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FREE VIBRATION OF THE DISCRETE-CONTINUOUS SYSTEM WITH DAMPING

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1. Introduction

The mathematical models of complex constructions often consist of elements with continuous mass distribution and discrete oscillators. The analysis of such discrete-continuous systems leads to the necessity of solving a set of conjugated ordinary and partial differential equations. It arises new kinds of problems in comparison with the analysis of continuous or discrete systems. Because of that, in many works concerning discrete-continuous systems the following method of analysis is proposed. The continuous subsystem is discretized and the calculations are made for given parameters of the system. Such a method was given by J.Kruszewski (1975). An exact solution of the initial-boundary problem concerning the vibration of a conservative discretecontinuous system is given by S.Kasprzyk and Dan-Tinh (1979). The exact equations for the frequencies and modes of vibration of the system were obtained there, orthogonality of modes of vibration was also proved and finally, the problem of free vibration of the system with arbitrary initial conditions was solved. The method given in the above mentioned work can be used to describe the vibration of many discrete-continuous systems with negligible damping. But in a great number of problems , especially those concerning vibroisolation damping is of great importance. The aim of this paper is to give an exact and general method for the description of free vibration of a one-dimensional discrete- continuous system in which concentrated masses are connected to the continuous subsystem by visco-elastic elements discribed by the Voigt-Kelvin model.

2. Formulation of the problem

The calculations were carried out for a beam hinge at the ends with 'n' masses connected to it by visco-elastic elements described by the Voigt-Kelvin model. This particular system was chosen because of some practical reasons related to the damping of the vibration of the beam with dynamic dampers. The following notation is used

- ℓ [m] length of the beam,
- μ [kg/m] linear mass density (the beam is prismatic),
- J [m^4] cross-sectional moment of inertia,
- E [N/m^2] Young's modulus for the material of the beam,
- ℓ_j [m] the distance from the end of the beam to thepoint of mounting of j-th oscillator,
- c_j [N/m] the stiffness of the visco-elastic element,
- η_j [Ns/m] the damping coefficient of the visco-elastic element.

The coordinates in which the vibration of the system will be described are shown in Figure 1.

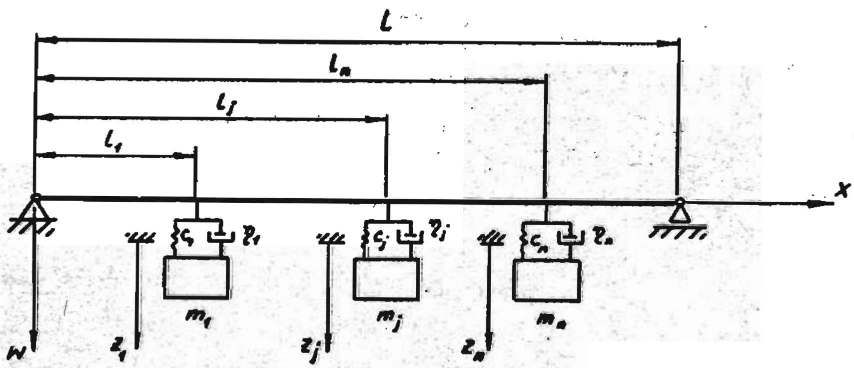

Fig. 1. Model of the system.

We consider small vibration of the system about the equilibrium. The following dimensionless parameters are introduced:

$$w = \mathcal{w}/\ell, \quad x = \mathcal{x}/\ell, \quad l_j = \ell_j/\ell, \quad z_j = \mathcal{z}_j/\ell,$$

$$m_j = \mathcal{m}_j/(\mu l), \quad \omega_{0j} = \phi_{0j}/\Omega, \quad t = \tau\Omega, \quad \zeta_j = \eta_j/(2m_j\phi_{0j}),$$

where

$\Omega = (\pi/\ell)^2\sqrt{(EJ/\mu)}$ - fundamental frequency of free vibration of the beam without oscillators,

$\phi_{0j} = \sqrt{(c_j/m_j)}$ - the frequency of free vibration ofthe j-th oscillator.

Taking the above introduced dimensionless parameters into account the equations of motion for the system and the boundary conditions can be written as follows.

$$
\begin{aligned}
&\frac{\partial^2 w}{\partial t^2} + 1/\pi^4 \frac{\partial^4 w}{\partial x^4} = \sum_{j=1}^{n} \left(-m_j \frac{\partial^2 z_j}{\partial t^2} \delta(x-l_j)\right) , \\
&\frac{\partial^2 z_j}{\partial t^2} = -\omega_{0j}^2 \, [z_j - w(l_j,t)] - 2\,\omega_{0j}\,\zeta_j \left[\frac{dz_j}{dt} - \left.\frac{\partial w}{\partial t}\right|_{x=l_j} \right] ,
\end{aligned}
\tag{2.1}
$$

$$
\begin{aligned}
&w(0,t) = 0, && w(1,t) = 0, \\
&\left.\frac{\partial^2 w}{\partial x^2}\right|_{x=0} = 0, && \left.\frac{\partial^2 w}{\partial x^2}\right|_{x=1} = 0.
\end{aligned}
\tag{2.2}
$$

The initial conditions for t=0 have the form

$$
\begin{aligned}
&w(x,0) = f(x), && \left.\frac{\partial w}{\partial t}\right|_{t=0} = \varphi(x) , \\
&z_j(0) = z_{j0} , && \left.\frac{dz_j}{dt}\right|_{t=0} = v_{j0} .
\end{aligned}
\tag{2.3}
$$

Where $f(x)$ and $\varphi(x)$ are dimensionless functions defining the displace ments and velocities of the points of the beam and z_{j0}, v_{j0} define the displacement and velocity of the mass of the j-th oscillator for t=0. The dimensionless quantities in equations (2.3) are defined as follows

$$
\begin{aligned}
&f(x) = \mathfrak{f}(x)/\ell , && \varphi(x) = \tilde{\varphi}(x)/(\ell\Omega) , \\
&z_{j0} = \mathfrak{z}_{j0}/\ell , && v_{j0} = \mathfrak{v}_{j0}/(\ell\Omega) ,
\end{aligned}
\tag{2.4}
$$

where $\mathfrak{f}(x)$, $\tilde{\varphi}(x)$, $\mathfrak{z}_{j0}$, $\mathfrak{v}_{j0}$ are dimensional quantities.

3. Complex freguencies and modes of vibration

Further analysis will be based on the use of complexfunctions of real variable. In the set of these functions the separation of the spatial and time variables is possible. Let us assume

$$w(x,t) = X(x)\, e^{\gamma t} ,$$
$$z_j(t) = Z_j\, e^{\gamma t} . \tag{3.1}$$

The vector $[\ X(x),\ Z_1 \ldots Z_n\ ]$ which is the eigenvector of the boundary value problem under consideration, will be assumed as a complex mode of vibration. The complex frequency of vibration can be writtenas follows

$$\gamma = \alpha + i\omega . \tag{3.2}$$

The real part of "γ" is the damping of a given mode, and the imaginary part is the frequency.
In the sequence of eigenvalues $\{\gamma_n\}$ pairs of elements are conjugate. The modes of vibration corresponding to each pair are conjugate functions.
We rearrange the equations of complex modes and the boundary condition equations changing the expressions on both sides of these equations into conjugate expressions. As a result we get

$$[X^{IV} - k^4 X]^{*} = [\ \sum_j m_j k^4\, Z_j\, \delta(\, x-l_j)\ ]^{*} ,$$
$$[\gamma^2\, Z_j + (\omega_{0j}^2 + 2\omega_{0j}\zeta_j\gamma)Z_j]^{*} = [(\omega_{0j}^2 + 2\,\omega_{0j}\zeta_j\gamma)X(l_j)]^{*} , \tag{3.3}$$

$$[X(0)]^{*} = 0 , \quad [X^{II}(0)]^{*} = 0 ,$$
$$[X(1)]^{*} = 0 , \quad [X^{II}(1)]^{*} = 0 . \tag{3.4}$$

Applying the rules of complex algebra , we rearrange the equations (3.3) in to the form

$$(X^{*})^{IV} - (k^{*})^4\, X^{*} = \sum_j m_j (k^{*})^4\, Z_j^{*}\, \delta(\, x-l_j) ,$$
$$(\gamma^{*})^2\, Z_j^{*} + (\omega_{0j}^2 + 2\,\omega_{0j}\zeta_j\, \gamma^{*})\, Z_j^{*} = (\omega_{0j}^2 + 2\,\omega_{0j}\zeta_j\, \gamma^{*})\, X^{*}(l_j) . \tag{3.5}$$

The equations (3.4)and (3.5) are identical to the original equations after substituting the quantities γ^* and $[X^*(x),\ Z_1^*...Z_n^*]$ for the conjugate quantities γ and $[X(x),\ Z_1...Z_n]$. So it is evident that if γ and $[X(x),\ Z_1...Z_n]$ is the solution of the boudary problem then γ^* and $[X^*(x),\ Z_1^*...Z_n^*]$ is also the solution of that problem. The sequence $\{\gamma_n\}$ contains conjugate terms. The modes of vibration corresponding to them are also conjugate.

Orthogonality is of great importance in the problem of free vibration. Orthogonality relations can be derivated for the complex modes of vibration already introduced. The idea of introducing the orthogonal modes of vibrations into damped discrete systems was given by F.Tse, I.Morse and R.Hinkle (1978). The derivation presented below combines this idea and the method of derivation of orthogonal modes in conservative discrete-continuous systems.

Let $[U_i(x),\ V_{1i}...V_{ni}]$ be the velocity mode during the vibration of the system with the i-th mode of vibration. The following relations are valid

$$\frac{\delta w}{\delta t} = U_i e^{\gamma_i t}, \qquad \frac{d z_j}{d t} = V_{ji} e^{\gamma_i t} . \tag{3.6}$$

It is evident that the following relation holds true linking the velocity mode and the displacement mode

$$U_i = \gamma_i X_i, \qquad V_{ji} = \gamma_i Z_{ji} . \tag{3.7}$$

The equations (2.1) after applying the equations (3.6) and (3.1) for the i-th mode have the form:

$$\begin{aligned} &\gamma_i U_i + 1/\pi^4 \frac{d^4 X_i}{d x^4} + \gamma_i \delta \int_0^1 X_i \delta_b^T dx - \gamma_i \delta_b Z_i + \delta \int_0^1 X_i \delta_c^T dx - \delta_c Z_i = 0 , \\ &\gamma_i M V_i - \gamma_i \int_0^1 X_i \delta_b^T dx + \gamma_i B Z_i - \int_0^1 X_i \delta_c^T dx + C Z_i = 0 , \end{aligned} \tag{3.8}$$

where the following denotations are used:

$$Z_i = \left[Z_{1i}, \ \dots\ , \dot{Z}_{ni} \right]^T , \quad V_i = \left[V_{1i}, \ \dots\ , \dot{V}_{ni} \right]^T . \tag{3.9}$$

$$\delta = \left[\delta(x-l_1),\ \dots\ ,\ \delta(x-l_n)\right] , \tag{3.10}$$

$$\delta_b = \left[2m_1\omega_{01}\zeta_1\delta(x-l_1),\ \dots\ ,\ 2m_n\omega_{0n}\zeta_n\delta(x-l_n)\right] , \tag{3.11}$$

$$\delta_c = \left[m_1\omega_{01}^2\delta(x-l_1),\ \dots\ ,\ m_n\omega_{0n}^2\delta(x-l_n)\right] , \tag{3.12}$$

$$M = \begin{bmatrix} m_1 & . & 0 \\ . & . & . \\ 0 & . & m_n \end{bmatrix} , \tag{3.13}$$

$$B = \begin{bmatrix} 2m_1\omega_{01}\zeta_1 & . & 0 \\ . & . & . \\ 0 & . & 2m_n\omega_{0n}\zeta_n \end{bmatrix} , \tag{3.14}$$

$$C = \begin{bmatrix} m_1\omega_{01}^2 & . & 0 \\ . & . & . \\ 0 & . & m_n\omega_{0n}^2 \end{bmatrix} , \tag{3.15}$$

By putting together the equations (3.7) and (3.8) and after rearranging them we arrive at the following matrix equation

$$\begin{gathered}
\gamma_i \begin{bmatrix} 0 & 1 \\ 1 & \delta\int_0^1(.)\delta_b^T dx \end{bmatrix} \begin{bmatrix} U_i \\ X_i \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 1/\pi^4 \dfrac{d^4(.)}{d x^4} + \delta\int_0^1(.)\delta_c^T dx \end{bmatrix} \begin{bmatrix} U_i \\ X_i \end{bmatrix} + \\
- \gamma_i \begin{bmatrix} 0 & 0 \\ 0 & \delta_b \end{bmatrix} \begin{bmatrix} V_i \\ X_i \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & \delta_c \end{bmatrix} \begin{bmatrix} V_i \\ X_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} , \\
\gamma_i \begin{bmatrix} 0 & M \\ M & B \end{bmatrix} \begin{bmatrix} V_i \\ Z_i \end{bmatrix} + \begin{bmatrix} -M & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} V_i \\ Z_i \end{bmatrix} - \gamma_i \begin{bmatrix} 0 & 0 \\ 0 & \int_0^1(.)\delta_b^T dx \end{bmatrix} \begin{bmatrix} U_i \\ X_i \end{bmatrix} + \\
- \begin{bmatrix} 0 & 0 \\ 0 & \int_0^1(.)\delta_c^T dx \end{bmatrix} \begin{bmatrix} U_i \\ X_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} .
\end{gathered} \tag{3.16}$$

Multiplying the first equation by the vector $[U_k, X_k]$ integrating in the interval <0;1> and multiplying the second equation by $[V_k^T, Z_k^T]$ and then adding such rearranged equations we obtain

$$\gamma_i\Big[\int_0^1 U_k X_i dx + \int_0^1 X_k U_i dx + \int_0^1 X_k \delta\, dx \int_0^1 X_i \delta_b^T dx - \Big(\int_0^1 X_k \delta_b dx\Big) Z_i + V_k^T M Z_i +$$

$$+ Z_k^T M V_i + Z_k^T B Z_i - Z_k^T \int_0^1 X_i \delta_b^T dx\Big] - \int_0^1 U_k U_i dx + 1/\pi^4 \int_0^1 X_k \frac{d^4 X_i}{d x^4} dx +$$

$$+ \int_0^1 X_k \delta\, dx \int_0^1 X_i \delta_c^T dx - \Big(\int_0^1 X_k \delta_c dx\Big) Z_i - V_k^T M V_i + Z_k^T C Z_i +$$

$$- Z_k^T \int_0^1 X_i \delta_c^T dx = 0 \;. \tag{3.17}$$

Similarly we can write:

$$\gamma_k\Big[\int_0^1 U_i X_k dx + \int_0^1 X_i U_k dx + \int_0^1 X_i \delta\, dx \int_0^1 X_k \delta_b^T dx - \Big(\int_0^1 X_i \delta_b dx\Big) Z_k + V_i^T M Z_k + Z_i^T M V_k +$$

$$+ Z_i^T B Z_k - Z_i^T \int_0^1 X_k \delta_b^T dx\Big] - \int_0^1 U_i U_k dx + 1/\pi^4 \int_0^1 X_i \frac{d^4 X_k}{d x^4} dx + \int_0^1 X_i \delta\, dx \int_0^1 X_k \delta_c^T dx +$$

$$- \Big(\int_0^1 X_i \delta_c dx\Big) Z_k - V_i^T M V_k + Z_i^T C Z_k - Z_i^T \int_0^1 X_k \delta_c^T dx = 0 \;. \tag{3.18}$$

Subtracting the equation (3.18) from (3.17) and after substituting the boundary conditions we get

$$(\gamma_i - \gamma_k)\Big[\int_0^1 U_k X_i dx + \int_0^1 X_k U_i dx + \int_0^1 X_k \delta\, dx \int_0^1 X_i \delta_b^T dx - \Big(\int_0^1 X_k \delta_b dx\Big) Z_i +$$

$$+ V_k^T M Z_i + Z_k^T M V_i + Z_k^T B Z_i - Z_k^T \int_0^1 X_i \delta_b^T dx\Big] = 0 \;. \tag{3.19}$$

So if $i \neq k$ then $\gamma \neq \gamma$ and as a result

$$\int_0^1 U_k X_i dx + \int_0^1 X_k U_i dx + \int_0^1 X_k \delta\, dx \int_0^1 X_i \delta_b^T dx - \Big(\int_0^1 X_k \delta_b dx\Big) Z_i +$$

$$+ V_k^T M Z_i + Z_k^T M V_i + Z_k^T B Z_i - Z_k^T \int_0^1 X_i \delta_b^T dx = 0 \;, \tag{3.20}$$

or after rearranging

$$\int_0^1 (U_k X_i + X_k U_i)dx + \sum_{j=1}^{n} \Big[m_i (V_{jk} Z_{ji} + Z_{jk} V_{ji}) +$$

$$+ 2m_j \omega_{0j} \zeta_j (X_k(l_j) - Z_{jk}) (X_i(l_j) - Z_{ji}) \Big] = 0 \tag{3.21}$$

It is the orthogonality relation we have been seeking .

The method described above was applied to calculate the frequencies and damping of the modes of vibrations of a beam with two oscillators. It was assumed that:

$m_1 = m_2 = m = 0.2$, $\zeta_1 = \zeta_2 = \zeta$,

$\omega_{01} = \omega_{02} = \omega_0 = 0.5$, $l_1 = 0.4$, $l_2 = 0.6$.

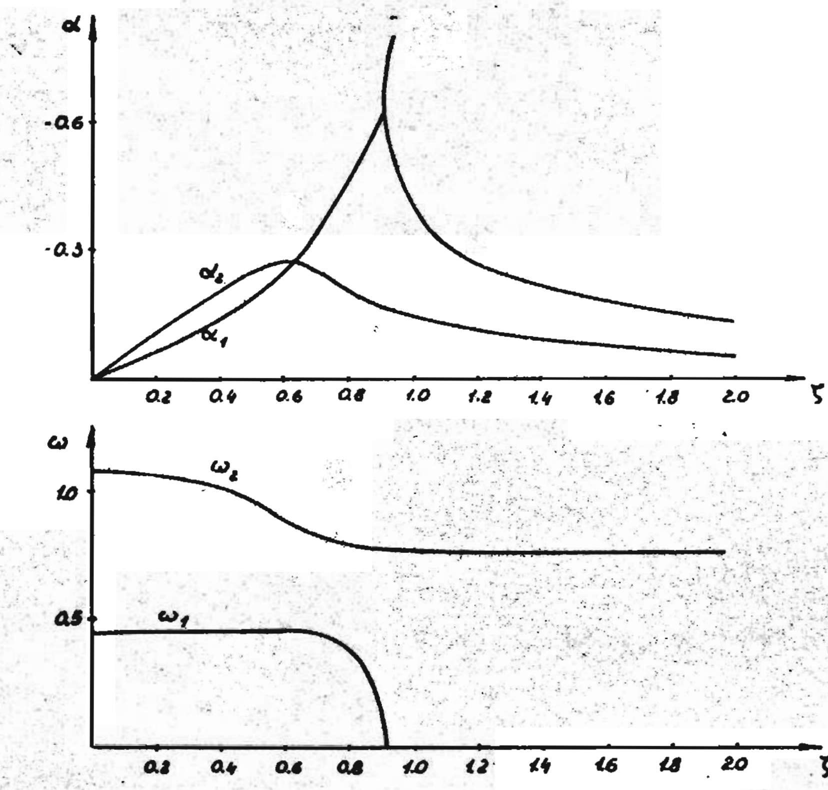

Fig.2. Damping α and frequency ω of the first two symmetricalmodes.

Considering the symmetry of the system we get two classes of modes: sym metric and antisymmetric. The results are shown in Figure 2.

4. Free vibration

Making use of the relations given earlier which defined the mode of displacement (3.1) and the mode of velocity (3.6) we get the solution in the form

$$\begin{bmatrix} \frac{dw}{dt} \\ w \end{bmatrix} = \sum_i C_i \begin{bmatrix} U_i(x) \\ X_i(x) \end{bmatrix} e^{\gamma_i t} , \tag{4.1}$$

$$\begin{bmatrix} \frac{dz_j}{dt} \\ z_j \end{bmatrix} = \sum_i C_i \begin{bmatrix} V_{ji} \\ Z_{ji} \end{bmatrix} e^{\gamma_i t} . \tag{4.2}$$

For t=0 substituting the initial conditions (2.3) in the equations (4.1) and (4.2) we get

$$\begin{bmatrix} \varphi(x) \\ f(x) \end{bmatrix} = \sum_i C_i \begin{bmatrix} U_i(x) \\ X_i(x) \end{bmatrix} , \tag{4.3}$$

$$\begin{bmatrix} v_0 \\ z_0 \end{bmatrix} = \sum_i C_i \begin{bmatrix} V_i \\ Z_i \end{bmatrix} , \tag{4.4}$$

where

$$z_0 = \begin{bmatrix} z_{01} \\ \cdot \\ z_{0n} \end{bmatrix} , \tag{4.5}$$

$$v_0 = \begin{bmatrix} v_{01} \\ \cdot \\ v_{0n} \end{bmatrix} . \tag{4.6}$$

We multiply the equation (4.3) by $[U_k \ X_k] \begin{bmatrix} 0 & 1 \\ 1 & \delta \int_0^1 (.) \delta_b^T dx \end{bmatrix}$ integrate the result in the interval <0;1>, multiply the equation (4.4) by $[V_k^T \ Z_k^T] \begin{bmatrix} 0 & M \\ M & B \end{bmatrix}$ and add the resulting formulae obtaining

$$\int_0^1 [U_k\ X_k] \begin{bmatrix} 0 & 1 \\ 1 & \delta\int_0^1 (.)\delta_b^T dx \end{bmatrix} \begin{bmatrix} \varphi \\ f \end{bmatrix} dx + [V_k^T Z_k^T] \begin{bmatrix} 0 & M \\ M & B \end{bmatrix} \begin{bmatrix} v_0 \\ z_0 \end{bmatrix} =$$

$$= \sum_i C_i \left\{ \int_0^1 [U_k\ X_k] \begin{bmatrix} 0 & 1 \\ 1 & \delta\int_0^1 (.)\delta_b^T dx \end{bmatrix} \begin{bmatrix} U_i \\ X_i \end{bmatrix} dx + [V_k^T Z_k^T] \begin{bmatrix} 0 & M \\ M & B \end{bmatrix} \begin{bmatrix} V_i \\ Z_i \end{bmatrix} \right\}. \tag{4.7}$$

Similarly we multiply the equation (4.3) by $[V_k^T\ Z_k^T] \begin{bmatrix} 0 & 0 \\ 0 & \int_0^1 (.)\delta_b^T dx \end{bmatrix}$ and also multiply the equation (4.4) by $[U_k\ X_k] \begin{bmatrix} 0 & 0 \\ 0 & \delta_b \end{bmatrix}$, integrate the last result in the interval <0; 1>and add the resulting formulae we obtaining

$$[V_k^T\ Z_k^T] \begin{bmatrix} 0 & 0 \\ 0 & \int_0^1 (.)\delta_b^T dx \end{bmatrix} \begin{bmatrix} \varphi \\ f \end{bmatrix} + \int_0^1 [U_k\ X_k] \begin{bmatrix} 0 & 0 \\ 0 & \delta_b \end{bmatrix} \begin{bmatrix} v_0 \\ z_0 \end{bmatrix} dx =$$

$$= \sum_i C_i \left\{ \int_0^1 [U_k\ X_k] \begin{bmatrix} 0 & 0 \\ 0 & \delta_b \end{bmatrix} \begin{bmatrix} V_i \\ Z_i \end{bmatrix} dx + [V_k^T Z_k^T] \begin{bmatrix} 0 & 0 \\ 0 & \int_0^1 (.)\delta_b^T dx \end{bmatrix} \begin{bmatrix} U_i \\ X_i \end{bmatrix} \right\}. \tag{4.8}$$

Subtracting equations (4.8) from (4.7) after substituting the orthogonality condition (3.20) and rearranging we get the formula for the constants C_k.

$$C_k = \frac{\int_0^1 (U_k\varphi + X_k f)dx + \sum_{j=1}^{n} [m_j(V_{jk}z_{0j} + Z_{jk}v_{0j}) + }{2\int_0^1 U_k X_k dx + 2\sum_{j=1}^{n} [m_j V_{jk} Z_{jk} +} \; \frac{+ 2m_j\omega_{0j}\zeta_j(Z_{jk} - X_k(l_j))(z_{0j} - f(l_j))]}{+ m_j\omega_{0j}\zeta_j(X_k(l_j) - Z_{jk})^2\ ]} . \tag{4.9}$$

Substituting the constants C_k in equations (4.1) we get the solution of the initial-boundary value problem and the exact description of free vibration of the system. Since the sequence of eigenvalues contains the conjugate values

$$\gamma_k = \alpha_k + i\omega_k , \tag{4.10}$$

$$\gamma_{k'} = \gamma_k^* = \alpha_k - i\omega_k , \tag{4.11}$$

and the corresponding modes of vibration $[X_k(x),\ Z_{1k} \ldots Z_{nk}]$ and $[X_{k'}(x),\ Z_{1k'} \ldots Z_{nk'}]$ are also conjugate, it can be seen from the form of equation (4.9) that $C_{k'}$ is conjugate to C_k . So we canwrite

$$w(x,t) = \sum_k |C_k||X_k(x)|\, e^{\alpha_k t} \cos(\varphi_k + \theta_k(x) + \omega_k t) , \tag{4.12}$$

$$z_j(t) = \sum_k |C_k||Z_{jk}|\, e^{\alpha_k t} \cos(\varphi_k + \Psi_{jk} + \omega_k t) , \tag{4.13}$$

where

$$\varphi_k = \arg C_k , \quad \theta_k = \arg X_k(x) , \quad \Psi_{jk} = \arg Z_{jk}. \tag{4.14}$$

From the form of equations (4.12) and (4.13) it is seen that:

- the damping of vibration at a particular modes of vibration depends on 'α_k'
- the vibration of two different points of the beam 'x_1' and 'x_2' with an arbitrarily chosen mode 'k' is out of phase.

5. Conclusions

From the calculations carried out in this paper it is seen that the problem of free vibration of a discrete-continuous system with arbitrary initial conditions can be solved using properly introduced complex modes of vibration. Superposition of these modes enables one to obtain the solution of the stated problem. Orthogonality of the complex modes of vibration plays a crucial role in the calculations. The calculation of the complex modes is connected with solving a complicated non-linearalgebraic equation in the complex domain from which a sequence of complex frequencies of free vibrations is obtained.

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Summary

DRGANIA WŁASNE UKŁADU DYSKRETNO-CIĄGŁEGO Z TŁUMIENIEM

W pracy przedstawiono metodę rozwiązania problemu drgań własnych układu dyskretno-ciągłego z tłumieniem. Metoda sprowadza się do analizy drgań w zbiorze funkcji zespolonych zmiennych rzeczywistych. Zespolone formy drgań z wprowadzonym warunkiem ortogonalności są podstawą opisu drgań własnych przy dowolnych warunkach początkowych.