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HOMOGENIZATION OF INHOMOGENEOUS BEAM UNDER A MOVING LOAD

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The paper deals with the asymptotic wave solutions of displacement of an inhomogeneous beam under a load distributed over a given length and moving at a constant velocity. The beam rests on a viscoelastic foundation. For a homogeneous beam the respective solutions are well known. Heterogeneity of the beam structure leads to a non–stationary motion. In this case do not exist precise solutions, so the method of 2 scale asymptotic expansion is applied. The analytical calculations are carried out for two particular cases of periodic inhomogeneous beam structure:
1. "discrete" – the beam is made of 2 different materials
2. "continuous" – the inhomogeneities are described by some continuous functions of position.

1. Formulation of the problem

Let us consider an infinite Euler–Bernoulli beam resting on a viscoelastic foundation with the elastic coefficient q. EI denotes flexural rigidity, ρ – mass density and A – cross sectional area with the moment of inertia I. We assume that all quantities given above are described by some periodic functions of one variable changing along the beam. The external load F is defined by

$$F(x,t) \equiv F(x-vt) = F(z) = \begin{cases} F_0, & z \in (0,L) \\ 0, & z \notin (0,L) \end{cases} . \tag{1.1}$$

It means that the load is uniformly distributed on the length L and it moves in positive direction of x variable (along the beam) with a constant velocity v, and t – denotes the time variable.

L is assumed to be large compared with the length of period describing inhomogeneity of the beam structure i.e.

$$\frac{l}{L} = \varepsilon \ll 1, \tag{1.2}$$

where l is the length of the period of the following functions:

$$(EI)^{\varepsilon}(x), \qquad \rho^{\varepsilon}(x), \qquad A^{\varepsilon}(x). \tag{1.3}$$

If we introduce a new variable (microvariable):

$$y = \frac{x}{\varepsilon}, \tag{1.4}$$

the functions

$$\begin{aligned} (EI)^{\varepsilon}(x) &\equiv EI(y), \\ \rho^{\varepsilon}(x) &\equiv \rho(y), \\ A^{\varepsilon}(x) &\equiv A(y), \end{aligned} \tag{1.5}$$

are periodic functions of y with the period:

$$L = \frac{l}{\varepsilon}. \tag{1.6}$$

The assumption that ε is small means that the structure of the beam is dense along the characteristic length of the external load.

The motion of the beam under the load (1.1) is described by the following equation:

$$\begin{aligned} F(x - vt) &= \frac{\partial^2}{\partial x^2}(EI)^{\varepsilon}(x)\frac{\partial^2}{\partial x^2}w^{\varepsilon}(x,t) + (\rho A)^{\varepsilon}(x)\frac{\partial^2}{\partial t^2}w^{\varepsilon}(x,t) + \\ &+ \eta\frac{\partial}{\partial t}w^{\varepsilon}(x,t) + qw^{\varepsilon}(x,t), \end{aligned} \tag{1.7}$$

where η is the external damping coefficient.

The aim of the paper is to find the deflection of the beam $w^{\varepsilon}(x,t)$. The extact solution of the problem (1.7) with the appropriate boundary conditions (see [2]) does not exist; we will seek for an approximate solution using the method of two–scale asymptotic expansion.

2. Two–scale asymptotic expansion

We assume that

$$\begin{aligned} w^{\varepsilon}(x,t) &\cong w_0(x,y,t) + \varepsilon^2 w_2(x,y,t) + \varepsilon^3 w_3(x,y,t) + ..., \\ y &= \frac{x}{\varepsilon}, \end{aligned} \tag{2.1}$$

where w_i, $i = 0, 2, 3$, are y–periodic functions.

Following ([3], [4]) we put

$$\frac{\partial}{\partial x} \longrightarrow \frac{\partial}{\partial x} + \frac{1}{\varepsilon}\frac{\partial}{\partial y}. \tag{2.2}$$

Putting (2.1) in (1.7) and using (2.2) we obtain the terms of order: ε^{-4}, ε^{-2}, ε^{-1} respectively.

Combining the terms of the same order in ε we get the hierarchy of the formula for w_0, w_2, w_3 etc.

From equation:

$$\varepsilon^{-4} \Big| \qquad \frac{\partial^2}{\partial y^2}(EI)(y)\frac{\partial^2}{\partial y^2}w_0(x,y,t) = 0, \tag{2.3}$$

we conclude that w_0 (x,y,t) is constant as a function of y (here is only one periodic solution of the equation (2.3) see [3]).

It is

$$w_0(x,y,t) = w_0(x,t). \tag{2.4}$$

Using (2.4) we obtain the formula for $(w_2(x,y,t)$ in the form:

$$\varepsilon^{-2} \Big| \qquad \Big[\frac{\partial^2}{\partial y^2}(EI)(y)\Big]\Big[\frac{\partial^2}{\partial x^2}w_0(x,t)\Big] + \\ + \frac{\partial^2}{\partial y^2}(EI)(y)\frac{\partial^2}{\partial y^2}w_2(x,y,t) = 0, \tag{2.5}$$

or

$$\frac{\partial^2}{\partial y^2}(EI)(y)\frac{\partial^2}{\partial y^2}w_2(x,y,t) = -\Big[\frac{\partial^2}{\partial y^2}(EI)(y)\Big]\frac{\partial^2}{\partial x^2}w_0(x,t). \tag{2.6}$$

Because of:

$$\frac{1}{L}\int_0^L \frac{\partial^2}{\partial y^2}(EI)(y)dy = 0, \tag{2.7}$$

and then using the theorem of the existance and uniqueness of periodic solutions of equations with periodic coefficients [3] we get only one periodic solution (with an accuracy up to a constant) in y variable. We are looking for $w_2(x,y,t)$ in the way of separating the variables:

$$w_2(x,y,t) = -\mathcal{X}(y)\frac{\partial^2}{\partial x^2}w_0(x,t), \tag{2.8}$$

where $\mathcal{X}(y)$ is a periodic solution of the following equation:

$$\frac{\partial^2}{\partial y^2}(EI)(y)\frac{\partial^2}{\partial y^2}\mathcal{X}(y) = \frac{\partial^2}{\partial y^2}(EI)(y), \qquad y \in (0,L), \tag{2.9}$$

Equation (2.9) describes so–called "cell – problem" for the inhomogeneous beam ([5]).

Similarly we get the formula for $w_3(x,y,t)$:

$$w_3(x,y,t) = -\psi(y)\frac{\partial^3 w_0(x,t)}{\partial x^3}, \tag{2.10}$$

where $\psi(y)$ is a periodic solution of the equation:

$$\begin{aligned}\frac{\partial^2}{\partial y^2}(EI)(y)\frac{\partial^2}{\partial y^2}\psi(y) &= -2\Big[\frac{\partial}{\partial y}EI(y)\frac{\partial^2}{\partial y^2}\mathcal{X}(y) + \\ &+ \frac{\partial^2}{\partial y^2}EI(y)\frac{\partial}{\partial y}\mathcal{X}(y)\Big].\end{aligned} \tag{2.11}$$

The equation for $w_4(x,y,t)$ has the form

$$\begin{aligned}&-\frac{\partial^2}{\partial y^2}EI(y)\frac{\partial^2}{\partial y^2}w_4(x,y,t) = EI(y)\frac{\partial^4}{\partial x^4}w_0(x,t) + \\ &+EI(y)\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial y^2}w_2(x,y,t) + 4\frac{\partial}{\partial y}\Big[EI(y)\frac{\partial^3}{\partial x^2\partial y}w_2(x,y,t)\Big] + \\ &+\frac{\partial^2}{\partial y^2}EI(y)\frac{\partial^2}{\partial x^2}w_2(x,y,t) + 2\frac{\partial}{\partial y}\Big[EI(y)\frac{\partial}{\partial x}\frac{\partial^2}{\partial y^2}w_3(x,y,t)\Big] + \\ &+2\frac{\partial^2}{\partial y^2}\Big[EI(y)\frac{\partial^2}{\partial x\partial y}w_3(x,y,t)\Big] + \rho A(y)\frac{\partial^2}{\partial t^2}w_0(x,t) + \\ &+\eta\frac{\partial}{\partial t}w_0(x,t) + qw_0(x,t) - F(x-vt).\end{aligned} \tag{2.12}$$

We obtain w_0 from the condition of uniqueness of w_4 i.e. from the fact that the integral over $(0,L)$ of the right hand side of (2.12) is equal to zero i.e.,

$$\begin{aligned}F(x-vt) &= (EI)^{ef}\frac{\partial^4}{\partial x^4}w_0(x,t) + (\rho A)^{ef}\frac{\partial^2}{\partial t^2}w_0(x,t) + \\ &+ \eta\frac{\partial}{\partial t}w_0(x,t) + qw_0(x,t),\end{aligned} \tag{2.13}$$

where

$$\begin{aligned}(EI)^{ef} &= \frac{1}{L}\int_0^L \Big[(EI)(y) - (EI)(y)\frac{\partial^2}{\partial y^2}\mathcal{X}(y)\Big]dy, \\ (\rho A)^{ef} &= \frac{1}{L}\int_0^L (\rho A)(y)dy.\end{aligned} \tag{2.14}$$

The equation (2.13) describes the motion of the beam under moving load F, but the beam is now a homogeneous one. The coefficient (EI) is the effective rigidity of the beam in the case of "infinitely dense" periodic structure i.e. $\varepsilon = 0$.

Denoting

$$\frac{1}{L}\int_0^L (\cdot)dy = < \cdot >, \tag{2.15}$$

after same simple calculations we get

$$(EI)^{ef} = \left\langle \frac{1}{(EI)(y)} \right\rangle^{-1}. \tag{2.16}$$

This expression is identical to so-called Voight bound for the effective inhomogeneous elastic constant in material with microstructure [6].

The asymptotic expansion of real deflection $w^\varepsilon(x,t)$ with the accuracy up to the terms of order ε^2 has the form

$$w^\varepsilon(x,t) \cong w_0(x,t) - \varepsilon^2 \mathcal{X}(y)\frac{\partial^2}{\partial x^2} w_0(x,t), \qquad y = \frac{x}{\varepsilon}, \tag{2.17}$$

where $\mathcal{X}(y)$ is such a solution of (2.9) that

$$\int_0^L \mathcal{X}(y)dy = 0. \tag{2.18}$$

and w_0 is a soluton of (2.13) which is known from literature [2].

3. Examples

A. Let us consider the "discrete" periodic structure of the beam having the following form:

$$EI(y) = \begin{cases} a_1, & y \in (0,b) \quad b \in (0,L) \\ a_2, & y \in (b,L) \end{cases} \tag{3.1}$$

where a_1, a_2 are the elastic constants for two different materials of which the beam is made.

Then the equation (2.9) takes the form:

$$\begin{aligned} \frac{d^2}{dy^2} a_1 \frac{d^2}{dy^2} \chi_1(y) &= \frac{d^2}{dy^2} a_1, \\ \frac{d^2}{dy^2} a_2 \frac{d^2}{dy^2} \chi_2(y) &= \frac{d^2}{dy^2} a_2, \end{aligned} \tag{3.2}$$

and the conditions of periodicity and continuity are:

$$\begin{aligned} &\chi_1(b) = \chi_2(b), && \chi_1(0) = \chi_2(L), \\ &\dot\chi_1(b) = \dot\chi_2(b), && \dot\chi_1(0) = \dot\chi_2(L), \\ &a_1\ddot\chi_1(y) = a_2\ddot\chi_2(y). && \end{aligned} \tag{3.3}$$

To complete the system of equations (3.2), (3.3) we add the condition of uniqueness of the periodic solution as follows:

$$\int_0^b \chi_1(y)dy + \int_b^L \chi_2(y)dy = 0. \tag{3.4}$$

The general solutions of (3.2) are:

$$\begin{aligned} \chi_1(y) &= A_1y^2 + B_1y + C_1, \qquad y \in (0,b), \\ \chi_2(y) &= A_2y^2 + B_2y + C_2, \qquad y \in (b,L). \end{aligned} \tag{3.5}$$

Putting (3.5) into (3.3) and (3.4) we can find the coefficients A_i, B_i, C_i, $i = 1,2$. Finnaly, we get:

$$\begin{aligned} A_1 &= \frac{a_1 - a_2}{2a_1} + \frac{a_2}{a_1}\Big[\frac{(a_1 - a_2)b}{2(a_1 - a_2)b - a_1L}\Big], \\ B_1 &= \frac{b(a_1 - a_2)(L - b)}{2(a_1 - a_2)b - a_1L}, \\ C_1 &= \frac{1}{6}(L^2 - 3Lb + b^2)\frac{(a_1 - a_2)b}{2(a_1 - a_2)b - a_1L}, \\ A_2 &= \frac{(a_1 - a_2)b}{2(a_1 - a_2)b - a_1L}, \\ B_2 &= -\frac{b(b + L)(a_1 - a_2)}{2(a_1 - a_2)b - a_1L}, \\ C_2 &= \frac{b(a_1 - a_2)(L^2 + 3Lb + b^2)}{6[2(a_1 - a_2)b - a_1L]}. \end{aligned} \tag{3.6}$$

B. "continuous". We assume that the elastic modulus of the beam is described by the following continuous function:

$$EI(y) = \frac{1}{\sin y + a}. \tag{3.7}$$

Then the equation (2.9) takes the form

$$\frac{d^2}{dy^2}\Big(\frac{1}{\sin y + a}\Big)\frac{d^2}{dy^2}\mathcal{X}(y) = \frac{d^2}{dy^2}\Big(\frac{1}{\sin y + a}\Big). \tag{3.8}$$

Integrating (3.8) and making use of the condition (2.18) we obtain

$$\mathcal{X}(y) = -\frac{\sin y}{a}. \tag{3.9}$$

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Streszczenie

Praca dotyczy asymptotycznych rozwiązań falowych dla ugięcia niejednorodnej belki powstałego pod wpływem obciążenia przyłożonego na pewnej długości i poruszającego się wzdłuż osi belki ze sta ą prędkością. Belka spoczywa na lepkosprężystym podłożu. Dla belki o jednorodnej strukturze rozwiązanie tego zagadnienia jest znane. Niejednorodność materiałowa prowadzi do niestacjonarnego ruchu. Ponieważ w tym przypadku nie ma ścisłych rozwiązań zaproponowano 2–skalowe rozwinięcia asymptotyczne w celu aproksymacji funkcji ugięcia. Podano 2 przykłady, w których periodyczna struktura belki jest dana jako:

1. "dyskretna" – belka składa się z 2 różnych materiałów
2. "ciągła" – niejednorodności są opisane przez ciągłe funkcje położenia.

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