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VARIATIONAL FORMULATION FOR BUCKLING OF MULTI-WALLED CARBON NANOTUBES MODELLED AS NONLOCAL TIMOSHENKO BEAMS

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Variational formulation for multi-walled carbon nanotubes subject to buckling is derived by the semi-inverse method with governing equations based on the nonlocal Timoshenko beam theory which takes small scale effects and shear deformation into account. The nonlocal theory improves the range and applicability of the physical model by modelling the nano-scale phenomenon more accurately. The natural and geometric boundary conditions are derived, which lead to a set of coupled boundary conditions for multi-walled nanotubes as opposed to uncoupled boundary conditions in the case of simply supported and clamped boundaries and also in the case of a local theory. The variational principle and the corresponding Rayleigh quotient facilitate the application of approximate and numerical methods of solution.

Key words: carbon nanotubes, variational formulation, nonlocal theory

1. Introduction

The laws of continuum mechanics are known to be robust enough to treat intrinsically discrete objects only a few atoms in diameter (Yakobson and Smalley, 1997). Subsequent studies established the accuracy of continuum based approaches to the mechanics of nanotubes. A study on the range of applicability of elastic beam theory to model nanotubes and nanorods was given by Harik (2001). Beam models used to study the buckling behavior of carbon nanotubes (CNTs) mostly employed the Euler-Bernoulli or Timoshenko beam theories. The equation governing the buckling of an Euler-Bernoulli beam is expressed in terms of only one unknown, namely, the deflection of the beam and neglects the effect of transverse shear deformation. However, for nanotubes with low length-to-diameter ratio, the shear deformation can have a substantial effect on the buckling load and can be taken into account using the Timoshenko beam model. In this case, the governing equations have two dependent variables, namely, the slope and deflection of the beam and are able to predict the mechanical behavior of CNTs more accurately. Several studies on the buckling of nanotubes used these two beam models with the Euler--Bernoulli beam model used by Ru (2000), Wang and Varadan (2005), Wang *et al.* (2005), Sears and Batra (2006), Zhang *et al.* (2008) and the Timoshenko model by Zhang *et al.* (2006).

However, small scale effects were not taken into account in these papers. The importance of size effects for nano-sized structures was emphasized in Miller and Shenoy (2000), Chang and Gao (2003), Sun and Zhang (2003), Lima and Heb (2004) and Huang (2008) where properties of nano materials were obtained. Beam theories capable of taking the small scale effects into account are based on the nonlocal theory of elasticity which was developed in early seventies (Edelen and Laws, 1971; Eringen, 1972). The nonlocal theory was applied to the study of nano-scale Timoshenko beams in a number of papers (Wang *et al.*, 2006; Reddy, 2007, 2008; Wang *et al.*, 2007, 2008; Hsu *et al.*, 2008). The nonlocal Euler-Bernoulli and Timoshenko beam models were employed to investigate the buckling and vibration characteristics of CNTs by Sudak (2003), Wang (2005), Wang and Hu (2005), Wang *et al.* (2006), Lu *et al.* (2007), Heireche *et al.* (2008) and Murmu and Pradhan (2009) and comparisons between the local and nonlocal models were given in these papers. These studies considered single and double-walled nanotubes involving mostly simply supported boundary conditions leading to analytical solutions of the differential equations in terms of sine and cosine functions. As such, they covered a limited set of configurations with respect to boundary conditions and with respect to number of nanotubes, mostly due to the complicated solutions which arise for other boundary conditions and also with the increasing number of nanotubes.

Variational formulations allow the implementation of approximate and numerical methods of solutions and facilitate the consideration of complicated boundary conditions, especially in the case of multi-walled nanotubes which are governed by a system of differential equations. Recently, variational formulations were employed in derivations of governing equations for various beam theories applicable to nano-sized beams by Reddy (2007, 2008). Variational principles and natural boundary conditions were derived for multi-walled CNTs by Adali (2008, 2009a) where CNTs were modelled as nonlocal Euler-Bernoulli beams subject to a buckling load (Adali, 2008) and undergoing vibrations (Adali, 2009a). The corresponding results for multi-walled

CNTs undergoing nonlinear vibrations were obtained by Adali (2009b) again using the Euler-Bernoulli model. Variational principles were derived by Kucuk *et al.* (2010) for multi-walled nanotubes undergoing transient vibrations with the model based on the nonlocal Timoshenko beam theory. The present study extends the results of Adali (2008) to the shear deformable case and the results of Kucuk *et al.* (2010) to the buckling case by using nonlocal Timoshenko beams to model the multi-walled CNTs. In this case, the formulation involves two independent variables for each nanotube as opposed to one independent variable for the Euler-Bernoulli beam model. The approach used in the present study to derive the variational principles is the semi-inverse method developed by He (1997, 2004). Several examples of variational principles for systems of differential equations obtained by this method can be found in the papers (He, 2005, 2006, 2007; Liu, 2005; Zhou, 2006) and in the references therein.

In the present study, first the coupled differential equations governing the buckling of multi-walled nanotubes based on the nonlocal Timoshenko beam theory are given. Next, a trial variational functional is formulated, and an unknown functional is introduced. Finally, this functional is determined, and the variational principle and the Rayleigh quotient are obtained by the semi-inverse method. The variational formulation developed for the multi-walled nanotubes is employed to derive the natural and geometric boundary conditions of the problem and the coupled nature of natural boundary conditions are noted.

2. Multi-walled carbon nanotubes

We consider a concentric multi-walled carbon nanotube system consisting of n nanotubes of cylindrical shape. The multi-walled nanotube lies on a Winkler foundation with elasticity modulus k, has length L and is under compressive stress σ_x. We define a difference operator given by

$$\Delta w_{ij} = w_i - w_j \tag{2.1}$$

where w_i and w_j are the deflections of the i-th and j-th nanotubes. The differential equations governing the buckling of multi-walled nanotubes based on the nonlocal Timoshenko beam theory can be expressed as

$$\begin{aligned}
&D_{a1}(w_1,\varphi_1,w_2) = L_{a1}(w_1,\varphi_1) - c_{12}\Delta w_{21} + \eta^2 c_{12}\frac{d^2\Delta w_{21}}{dx^2} = 0\\
&D_{b1}(w_1,\varphi_1) = L_{b1}(w_1,\varphi_1) = 0
\end{aligned}$$

$$
\begin{aligned}
&D_{a2}(w_1, w_2, \varphi_2, w_3) = L_{a2}(w_2, \varphi_2) + c_{12}\Delta w_{21} - c_{23}\Delta w_{32} \\
&\quad + \eta^2\Big(-c_{12}\frac{d^2\Delta w_{21}}{dx^2} + c_{23}\frac{d^2\Delta w_{32}}{dx^2}\Big) = 0 \\
&D_{b2}(w_2, \varphi_2) = L_{b2}(w_2, \varphi_2) = 0 \\
&\vdots \\
&D_{ai}(w_{i-1}, w_i, \varphi_i, w_{i+1}) = L_{ai}(w_i, \varphi_i) + c_{(i-1)i}\Delta w_{i(i-1)} - c_{i(i+1)}\Delta w_{(i+1)i} \\
&\quad -\eta^2 c_{(i-1)i}\frac{d^2\Delta w_{i(i-1)}}{dx^2} + \eta^2 c_{i(i+1)}\frac{d^2\Delta w_{(i+1)i}}{dx^2} = 0 \\
&D_{bi}(w_i, \varphi_i) = L_{bi}(w_i, \varphi_i) = 0 \qquad \text{for} \quad i = 3, 4, \ldots, n-1 \\
&\vdots \\
&D_{an}(w_{n-1}, w_n, \varphi_n) = L_{an}(w_n, \varphi_n) + c_{(n-1)n}\Delta w_{n(n-1)} \\
&\quad -\eta^2 c_{(n-1)n}\frac{d^2\Delta w_{n(n-1)}}{dx^2} = 0 \\
&D_{bn}(w_n, \varphi_n) = L_{bn}(w_n, \varphi_n) = 0
\end{aligned}
\tag{2.2}
$$

where the operators $L_{ai}(w_i, \varphi_i)$ and $L_{bi}(w_i, \varphi_i)$ are given by

$$
\begin{aligned}
&L_{ai}(w_i, \varphi_i) = \kappa G A_i \frac{d}{dx}\Big(\varphi_i - \frac{dw_i}{dx}\Big) + A_i\sigma_x\frac{d^2 w_i}{dx^2} - \eta^2 A_i \sigma_x \frac{d^4 w_i}{dx^4} \\
&\quad + \delta_{in}\Big(k w_n - k\eta^2\frac{d^2 w_n}{dx^2}\Big) \\
&L_{bi}(w_i, \varphi_i) = \kappa G A_i\Big(\varphi_i - \frac{dw_i}{dx}\Big) - EI_i\frac{d^2\varphi_i}{dx^2}
\end{aligned}
\tag{2.3}
$$

where the index $i = 1, 2, \ldots, n$ refers to the order of the nanotubes with the innermost nanotube indicated by $i = 1$ and the outermost nanotube by $i = n$ with $0 \leqslant x \leqslant L$. In Eq. $(2.3)_1$, δ_{in} is the Kronecker delta with $\delta_{in} = 0$ for $i \neq n$ and $\delta_{nn} = 1$. In Eqs. (2.3), E is the Young modulus, G is the shear modulus, κ is the shear correction factor, I_i is the moment of inertia, A_i is the cross-sectional area of the i-th carbon nanotube and σ_x is the buckling stress. The coefficient $c_{(i-1)i}$ is the interaction coefficient of the van der Waals forces between the $(i-1)$-th and i-th nanotube with $i = 2, \ldots, n$. The small scale effect is reflected by the parameter $\eta = e_0 a$, where e_0 is a constant for adjusting the model by experimental results, and a is an internal characteristic length.

3. Variational principle

According to the semi-inverse method (He, 1997, 2004), a variational trial-functional $V(w_i,\varphi_i)$ can be constructed as follows

$$
\begin{aligned}
V(w_i,\varphi_i) &= V_1(w_1,\varphi_1,w_2) + V_2(w_1,w_2,\varphi_2,w_3) + \dots \\
&\quad + V_{n-1}(w_{n-2},w_{n-1},\varphi_{n-1},w_n) + V_n(w_{n-1},w_n,\varphi_n)
\end{aligned}
\tag{3.1}
$$

where

$$
\begin{aligned}
V_1(w_1,\varphi_1,w_2) &= U_1(w_1,\varphi_1) + \int\limits_0^L F_1(w_1,w_2)\,dx \\
V_2(w_1,w_2,\varphi_2,w_3) &= U_2(w_2,\varphi_2) + \int\limits_0^L F_2(w_1,w_2,w_3)\,dx \\
V_i(w_{i-1},w_i,\varphi_i,w_{i+1}) &= U_i(w_i,\varphi_i) + \int\limits_0^L F_i(w_{i-1},w_i,w_{i+1})\,dx \\
&\qquad \text{for} \quad i=3,4,\dots,n-1 \\
V_n(w_{n-1},w_n,\varphi_n) &= U_n(w_n,\varphi_n) + \frac{1}{2}\int\limits_0^L \Big[kw_n^2 + k\eta^2\Big(\frac{dw_n}{dx}\Big)^2\Big]\,dx \\
&\quad + \int\limits_0^L F_n(w_{n-1},w_n)\,dx
\end{aligned}
\tag{3.2}
$$

with $U_i(w_i,\varphi_i)$ given by

$$
\begin{aligned}
U(w_i,\varphi_i) &= \frac{1}{2}\int\limits_0^L \Big[\kappa G A_i\Big(\varphi_i - \frac{dw_i}{dx}\Big)^2 + EI_i\Big(\frac{d\varphi_i}{dx}\Big)^2 - A_i\sigma_x\Big(\frac{dw_i}{dx}\Big)^2 \\
&\quad - \eta^2 A_i\sigma_x\Big(\frac{d^2w_i}{dx^2}\Big)^2\Big]\,dx
\end{aligned}
\tag{3.3}
$$

where $i=1,2,\dots,n$ and $F_i(w_{i-1},w_i,w_{i+1})$ denotes the unknown functions of w_i and its derivatives to be determined such that differential Eqs. (2.2) and (2.3) correspond to the Euler-Lagrange equations of variational functional (3.1).

The Euler-Lagrange equations of the variational functional in Eq. (3.1) are given by

$$
\begin{aligned}
&L_{a1}(w_1,\varphi_1)+\sum_{j=1}^{2}\frac{\delta F_j}{\delta w_1}=L_{a1}(w_1,\varphi_1)+\sum_{j=1}^{2}\frac{\partial F_j}{\partial w_1}-\sum_{j=1}^{2}\frac{d}{dx}\Big(\frac{\partial F_j}{\partial w_{1x}}\Big)=0\\
&L_{a2}(w_2,\varphi_2)+\sum_{j=1}^{3}\frac{\delta F_j}{\delta w_2}=L_{a2}(w_2,\varphi_2)+\sum_{j=1}^{3}\frac{\partial F_j}{\partial w_2}-\sum_{j=1}^{3}\frac{d}{dx}\Big(\frac{\partial F_j}{\partial w_{2x}}\Big)=0\\
&L_{ai}(w_i,\varphi_i)+\sum_{j=i-1}^{i+1}\frac{\delta F_j}{\delta w_i}=L_{ai}(w_i,\varphi_i)+\sum_{j=i-1}^{i+1}\frac{\partial F_j}{\partial w_i}-\sum_{j=i-1}^{i+1}\frac{d}{dx}\Big(\frac{\partial F_j}{\partial w_{ix}}\Big)=0\\
&\qquad \text{for} \qquad i=3,4,\dots,n-1\\
&L_{an}(w_n,\varphi_n)+\sum_{j=n-1}^{n}\frac{\delta F_j}{\delta w_n}=L_{an}(w_n,\varphi_n)+\sum_{j=n-1}^{n}\frac{\partial F_j}{\partial w_n}-\sum_{j=n-1}^{n}\frac{d}{dx}\Big(\frac{\partial F_j}{\partial w_{nx}}\Big)=0\\
&L_{bi}(w_i,\varphi_i)=0 \qquad \text{for} \qquad i=1,2,\dots,n
\end{aligned}
\tag{3.4}
$$

where the subscript x denotes differentiation with respect to x and

$$
\frac{\delta F_i}{\delta w_i}=\frac{\partial F_i}{\partial w_i}-\frac{d}{dx}\Big(\frac{\partial F_i}{\partial w_{ix}}\Big)+\frac{d^2}{dx^2}\Big(\frac{\partial F_i}{\partial w_{ixx}}\Big)+\dots
\tag{3.5}
$$

is the variational derivative of F_i with respect to w_i as defined in the semi-inverse method (He, 1997, 2004). Comparing Eqs. (3.4) with (2.2) and (2.3), we observe that the following equations have to be satisfied for the Euler-Lagrange equations to represent governing Eqs. (2.2) and (2.3), viz.

$$
\begin{aligned}
&\sum_{j=1}^{2}\frac{\delta F_j}{\delta w_1}=-c_{12}\Delta w_{21}+\eta^2 c_{12}\frac{d^2\Delta w_{21}}{dx^2}\\
&\sum_{j=i-1}^{i+1}\frac{\delta F_j}{\delta w_i}=c_{(i-1)i}\Delta w_{i(i-1)}-c_{i(i+1)}\Delta w_{(i+1)i}-\eta^2 c_{(i-1)i}\frac{d^2\Delta w_{i(i-1)}}{dx^2}\\
&\qquad +\eta^2 c_{i(i+1)}\frac{d^2\Delta w_{(i+1)i}}{dx^2}\\
&\sum_{j=n-1}^{n}\frac{\delta F_j}{\delta w_n}=c_{(n-1)n}\Delta w_{n(n-1)}-\eta^2 c_{(n-1)n}\frac{d^2\Delta w_{n(n-1)}}{dx^2}
\end{aligned}
\tag{3.6}
$$

where $i = 2, 3, \ldots, n-1$. From Eqs. (3.6), it follows that

$$
\begin{aligned}
F_1(w_1, w_2) &= \frac{c_{12}}{4}\Delta w_{21}^2 + \frac{c_{12}}{4}\eta^2\Big(\frac{d\Delta w_{21}}{dx}\Big)^2 \\
F_i(w_{i-1}, w_i, w_{i+1}) &= \frac{c_{(i-1)i}}{4}\Delta w_{i(i-1)}^2 + \frac{c_{i(i+1)}}{4}\Delta w_{(i+1)i}^2 \\
&\quad + \frac{\eta^2 c_{(i-1)i}}{4}\Big(\frac{d\Delta w_{i(i-1)}}{dx}\Big)^2 + \frac{\eta^2 c_{i(i+1)}}{4}\Big(\frac{d\Delta w_{(i+1)i}}{dx}\Big)^2 \\
&\quad \text{for} \quad i = 2, 3, \ldots, n-1 \\
F_n(w_{n-1}, w_n) &= \frac{c_{(n-1)n}}{4}\Delta w_{n(n-1)}^2 + \frac{\eta^2 c_{(n-1)n}}{4}\Big(\frac{d\Delta w_{n(n-1)}}{dx}\Big)^2
\end{aligned}
\tag{3.7}
$$

with F_i, $i = 1, 2, \ldots, n$ given by Eqs. (3.7), we observe that Eqs. (3.4) are equivalent to Eqs. (2.2) and (2.3).

3.1. Rayleigh quotient

Next the Rayleigh quotient is obtained for the buckling stress noting that

$$
U(w_i, \varphi_i) = \frac{1}{2}\int_0^L \Big[\kappa G A_i\Big(\varphi_i - \frac{dw_i}{dx}\Big)^2 + EI_i\Big(\frac{d\varphi_i}{dx}\Big)^2\Big]\,dx - Y(w_i)\sigma_x \tag{3.8}
$$

where

$$
Y_i(w_i) = \frac{1}{2}\int_0^L \Big[A_i\Big(\frac{dw_i}{dx}\Big)^2 + \eta^2 A_i\Big(\frac{d^2 w_i}{dx^2}\Big)^2\Big]\,dx \tag{3.9}
$$

From Eqs. (3.1), (3.8) and (3.9), the Rayleigh quotient is obtained as

$$
\begin{aligned}
\sigma_x = \min_{w_i, \varphi_i} \frac{1}{\sum_{i=1}^n Y_i(w_i)} \Big\{ &\sum_{i=1}^n \frac{1}{2}\int_0^L \Big[\kappa G A_i\Big(\varphi_i - \frac{dw_i}{dx}\Big)^2 + EI_i\Big(\frac{d\varphi_i}{dx}\Big)^2\Big]\,dx \\
&+ \frac{k}{2}\int_0^L \Big[w_n^2 + \eta^2\Big(\frac{dw_n}{dx}\Big)^2\Big]\,dx + \sum_{i=1}^n \int_0^L F_i\,dx \Big\}
\end{aligned}
\tag{3.10}
$$

where F_i, $i = 1, 2, \ldots, n$ are given by Eqs. (3.7) and $w_i(x) \in C^2(0, L)$, $\phi_i(x) \in C(0, L)$.

4. Boundary conditions

Next, we take the variations of the functional $V(w_i, \varphi_i)$ in Eq. (3.1) using Eqs. (3.2) and (3.3) with respect to w_i and φ_i in order to derive the natural and geometric boundary conditions. Let δw_i and $\delta\varphi_i$ denote variations of w_i and φ_i. We observe that first variations of $V(w_i, \varphi_i)$ with respect to w_i and φ_i, denoted by $\delta_{w_i} V$ and $\delta_{\varphi_i} V$, respectively, can be obtained by integration by parts and expressed as

$$\begin{aligned}
&\delta_{w_1} V = \delta_{w_1} V_1 + \delta_{w_1} V_2 = \int_0^L D_{a1}(w_1, \varphi_1, w_2)\delta w_1 \; dx + \partial\Omega_{a1}(0, L)\\
&\delta_{\varphi_1} V = \delta_{\varphi_1} V_1 = \int_0^L D_{b1}(w_1, \varphi_1)\delta\varphi_1 \; dx + \partial\Omega_{b1}(0, L)\\
&\delta_{w_i} V = \sum_{j=i-1}^{i+1} \delta_{w_i} V_j = \int_0^L D_{ai}(w_{i-1}, w_i, \varphi_i, w_{i+1})\delta w_i \; dx + \partial\Omega_{ai}(0, L)\\
&\qquad \text{for} \quad i = 2, \dots, n-1\\
&\delta_{\varphi_i} V = \delta_{\varphi_i} V_i = \int_0^L D_{bi}(w_i, \varphi_i)\delta\varphi_i \; dx + \partial\Omega_{bi}(0, L)\\
&\qquad \text{for} \quad i = 2, \dots, n-1\\
&\delta_{w_n} V = \delta_{w_n} V_{n-1} + \delta_{w_n} V_n = \int_0^L D_{an}(w_{n-1}, w_n, \varphi_n)\delta w_n \; dx + \partial\Omega_{an}(0, L)\\
&\delta_{\varphi_n} V = \delta_{\varphi_n} V_n = \int_0^L D_{bn}(w_n, \varphi_n)\delta\varphi_n \; dx + \partial\Omega_{bn}(0, L)
\end{aligned} \tag{4.1}$$

where $\partial\Omega_{ia}(0, L)$ and $\partial\Omega_{ib}(0, L)$ are the boundary terms defined as

$$\begin{aligned}
\partial\Omega_{a1}(0, L) = &-\eta^2 A_1 \sigma_x \frac{d^2 w_1}{dx^2} \delta w_1' \Big|_{x=0}^{x=L} + \eta^2 A_1 \sigma_x \frac{d^3 w_1}{dx^3} \delta w_1 \Big|_{x=0}^{x=L}\\
&+ \Big[-\kappa G A_1 \Big(\varphi_1 - \frac{dw_1}{dx}\Big) + (-A_1\sigma_x + \eta^2 c_{12}) \frac{dw_1}{dx} - \eta^2 c_{12} \frac{dw_2}{dx}\Big] \delta w_1 \Big|_{x=0}^{x=L}
\end{aligned}$$

$$
\begin{aligned}
\partial\Omega_{ai}(0,L) &= -\eta^2 A_i\sigma_x \frac{d^2 w_i}{dx^2}\delta w_i'\Big|_{x=0}^{x=L} + \eta^2 A_i\sigma_x \frac{d^3 w_i}{dx^3}\delta w_i\Big|_{x=0}^{x=L} \\
&\quad + \Big\{-\kappa G A_i\Big(\varphi_i - \frac{dw_i}{dx}\Big) + [-A_i\sigma_x + \eta^2(c_{(i-1)i} + c_{i(i+1)})]\frac{dw_i}{dx} \\
&\quad - \eta^2\Big(c_{(i-1)i}\frac{dw_{i-1}}{dx} + c_{i(i+1)}\frac{dw_{i+1}}{dx}\Big)\Big\}\delta w_i\Big|_{x=0}^{x=L} \qquad \text{for} \quad i = 2,3,\dots,n-1 \\
\partial\Omega_{an}(0,L) &= -\eta^2 A_n\sigma_x \frac{d^2 w_n}{dx^2}\delta w_n'\Big|_{x=0}^{x=L} + \eta^2 A_n\sigma_x \frac{d^3 w_n}{dx^3}\delta w_n\Big|_{x=0}^{x=L} \\
&\quad + \Big\{-\kappa G A_n\Big(\varphi_n - \frac{dw_n}{dx}\Big) + [-A_n\sigma_x + \eta^2(c_{(n-1)n} + k)]\frac{dw_n}{dx} \\
&\quad - \eta^2 c_{(n-1)n}\frac{dw_{n-1}}{dx}\Big\}\delta w_n\Big|_{x=0}^{x=L} \\
\partial\Omega_{bi}(0,L) &= EI_i\frac{d\varphi_i}{dx}\delta\varphi_i\Big|_{x=0}^{x=L} \qquad \text{for} \quad i = 1,2,\dots,n
\end{aligned}
\tag{4.2}
$$

where $\delta w_i'$ is the derivative of δw_i with respect to x. Thus the boundary conditions at $x = 0, L$ are given by

$$
\begin{aligned}
& EI_i\frac{d\varphi_i}{dx} = 0 \qquad \text{or} \qquad \varphi_i = 0 \qquad \text{for} \quad i = 1,2,\dots,n \\
& (-\eta^2 A_i\sigma_x)\frac{d^2 w_i}{dx^2} = 0 \qquad \text{or} \qquad \frac{dw_i}{dx} = 0 \qquad \text{for} \quad i = 1,2,\dots,n \\
& \eta^2 A_1\sigma_x\frac{d^3 w_1}{dx^3} - \kappa G A_1\Big(\varphi_1 - \frac{dw_1}{dx}\Big) + (-A_1\sigma_x + \eta^2 c_{12})\frac{dw_1}{dx} - \eta^2 c_{12}\frac{dw_2}{dx} = 0 \\
& \qquad \text{or} \qquad w_1 = 0 \\
& \eta^2 A_i\sigma_x\frac{d^3 w_i}{dx^3} - \kappa G A_i\Big(\varphi_i - \frac{dw_i}{dx}\Big) + [-A_i\sigma_x + \eta^2(c_{(i-1)i} + c_{i(i+1)})]\frac{dw_i}{dx} \\
& \qquad - \eta^2\Big(c_{(i-1)i}\frac{dw_{i-1}}{dx} + c_{i(i+1)}\frac{dw_{i+1}}{dx}\Big) = 0 \\
& \qquad \text{or} \qquad w_i = 0 \qquad \text{for} \quad i = 2,\dots,n-1 \\
& \eta^2 A_n\sigma_x\frac{d^3 w_n}{dx^3} - \kappa G A_n\Big(\varphi_n - \frac{dw_n}{dx}\Big) + [-A_n\sigma_x + \eta^2(c_{(n-1)n} + k)]\frac{dw_n}{dx} \\
& \qquad - \eta^2 c_{(n-1)n}\frac{dw_{n-1}}{dx} = 0 \qquad \text{or} \qquad w_n = 0
\end{aligned}
\tag{4.3}
$$

It is observed that for the small scale parameter $\eta > 0$ (nonlocal theory), the natural boundary conditions are coupled, and these boundary conditions uncouple for $\eta = 0$ (local theory).

5. Conclusions

Variational principles are derived using a semi-inverse variational method for multi-walled CNTs under buckling loads with the model formulation based on the nonlocal theory of Timoshenko beams. The nonlocal elasticity theory accounts for small scale effects applicable to nano-sized objects, and the Timoshenko beam model takes shear deformation into account which is not negligible in the case of nanotubes with a small length-to-diameter ratio. As such, the formulation used improves the accuracy of the results. The corresponding Rayleigh quotient as well as the natural and geometric boundary conditions are derived. It is observed that the natural boundary conditions are coupled at the free end due to small scale effects being taken into account. The variational principle facilitates the application of approximate and numerical methods of solution and allows the computation of buckling loads for complicated boundary conditions.

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Wariacyjne sformułowanie problemu wyboczenia wielościennych nanorurek węglowych modelowanych jako nielokalne belki Timoszenki

Streszczenie

W pracy przedyskutowano wariacyjne sformułowanie zagadnienia wyboczenia wielościennych nanorurek węglowych wyprowadzone metodą pół-odwrotną z równaniami konstytutywnymi opartymi na nielokalnej teorii belki Timoszenki uwzględniającej efekty małoskalowe i odkształcenia postaciowe. Teoria nielokalna rozszerza zakres stosowalności modelu fizycznego belki poprzez dokładniejsze odwzorowanie zjawisk nanoskalowych. Wprowadzono naturalne i geometryczne warunki brzegowe dla wielościennych nanorurek, które ostatecznie ujęto jako warunki brzegowe sprzężone, w odróżnieniu do warunków rozprzężonych w przypadku prostego podparcia lub zamurowania brzegów oraz zastosowania teorii lokalnej. Wykazano, że wykorzystane zasady wariacyjne i wynikający iloraz Rayleigha podnoszą wydajność przybliżonych metod numerycznych.

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