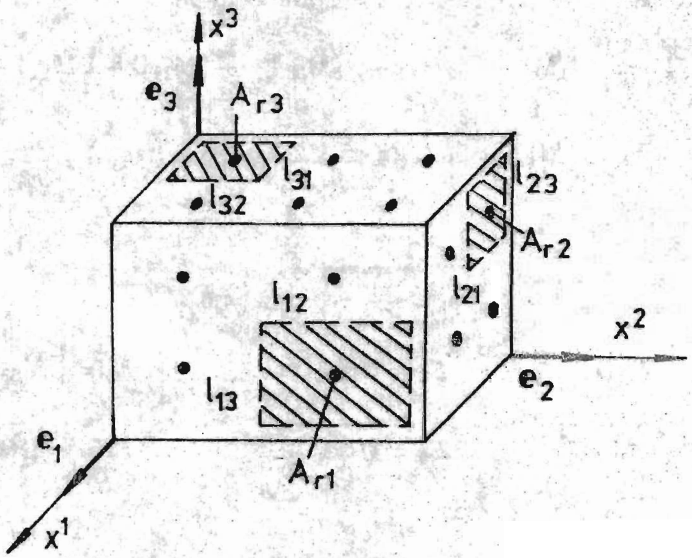

Fig. 2.

The reinforcement tensor is, following Jemioło et al. [9], defined as

$$\mathsf{R} = R_1 e_1 \otimes e_1 + R_2 e_2 \otimes e_2 + R_3 e_3 \otimes e_3 \tag{5.1}$$

where R_i are the intensities of reinforcement embedded in the directions x^i, Fig.2,

$$R_i = \frac{A_{ri}}{A_{mi}} \tag{5.2}$$

and A_{ri} denotes the cross-sectional area of reinforcing fibre in the direction x^i whereas A_{mi} is the area of matrix, perpendicular to e_i and belonging to the fibre with area A_{ri}. These areas can be calculated as

$$A_{m1} = l_{12} l_{13} \qquad A_{m2} = l_{21} l_{23} \qquad A_{m3} = l_{31} l_{32} \tag{5.3}$$

where l_{ij} are shown consistently in Fig.2.

5.1. Elasticity

Representation of the function (2.2) with parametric tensor R was discussed in detail in by Jemioło et al. [9] both for Cauchy and Green materials, respectively.

Nonlinear elasticity relationship for the latter has the form

$$\mathsf{T} = \alpha_1\mathsf{I} + \alpha_2\mathsf{E} + \alpha_3\Big(\mathsf{ER} + \mathsf{RE}\Big) + \alpha_4\mathsf{R} + \alpha_5\mathsf{E}^2 + \alpha_6\mathsf{R}^2 + \alpha_7\Big(\mathsf{ER}^2 + \mathsf{R}^2\mathsf{E}\Big) \tag{5.4}$$

$$\alpha_k = \alpha_k\Big(\mathrm{tr}\mathsf{E}, \mathrm{tr}\mathsf{E}^2, \mathrm{tr}\mathsf{E}^3, \mathrm{tr}\mathsf{ER}, \mathrm{tr}\mathsf{ER}^2, \mathrm{tr}\mathsf{E}^2\mathsf{R}, \mathrm{tr}\mathsf{E}^2\mathsf{R}^2, r_i\Big) = \alpha_k(I_l, r_i) \qquad k,l = 1,...,7 \tag{5.5}$$

The magnitudes $r_i = \mathrm{tr}\mathsf{R}^i$, $i = 1,2,3$ are treated here as the known ones since the reinforcement is considered to be prescribed. Eq (5.14) satisfies similar relationships as Eqs (4.6) and (4.7), in which W is a function of invariants I_l shown in Eq (5.5).

A linearized form of Eq (5.4) was derived elsewhere [9]

$$\mathsf{T} = \Big(\lambda\mathrm{tr}\mathsf{E} + \beta\mathrm{tr}\mathsf{RE}\Big)\mathsf{I} + 2\mu\mathsf{E} + 2\gamma\Big(\mathsf{ER} + \mathsf{RE}\Big) + \beta\Big(\mathrm{tr}\mathsf{E}\Big)\mathsf{R} \tag{5.6}$$

where

$$\lambda = \frac{E_c\nu_c}{(1+\nu_c)(1-2\nu_c)} \qquad\qquad \mu = \frac{E_c}{2(1+\nu_c)}$$

are Lame elastic constants for matrix and β, γ should be found from well planned tests on the composite material.

In the standard matrix notation of Hooke's law for orthotropic material we have

$$\mathsf{T}_{(6\times1)} = \mathsf{C}_{(6\times6)}\mathsf{E}_{(6\times1)} \tag{5.7}$$

where

$$\mathsf{C}_{(6\times6)} = \begin{bmatrix} e_1 & f_3 & f_2 & 0 & 0 & 0 \\ & e_2 & f_1 & 0 & 0 & 0 \\ & & e_3 & 0 & 0 & 0 \\ & & & g_3 & 0 & 0 \\ & \text{sym.} & & & g_2 & 0 \\ & & & & & g_1 \end{bmatrix} \tag{5.8}$$

Nine elasticity constants e_i, f_i, g_i, $i = 1,2,3$ depend on the constants α, μ, β, γ, R_i in the following manner

$$\begin{aligned} e_i &= \lambda + 2\mu + 2(2\gamma + \beta)R_i \\ f_i &= \lambda + \beta(R_j + R_k) \\ g_i &= \mu + \gamma(R_j + R_k) \end{aligned} \tag{5.9}$$

$$(i,j,k) = (1,2,3), (2,3,1), (3,1,2)$$

It is worth noting that the obtained representation (5.4) is identical with that shown by Vakulenko and Markov cf [17]. They considered various cases of isotropic scalar- and tensor-valued functions depending on two symmetric second-order tensors.

5.2. Perfect plasticity

General nonlinear relationship for ideally plastic behaviour of the composite body is found to have the form

$$\mathsf{T} = B_1\mathsf{I} + \frac{B_2}{E_2}\mathsf{D} + \frac{B_3}{E_2}\Big(\mathsf{DR}+\mathsf{RD}\Big) + B_4\mathsf{R} + \frac{B_5}{E_2^2}\mathsf{D}^2 + B_6\mathsf{R}^2 + \frac{B_7}{E_2}\Big(\mathsf{DR}^2+\mathsf{R}^2\mathsf{D}\Big) \tag{5.10}$$

where

$$\begin{gathered} B_k = B_k(p_l) \qquad k = 1,...,7 \qquad l = 1,...,6 \\ p_1 = \frac{E_1}{E_2} \qquad p_l = \frac{E_{l+1}}{E_2} \qquad E_2 \neq 0 \qquad l = 2,...,6 \end{gathered} \tag{5.11}$$

$$\begin{gathered} E_1 = \mathrm{tr}\mathsf{D} \qquad E_2 = \sqrt{\mathrm{tr}\mathsf{D}^2} \qquad E_3 = \sqrt[3]{\mathrm{tr}\mathsf{D}^3} \\ E_4 = \mathrm{tr}\mathsf{DR} \qquad E_5 = \mathrm{tr}\mathsf{DR}^2 \qquad E_6 = \sqrt{\mathrm{tr}\mathsf{D}^2\mathsf{R}} \\ E_7 = \sqrt{\mathrm{tr}\mathsf{D}^2\mathsf{R}^2} \end{gathered} \tag{5.12}$$

Inverse of Eq (5.10) supplies the flow rule in the form

$$\begin{aligned} \frac{\mathsf{D}}{\sqrt{\mathrm{tr}\mathsf{D}^2}} &= \nu_1\mathsf{I} + \nu_2\mathsf{R} + \nu_3\mathsf{T} + \nu_4\Big(\mathsf{TR}+\mathsf{RT}\Big) + \nu_5\mathsf{R}^2 + \\ &+ \nu_6\mathsf{T}^2 + \nu_7\Big(\mathsf{TR}^2+\mathsf{R}^2\mathsf{T}\Big) \end{aligned} \tag{5.13}$$

$$\nu_k = \nu_k\Big(\mathrm{tr}\mathsf{T}, \mathrm{tr}\mathsf{T}^2, \mathrm{tr}\mathsf{T}^3, \mathrm{tr}\mathsf{TR}, \mathrm{tr}\mathsf{TR}^2, \mathrm{tr}\mathsf{T}^2\mathsf{R}, \mathrm{tr}\mathsf{T}^2\mathsf{R}^2\Big) \tag{5.14}$$

together with the yield criterion

$$f\Big(\mathrm{tr}\mathsf{T}, \mathrm{tr}\mathsf{T}^2, \mathrm{tr}\mathsf{T}^3, \mathrm{tr}\mathsf{TR}, \mathrm{tr}\mathsf{TR}^2, \mathrm{tr}\mathsf{T}^2\mathsf{R}, \mathrm{tr}\mathsf{T}^2\mathsf{R}^2\Big) = 0 \tag{5.15}$$

Detailed form of Eq (5.13) and its derivation under the constraints of quasilinearity of Eq (5.10) and $B_5 = B_6 = B_7 = 0$ was given in [9] where particular relations between B_k and ν_k were also provided.

When the maximum of dissipation power is assumed in Eqs (5.10) and (5.13), only one independent function B_2 is present.

6. Simplified perfect plasticity

It was Boehler and Sawczuk ([7]) who introduced instead of transversally isotropic material in question, an equivalent isotropic body. The same procedure was followed by Jemiolo et al. [9]. Equivalent stress tensor $\overset{*}{\mathbf{T}}$ is assumed

$$\overset{*}{\mathbf{T}}=\overset{*}{\mathbf{C}}\,\mathbf{T} \tag{6.1}$$

where $\overset{*}{\mathbf{C}}$ is a fourth-order tensor responsible for anisotropic properties.

Constructing the tensor $\overset{*}{\mathbf{C}}$ in a manner suggested by Telega ([16]) and assuming its linear dependence on $\mathbf{R}$, we get in the index notation

$$\begin{aligned}\overset{*}{\mathbf{C}}_{ijkl} &= \bar{\lambda}\delta_{ij}\delta_{kl}+\bar{\mu}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})+\bar{\beta}(\delta_{ij}R_{kl}+\delta_{kl}R_{ij})+\\ &+ \bar{\gamma}(\delta_{ik}R_{jl}+\delta_{il}R_{jk}+\delta_{jl}R_{ik}+\delta_{jk}R_{il})\end{aligned} \tag{6.2}$$

where $\bar{\lambda}$, $\bar{\mu}$, $\bar{\beta}$, $\bar{\gamma}$ are material constants.

The equivalent stress tensor is expressed by

$$\overset{*}{\mathbf{T}}=\left(\bar{\lambda}\mathrm{tr}\mathbf{T}+\bar{\beta}\mathrm{tr}\mathbf{RT}\right)\mathbf{I}+2\bar{\mu}\mathbf{T}+2\bar{\gamma}\left(\mathbf{TR}+\mathbf{RT}\right)+\bar{\beta}\left(\mathrm{tr}\mathbf{T}\right)\mathbf{R} \tag{6.3}$$

Let us postulate a linear dependence of the equivalent stress tensor on the strain rate tensor $\mathbf{D}$

$$\overset{*}{\mathbf{T}}=\bar{\alpha}_1\mathbf{I}+\bar{\alpha}_2\mathbf{D} \tag{6.4}$$

where

$$\bar{\alpha}_l=\bar{\alpha}_l\left(\mathrm{tr}\mathbf{D},\mathrm{tr}\mathbf{D}^2,\mathrm{tr}\mathbf{D}^3\right)\qquad l=1,2 \tag{6.5}$$

On splitting up Eq (6.4) into its spherical and deviatoric parts, we get

$$\mathrm{tr}\,\overset{*}{\mathbf{T}}=3\bar{\alpha}_1+\bar{\alpha}_2\mathrm{tr}\mathbf{D}\qquad\qquad \overset{*}{\mathbf{S}}=\bar{\alpha}_1\mathbf{F} \tag{6.6}$$

where

$$\overset{*}{\mathbf{S}}=\overset{*}{\mathbf{T}}-\frac{1}{3}\left(\mathrm{tr}\,\overset{*}{\mathbf{T}}\right)\mathbf{I}\qquad\qquad \mathbf{F}=\mathbf{D}-\frac{1}{3}\left(\mathrm{tr}\mathbf{D}\right)\mathbf{I} \tag{6.7}$$

$\overset{*}{\mathbf{S}}$, $\mathbf{F}$ stand for the deviators of equivalent stress tensor and strain tensor, respectively.

On using the homogeneity requirement (2.6), we arrive at the flow law

$$\mathrm{tr}\,\overset{*}{\mathbf{T}}=g_1(s,t)\qquad\qquad \frac{\mathbf{F}}{\sqrt{\mathrm{tr}\mathbf{F}^2}}=\frac{\overset{*}{\mathbf{S}}}{g_2(s,t)} \tag{6.8}$$

where

$$s = \frac{\sqrt[3]{\mathrm{tr}\mathbf{F}^3}}{\sqrt{\mathrm{tr}\mathbf{F}^2}} \qquad t = \frac{\mathrm{tr}\mathbf{D}}{\sqrt{\mathrm{tr}\mathbf{F}^2}} \tag{6.9}$$

The yield criterion takes the form

$$f\left(\mathrm{tr}\,\overset{*}{\mathbf{T}}, \sqrt{\mathrm{tr}\,\overset{*}{\mathbf{S}}}, \frac{\sqrt[3]{\mathrm{tr}\,\overset{*}{\mathbf{S}}{}^3}}{\sqrt{\mathrm{tr}\,\overset{*}{\mathbf{S}}{}^2}}\right) = 0 \tag{6.10}$$

Inserting Eq (6.3) into Eq $(6.8)_2$, we obtain the flow rule in the form

$$\frac{\mathbf{F}}{\sqrt{\mathrm{tr}\mathbf{F}^2}} = \frac{\left(\bar{\lambda}\mathrm{tr}\mathbf{T} + \bar{\beta}\mathrm{tr}\mathbf{RT} - \frac{g_1}{3}\right)\mathbf{I} + 2\bar{\mu}\mathbf{T} + 2\bar{\gamma}\left(\mathbf{TR} + \mathbf{RT}\right) + \bar{\beta}\left(\mathrm{tr}\mathbf{T}\right)\mathbf{R}}{g_2} \tag{6.11}$$

The above equation is remarkably simpler than the quasilinear flow equation (5.13) since it contains only four unknown constants and two unknown functions of kinematic parameters s, t. Employing the postulate of maximum dissipation, an additional relation between g_1 and g_2 is revealed, namely

$$g_1(s,t) = g_2(s,t)t \tag{6.12}$$

Using Eqs (6.12), (6.8) and (6.11), we get the decomposed flow law for spherical and deviatoric parts as dependent on a single unknown scalar-valued function of kinematic arguments s, t or on three invariants of the equivalent stress tensor

$$\mathrm{tr}\,\overset{*}{\mathbf{T}} = g_2 t \qquad \mathrm{tr}\,\overset{*}{\mathbf{S}}{}^2 = g_2^2 \qquad \mathrm{tr}\,\overset{*}{\mathbf{T}}{}^3 = g_2^3 s^3 \tag{6.13}$$

The yield criterion (6.10) must not be forgotten.

Let us, for example, generalize the well-known Drucker-Prager yield condition to cover the case of an orthotropic composite material. Its standard form is

$$\alpha I_1 + \sqrt{J_2} - k = 0 \tag{6.14}$$

For anisotropy it becomes

$$\alpha\mathrm{tr}\,\overset{*}{\mathbf{T}} + \frac{1}{\sqrt{2}}\sqrt{\mathrm{tr}\,\overset{*}{\mathbf{S}}{}^2} - k = 0 \tag{6.15}$$

whereas for orthotropy it takes the form

$$\mathrm{tr}\,\overset{*}{\mathbf{T}} = \left(3\bar{\lambda} + 2\bar{\mu} + \bar{\beta}\mathrm{tr}\mathbf{R}\right)\mathrm{tr}\mathbf{T} + \left(3\bar{\beta} + 4\bar{\gamma}\right)\mathrm{tr}\mathbf{RT} \tag{6.16}$$

$$\mathrm{tr}\,\overset{*}{\mathbf{S}}{}^2 = \mathrm{tr}\,\overset{*}{\mathbf{T}}{}^2 - \frac{\mathrm{tr}^2\,\overset{*}{\mathbf{T}}}{2} \tag{6.17}$$

$$\begin{aligned}\mathrm{tr}\,\overset{*}{\mathbf{T}}{}^2 = & \Big[3\bar\lambda^2 + 4\bar\lambda\bar\mu + 2\bar\lambda\bar\beta\mathrm{tr}\mathbf{R} + (\bar\beta^2 - 4\bar\gamma^2)\mathrm{tr}\mathbf{R}^2\Big]\mathrm{tr}^2\mathbf{T} + \\ & +\Big\{2\Big[3\bar\lambda\bar\beta + 4\bar\mu\bar\beta + 4\bar\lambda\bar\gamma + (\bar\beta^2 - 8\bar\gamma^2)\mathrm{tr}\mathbf{R}\Big]\mathrm{tr}\mathbf{T} + \\ & +(3\bar\beta^2 + 8\bar\beta\bar\gamma + 8\bar\gamma^2)\mathrm{tr}\mathbf{RT}\Big\}\mathrm{tr}\mathbf{RT} + (\bar\mu^2 + \bar\gamma^2\mathrm{tr}\mathbf{R}^2)\mathrm{tr}\mathbf{T}^2 + \\ & +4\bar\gamma\Big[4\bar\mu\mathrm{tr}\mathbf{RT}^2 + (\bar\beta + 4\bar\gamma)\mathrm{tr}\mathbf{T}\mathrm{tr}\mathbf{R}^2\mathbf{T} - 2\bar\gamma\mathrm{tr}\mathbf{R}^2\mathbf{T}^2\Big]\end{aligned} \tag{6.18}$$

Extended Cayley–Hamilton theorem, shown in [4,9], was used to get the expression for $\mathrm{tr}\,\overset{*}{\mathbf{T}}{}^2$. The constants α, k, $\bar\lambda$, $\bar\mu$, $\bar\beta$, $\bar\gamma$ must be derived from appropriate tests.

7. Final remarks

The general flow rules (3.25), (4.12), (5.13), (6.8) derived in the paper are not associated with the yield criterion (3.20), (4.11), (5.15), (6.10) in the classical sense. They come from the definition of plastic material (2.6). The principle of maximum dissipation results in the fact that the flow rules contain one independent unknown scalar-valued function.

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Równanie konstytutywne sprężystości i plastyczności izotropowej matrycy wzmocnionej trzema rodzinami prostoliniowego zbrojenia

Streszczenie

W pracy zastosowano teorię niewielomianowych anizotropowych funkcji tensorowych oraz twierdzenia o ich reprezentacjach. Podejście to zapewniło zarówno matematyczną przejrzystość jak i wymaganą ogólność formułowanych relacji konstytutywnych. Rozpatrzono materiał składający się z izotropowej matrycy wzmocnionej trzema rodzinami prostoliniowego zbrojenia. Rozważono trzy przypadki struktury rodzin zbrojenia: a) nieortogonalną, b) ortogonalną, c) ortogonalna z rodzinami prętów o jednakowych

własnościach mechanicznych. Wprowadzenie włókien powoduje, że grupa symetrii materialnej jest grupą skończoną, którą scharakteryzowano odpowiednimi tensorami parametrycznymi. Wyprowadzono rwnania konstytutywne nieliniowej sprężystości oraz idealnej plastyczności. Prawa płynięcia i warunki plastyczności wyprowadzono z założenia o jednorodności stopnia zero relacji konstytutywnej względem prędkości odkształcenia. Wykorzystano koncepcję konstruowania równań teorii plastyczności zaproponowaną przez Sawczuka, Stutza i Boehlera. Podano także zlinearyzowane związki oraz sposób formułowania uproszczonej teorii idealnej plastyczności. Uproszczenie polega na wprowadzeniu zastępczego tensora naprężeń przetransformowanego za pomocą tensora czwartego rzędu zależnego od tensorów parametrycznych.

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