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NONHOLONOMIC VARIATIONAL PRLOBLEMS AND HEURISTICS OF CONTROL FORCES

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We discuss differences between mechanical and mathematical variational problems concerning nonholonomic constraints, i.e., between the d'Alembert principle and the procedure based on the Lusternik theorem of conditional extrema. As a technical example of application we discuss the frictional speed reducer. Although it is the d'Alembert principle that is applicable to realistic nonholonomic systems, we show that the mathematical variational problem and the Lusternik procedure are convenient heuristic tools in some mechanical problems of a control and a programme motion, respectively. The point is that the realistic and formally convenient models of controlling forces and controlling agents are being suggested.

1. D'Alembert and Lusternik principles

Let us consider a mechanical system with the Lagrange function $L(q, \dot{q})$; q^i and $\dot{q}^i$, $i = 1...n$, denote generalized coordinates and velocities respectively. Configuration space, i.e., the n–dimensional manifold of all admissible values of q, will be denoted by Q. On the obvious assumptions, the variational problem $\delta \int L dt = 0$ with fixed ends in the space of variables $(t, ..., q^i, ...)$ (i.e., in the configuration space–time of the problem) leads to the well–known Euler–Lagrange equations of motion:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0, \qquad i = 1, ..., n. \tag{1.1}$$

Besides interactions represented by L, we can take into account the additional ones, represented by covariant vectors $\Phi_i(q, \dot{q})$ of generalized forces conjugate to variables q^i; the resulting equations of motion have the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = \Phi_i, \qquad i = 1, ..., n. \tag{1.2}$$

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For example, the standard description of frictional phenomena is based on the phenomenological dissipative forces in the form of

$$\Phi_i = -a_{ij}(q,\dot{q})\dot{q}^j, \tag{1.3}$$

where the matrix $[a_{ij}]$ is symmetric and has such a form that the equation

$$\det\left[a_{ij} - \lambda \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right] = 0, \tag{1.4}$$

has only the non–negative solutions for λ (the positive values of λ would corespond to the "anti–frictional" self–acceleration; the vanishing values of λ correspond to the lack of friction in certain directions).

Let us consider the following constrained variational problem:

$$\delta \int L dt = 0, \qquad F_a(q,\dot{q}) = 0, \qquad a = 1, ..., m. \tag{1.5}$$

The Lusternik theorem implies the following system of differential equations in functions $q^i(t)$:

$$\begin{aligned} &a) \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = R_i, \qquad i = 1, ..., n, \\ &b) \qquad F_a(q,\dot{q}) = 0, \qquad a = 1, ..., m, \end{aligned} \tag{1.6}$$

where

$$\begin{aligned} R_i &= \mu^a(t)\frac{\partial F_a}{\partial q^i} - \frac{d}{dt}\left(\mu^a \frac{\partial F_a}{\partial \dot{q}^i}\right) = -\mu^a(t)\frac{\partial^2 F_a}{\partial \dot{q}^i \partial \dot{q}^j}\ddot{q}^j + \\ &- \mu^a(t)\frac{\partial^2 F_a}{\partial \dot{q}^i \partial q^j}\dot{q}^j + \mu^a(t)\frac{\partial F_a}{\partial q^i} - \dot{\mu}^a \frac{\partial F_a}{\partial \dot{q}^i}, \end{aligned} \tag{1.7}$$

$\ddot{q}^j$ denoting generalized accelerations and μ^a – Lagrange multipliers.

Equation (1.6) together with the "constitutive relation" (1.7) create a system of $(n+m)$ differential equations with regard to $(n+m)$ functions $q^i(t)$, $\mu^a(t)$. The system is of second differential order with respect to q and of first differential order with respect to μ. When considering the real mechanical problems, we are interested in q^i alone while the μ^a are to be eliminated.

A similar scheme may be formulated for the differential constraints of higher order $F_a(q,\dot{q},\ddot{q},...,q^{(n)}) = 0$, or even for more general functional constraints.

If constraints (1.6b) are holonomic, i.e., F_a do not depend on velocities $\dot{q}^i$, then $R_i = \mu^a \partial F_a / \partial q^i$, and equations (1.6) coincide with those obtained by applying the d'Alembert principle to mechanical systems defined by L and subject to ideal holonomic constraints described by $F_a = 0$, $a = 1, \ldots, m$. If some non-Lagrangian interactions are present, the whole system of equations of motion has the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = \Phi_i + \mu^a \frac{\partial F_a}{\partial q^i}, \qquad F_a = 0. \tag{1.8}$$

The quantities $R_i = \mu^a \partial F_a / \partial q^i$ are called reaction forces, they are responsible for realization of the programme of motion given by conditions $F_a = 0$, $a = 1, \ldots, m$. At the same time they do not affect the along-constraints modes of motion (the splitting into along-constraints directions and vertical directions is understood in the sense of the metric $\partial^2 L / \partial \dot{q}^i \partial \dot{q}^j$).

Let us now assume that constraints are non-holonomic, i.e., functions F_a depend on velocities in an essential way; by nonessential dependence we mean such one that

$$F_a(q, \dot{q}) = K_a^b \dot{G}_b(q) = K_a^b(q, \dot{q}) \frac{\partial G_b}{\partial q^i} \dot{q}^i,$$

where the matrix $\mathbf{K}$ is non-singular and depends smoothly on its arguments. It turns out that constrained variational conditions (1.5) are non-adequate in mechanical problems with non-holonomic constraints arising due to the contact-friction mechanism (e.g., non-sliding rolling of rigid bodies). In principle, all realistic non-holonomic constraints of this class are linear in velocities, i.e., their equations can be reduced to the form

$$F_a(q, \dot{q}) = \omega_{ai}(q) \dot{q}^i = 0. \tag{1.9}$$

Holonomic (or rather semi-holonomic) constraints $G_a = \text{const}$ may be formally represented in this form by putting

$$F_a = \frac{\partial G_a}{\partial q^i} \dot{q}^i, \quad \text{i.e.,} \qquad \omega_{ai} = \frac{\partial G_a}{\partial q^i}. \tag{1.10}$$

For realistic mechanical problems with constraints (1.9), equations of motion are derived from the d'Alembert principle, i.e., from the assumption that reactions R_i are dual to admissible velocities, i.e., $R_i u^i = 0$ if $\omega_{ai} u^i = 0$. This principle leads to equations of motion:

$$\begin{aligned} a) \qquad & \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = \lambda^a(t) \omega_{ai}, \\ b) \qquad & \omega_{ai} \dot{q}^i = 0, \end{aligned} \tag{1.11}$$

λ's being Lagrange multipliers. If there are non-Lagrangian interactions, then the right-hand side of (1.11a) should be complemented by adding an appropriate generalized force $\Phi_i(q, \dot{q})$.

At the same time, the constrained variational problem:

$$\delta \int L dt = 0, \qquad \omega_{ai}(q) \dot{q}^i = 0, \tag{1.12}$$

leads to the following Euler–Lagrange equations:

$$
\begin{aligned}
&a) \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = \mu^a\left(\frac{\partial \omega_{aj}}{\partial q^i} - \frac{\partial \omega_{ai}}{\partial q^j}\right)\dot{q}^j - \dot{\mu}^a\omega_{ai}, \\
&b) \qquad \omega_{ai}\dot{q}^i = 0,
\end{aligned} \tag{1.13}
$$

μ's being Lagrange multipliers.

The difference between (1.11) and (1.13) is obvious. The righthand side of (1.11) can be identified with the second term of the right–hand side of (1.13a) by putting $\lambda^a = -\dot{\mu}^a$, however, the first, curl–like, term on the right hand side of (1.13.a) has no analogue in (1.11). Variational reactions

$$
R_i^{\text{var}} = \mu^a\left(\omega_{aj,i} - \omega_{ai,j}\right)\dot{q}^j - \dot{\mu}^a\omega_{ai}, \tag{1.14}
$$

do not work along actual velocities $\dot{q}^i$ compatible with constraints, $R_i^{\text{var}}(q,\dot{q})\dot{q}^i = 0$, however, they fail to be dual to all possible virtual velocities satisfying (1.13.b), $R_i^{\text{var}}(q,\dot{q})u^i \neq 0$. In other words, R^{var} affects also the along–constraints motion.

Let us observe that in some sense the system (1.13) has more degrees of freedom than (1.11), because it contains both μ's and their time derivatives $\dot{\mu}$, thus, Lagrange multipliers cannot be eliminated in an algebraic way.

The difference between (1.11) and (1.13), and the non–variational character of (1.11) have been a subject of many discussion and speculations. Although (1.11), (1.13) are evidently different, one can not apriori exclude the possibility that for realistic systems and in realistic range of phenomena (i.e., in certain range of initial conditions) their quantitative predictions will be comparable, and, incidentally, there are examples of situations where certain families of motions ruled by (1.11) and (1.13) coincide. This was our idea in [6]. However, it turns out that it is just in realistic technical problems where the predictions of variational equations (1.13) are drastically different from those based on the d'Alembert principle (1.11).

2. Example

As a typical example let us quote the system pictured in figure 1; it is practically used as an element of the frictional speed reducer [4] and other similar intstruments. Rotation axes of rotors A, B are co–planar and mutually perpendicular. Besides of the rotational motion around its axis, the rotor B can undergo translational motions along the axis. The boundary of the rotor B is tangent to the target of the rotor A, and at the moving contact point the usual conditions of non–slidind rolling are satisfied. We assume that both rotors are symmetric, rotate about their vertical symmetry axes, and the centres of mass are placed on

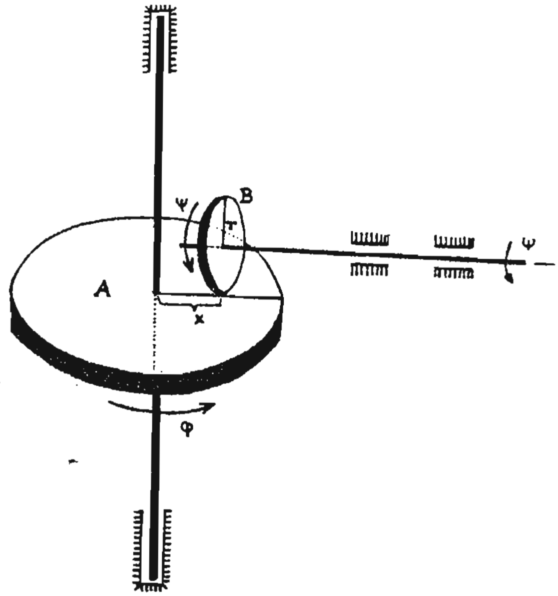

Fig. 1.

rotation axes. Let a, b denote the inertial moments of circles A, B, m – the mass of B, and r – the radius of the circle B. Let us assume that rotations are free and the only forces are those acting along the x–axis, i.e., along the translational degree of freedom of B.Thus, Lagrangian has the form:

$$L = T - V(x) = \frac{a}{2}\dot{\varphi}^2 + \frac{b}{2}\dot{\psi}^2 + \frac{m}{2}\dot{x}^2 - V(x). \tag{2.1}$$

Equations of non–holonomic constraints have the form

$$F = x\dot{\varphi} - r\dot{\psi} = 0.$$

D'Alembert principle leads to equations of motion

$$\begin{aligned} &a) && a\ddot{\varphi} = -\lambda x,\\ &b) && b\ddot{\psi} = -\lambda r,\\ &c) && m\ddot{x} = -V'(x),\\ &d) && x\dot{\varphi} - r\dot{\psi} = 0. \end{aligned} \tag{2.2}$$

Translational dynamics (2.2.c) is autonomous. We can easily eliminate λ using (2.2.b),

$$\lambda = -\frac{b}{r}\ddot{\psi}. \tag{2.3}$$

Substituting this to (2.2.a) and making use of (2.2.d) we obtain:

$$\left(a-\frac{b}{r^2}x^2\right)\ddot{\varphi}-\dot{x}\dot{\varphi}=0. \tag{2.4}$$

When (2.2.c) is solved, this becomes a closed time–dependent equation for φ and can be in principle solved. Then ψ is found from

$$\dot{\psi}-\frac{x}{r}\dot{\varphi}=0. \tag{2.5}$$

Variational equations have the form:

$$\begin{aligned} &a) && \frac{d}{dt}(a\dot{\varphi}+x\mu)=0,\\ &b) && \frac{d}{dt}(b\dot{\psi}+r\mu)=0,\\ &c) && m\ddot{x}+V'(x)-\mu\dot{\varphi}=0,\\ &d) && x\dot{\varphi}-r\dot{\psi}=0. \end{aligned} \tag{2.6}$$

The dynamics of x is non–autonomous and if $\mu\neq 0$, there is a rether strange helical force $\mu\dot{\varphi}$ acting along the x–axis. And equations (2.6) imply that with a possible exception of some trivial situations, μ does not vanish. Namely, integrating (2.6.a,b) and using (2.6.d) we obtain

$$\begin{aligned} &a) && a\dot{\varphi}+x\mu=A,\\ &b) && \frac{b}{r}x\dot{\varphi}+r\mu=B. \end{aligned} \tag{2.7}$$

A, B being integration constants. Performing a few eliminations of variables, based on (2.7), we can reduce the system (2.6) to the following sequence of equations:

$$\begin{aligned} &a) && m\ddot{x}+V'(x)+\frac{b}{r^2}\left(\frac{A-\frac{B}{r}x}{a-\frac{b}{r^2}x^2}\right)^2-\frac{B}{r}\frac{A-\frac{B}{r}x}{a-\frac{b}{r^2}x^2}=0,\\ &b) && \left(a-\frac{b}{r^2}x^2\right)\dot{\varphi}+\frac{B}{r}x=A,\\ &c) && \dot{\psi}=\frac{x}{r}\dot{\varphi},\\ &d) && \mu=\frac{B}{r}-\frac{b}{r^2}x\dot{\varphi}. \end{aligned} \tag{2.8}$$

(2.8.a) can be independently solved, and substituting its solution to (2.8.b,c,d) we solve the problem. Only for the special choice of angular constants of motion, $A=B=0$, translational equation (2.8.a) coincides with (2.2.c). However, there are no rotational motions in this case, $\varphi = \text{const}$, $\psi = \text{const}$. If $x = \pm r\sqrt{\frac{a}{b}}$ happens to be an equilibrium point of V, then (2.8.b, c) do not imply the constancy of φ, ψ, however, the time dependence of φ is completely nondetermined. If x_0 is an equilibrium point of V and A, B are chosen in such a way that $Ar - Bx_0 = 0$, then there are also stationary solutions $x = x_0$, $\varphi = \text{const}$, $\psi = \text{const}$.

Obviously, it is (2.2), not (2.6), what is compatible with experimental data. It is not only the very existence of the non-observed helical force in (2.6.c) what decides that the variational approach is non-realistic. Much more important is that the helical term is strongly singular. Thus, phase portraits of (2.2), (2.6) are completely different, topologically non-equivalent.

Although variational model (1.13) is more convincing from the point of view of mathematical aprioric ideas, it is d'Alembert model that provides an adequate description of realistic non-holonomic problems of the sliding-free rolling.

A deep mathematical analysis of certain differences between variational and mechanical (i.e., d'Alembertian) models of nonholonomic constraints may be found in [8], [2].

3. Constraints nonlinear in velocities

If non-holonomic constraints are non-linear in velocities, then variational problem (1.5) is still well-defined, however, there is no automatic generalization of the d'Alembert principle. Obviously, in practical problems with constraints generated by the direct contact mechanism, this has no practical meaning, because such constraints are always linear in velocities. Nevertheless, there were certain academic speculations concerning nonlinear nonholonomic constraints arising in a consequence of certain singular limit transitions from linear nonholonomic problems; the mentioned singularity consists in disappearing of certain degrees of freedom (Hamel's system). Besides, there is a motivation for such discussions from the theory of programme motion and control theory. When we investigate artificial systems of automatic regulation, in particular, systems stabilizing the absolute value of velocity or angular velocity of satellites and other spatial objects, the question concerning possible generalizations of the d'Alembert principle appears in a rather natural way, at least as a question of heuristic nature.

The most popular in literature nonlinear generalization of d'Alembert principle, namely, so-called Appell–Tshetajev principle, leads to the following equations of motion:

$$\begin{aligned} &a) \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = \Phi_i + \lambda^a \frac{\partial F_a}{\partial \dot{q}^i}, \\ &b) \qquad F_a(q, \dot{q}) = 0, \end{aligned} \tag{3.1}$$

$\lambda^a(t)$, $a = 1, ..., n$ denote Lagrange multipliers. Equations (3.1) reduce to (1.11) if constraints are linear in velocities, $F_a = \omega_{ai}(q)\dot{q}^i$. They reduce to (1.8) for holonomic constraints represented in anholonomic form,

$$\frac{\partial F_a}{\partial q^i}\dot{q}^i = 0. \tag{3.2}$$

The Appell–Tshetajev principle is a natural generalization of the d'Alembert principle. In any case, it has a well–defined geometric meaning. Namely, let for a fixed configuration $q \in Q$, $V_q \subset T_qQ$ denote the manifold of all virtual velocities at q, compatible with constraints, i.e., satisfying conditions:

$$F_a(q, v) = 0. \tag{3.3}$$

Obviously, V_q is an $(n-m)$–dimensional surface in the n–dimensional linear space T_qQ of all non–constrained virtual velocities at q. The Appell–Tshetajev principle states that for any $q \in Q$ and for any $v \in V_q$, the reaction force $R_i(q, v)$ maintaining constraints $F_a = 0$ is dual to $T_vV_q \subset T_qQ$, i.e., to the linear subspace of T_qQ tangent to the surface V_q, i.e., $R(q, v)_i u^i = 0$ for any $u \in T_vV_q$. This means exactly that

$$R_i = \lambda^a \frac{\partial F_a}{\partial \dot{q}^i}, \tag{3.4}$$

i.e., equations of motion have the form (3.1).

Comparing (3.4) with (1.6), (1.7) and putting $\lambda^a = -\dot{\mu}^a$ we again conclude that the Appell–Tshetajev reactions is a natural part of the Lusternik reaction (1.7) corresponding to the constrained variational problem (1.5). However, Lusternik reaction constains also three additional terms.

For example, let us consider the system with Lagrangian $L = \frac{m}{2}g_{ij}\dot{q}^i\dot{q}^j - V(q)$, g_{ij} being constants; and constraints stabilizing the absolute value of velocity, $F = \frac{1}{2}g_{ij}\dot{q}^i\dot{q}^j - \frac{c^2}{2}$ (c–constant). Lusternik equations (1.6), (1.7) have the form

$$\begin{aligned} &a) \qquad (m+\mu)\ddot{q}^i + \dot{\mu}\dot{q}^i + g^{ij}\frac{\partial V}{\partial q^j} = 0, \\ &b) \qquad g_{ij}\dot{q}^i\dot{q}^j - c^2 = 0. \end{aligned} \tag{3.5}$$

The corresponding Appell–Tshetajev equations read:

$$\begin{aligned} &a) \qquad m\ddot{q}^i - \lambda\dot{q}^i + g^{ij}\frac{\partial V}{\partial q^j} = 0, \\ &b) \qquad g_{ij}\dot{q}^i\dot{q}^j - c^2 = 0. \end{aligned} \tag{3.6}$$

Even if we identify λ with $-\dot{\mu}$, it is clear that there is an important difference of qualitative nature between (3.5) and (3.6). Namely, besides of the frictional reaction force $\dot{\mu}\dot{q}^i$, (3.5) contains the additional inertial term $\mu\ddot{q}^i$. Lagrange multiplier enters in (3.5) in a first–order differential way.

As mentioned, for natural systems with linear non–holonomic constraints, the Appell–Tshetajev–d'Alembert maintaining forces (reactions) are confirmed by experiment, whereas the Lusternik variational reactions turn out to be completely non–physical. Nevertheless, the formulas (1.7) are useful and provide some kind of heuristic guiding hints, when we go over to artificially generated constraints (servo–constraints), control problems and programme motion.

Namely, the problem is then formulated as follows: we have a system which under "natural" conditions moves according to equations (1.1) or (1.2), and we want to force it to move according to the programme described by certain scleronomic, $F_a(q,\dot{q}) = 0$, or rheonomic, $F_a(q,\dot{q},t) = 0$, conditions; $a = 1,...,m$.

If $m = n$ and the dependence of F_a on the time variable is nontrivial, this means that our demands are maximal, i.e., the system is to move along a fixed, prescribed trajectory.

The programme conditions $F_a = 0$ are in general incompatible with equations of motion and we must introduce certain additional programme forces R, which, when added to the right–hand side of (1.2), make equations (1.2), (1.6.b) compatible, or at least compatible to a certain degree of accuracy, with our programme conditions. Variational scheme based on (1.6), (1.7) is one of infinitely many possiblities. However, its main value is not the particular form (1.7) of R, but rather, its general structure as a superposition of certain characteristic expressions with the specific tensorial structure and physical interpretation.

The first term, $-\mu^a(\partial^2 F_a/\partial\dot{q}^i\partial\dot{q}^j)\ddot{q}^j$, suggested that one of possible ways of controlling mechanical motion is to introduce the controlling input into inertial properties of the body, i.e., to use controlling forces of the form

$$P_{ij}\ddot{q}^j, \tag{3.7}$$

where the matrix $\mathbf{P}$ depends on time directly (and then the coefficients P_{ij} themselves are control parameters) or through certain additional degrees of freedom [7]. This method is practically used, e.g., to stabilize the angular velocity of engine shafts.

The second term of (1.7) and the first term on the right hand side of (1.13.a) suggest controlling forces of the form

$$R_{ij}\dot{q}^j, \qquad S_{ij}\dot{q}^j, \qquad \mathbf{R}^T = \mathbf{R}, \qquad \mathbf{S}^T = -\mathbf{S}, \tag{3.8}$$

where again the matrices $\mathbf{R}$, $\mathbf{S}$ serve as inputs of controlling influences. $R_{ij}\dot{q}^j$ descibes damping forces if eigenvalues of $\mathbf{R}$ are negative. The term $S_{ij}\dot{q}^j$ describes controlling influences based on "gyroscopic", or "magnetic" forces. They are dual to velocity vectors and do not influence the energy balance. The remaining terms of (1.7) have no special structure as tensorial functions of $(q,\dot{q})$; they suggest the direct controlling influences of the form $F(t)$.

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Streszczenie

Przedyskutowano różnice między mechanicznymi a matematycznymi zagadnieniami wariacyjnymi z więzami, tzn., między zasadą d'Alemberta a procedurą opartą na twierdzeniu Lusternika o ekstremum związanym. Różnice te zilustrowano przy pomocy przykładu technicznego, jakim jest reduktor prędkości. Zwrócono uwagę, że chociaż wariacyjna zasada Lusternika nie da się zastosować w dynamice realistycznych układów nieholonomicznych, to jest ona heurystycznie użyteczna w zagadnieniach ruchu programowego i sterowania w mechanice. Rzecz w tym, że dostarcza ona pewnych sugestii odnośnie wyboru realistycznych i wygodnych matematycznie modeli sił i czynników sterujących.

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