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SOME BASIC PROBLEMS OF THE FINITE ELEMENT ANALYSIS OF ELASTOPLASTIC STRUCTURES (A SURVEY)

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1. General remarks

Plastic properties of materials are associated with permanent (plastic) deformations, i.e. such deformations which do not disappear in spite of vanishing of loading (exertion) factors which initiated the deformation process. Plastic deformations are analysed in the frames of the *theory of plasticity*that deals with idealized models of materials which have twomain features:

(1) The deformation process is *irreversible, history- dependent*, associated with plastic strains and dissipation of energy;

(2) The deformation process is *time-independent* and *rate- insensitive*. The first feature distinguishes the theory of plasticity from the theory of elasticity and in the case if the latter one does not appear the material (deformation process) is called viscoplastic. If the deformation process turns out to be partially reversible the material is *elastoplastic*.

We have started with the basic definitions which are at the beginning of a distinguish monograph by ŻYCZKOWSKI (1981) where the theory of plasticity is originally presented. We mention here only that the classical theory of plasticityrefers to the *phenomenological formulation* on the basis of continuum mechanics. Such a theory was realy developed in 40-50-ies and it was explored in various approaches and approximate methods, codes and instructions for design of engineering structures. Despite of that the elastic-plastic analysis turned out to be difficult and limited to simple problems and uncomplicated structures or only to their elements.

Appearance of computers and development of numerical methods opened the door to wider analysis of problems of the theory of plasticity and its applications. It is evident that the Finite Element Method (FEM) was used guite early. The first paper in this field was published byGALLAGHER et al.asearly as 1962. Since the end of 60-ies big computer codes and systems have been implemented in the field of plastic analysis,cf.ARMEN and PIFKO (1982), and have been successfully used to the analysis of various aerospace and naval structures, in reactor technology, in civil and mechanical engineering as well. The progress in this field is reported on various symposia, seminars and conferences. From among a great number of conference proceedings it is worth to turn attention to the latest ones, devoted to the computational plasticity, Eds. OWEN et al. (1987,1989), and also to the theoretical background, Eds. SAWCZUK and BIANCHI (1985), Eds. KHAN and TOKUDA (1989).

A characteristic feature of computational methods for the analysis of problems, founded on the theory of plasticity, is a wide utilization of numerical methods and the software which have been developed for the analysis of elastic and geometrically nonlinear problems. The needed modifications and supplements concern the constitutive eguations and consideration of the history-dependence of the plastic deformation process.

The comparison of precomputer and recent approaches points out a preference of elastic-plastic models over simpler rigid perfect plastic models and the common use of the incremental, plastic flow theory instead of the total strain deformation theory. It is also evident that the interest in the limit state analysis decreases in favour of the analysis of the full deformation process — starting from the first yielding up the limit state. Such a reorientation corresponds, of course, also to the common application at incremental techniques to the analysis of nonlinear problems.

The presented paper is based on the lecture-notes by the author which were printed in 1989 as a report of the Delft University of Technology. In the paper only selected problems of the elastoplastic analysis of structures are pointed out.

The main differences between the elastic and elastoplastic FE analyses are discussed. Levels of the analysis are defined. At the point level $\mathcal{P}$ the assumptions of the classical theory ot plasticity are assumed. Quadrature formulae are used to compute the generalized stresses on the cross-section level $\mathcal{S}$. The consistent approach, dependet the implicit integration scheme on the level $\mathcal{P}$, is especially efficient when it is associated with the Newton-Raphson method on the structural level $\mathcal{B}$. Comparison of varions methods are made on an example of the perforated tension strip to confirm the above conclusion.

2. Levels of the analysis

The distinction of the analysis levels, introduced by ŻYCZKOWSKI (1981), enables us to discuss precisely various problems under consideration. Similarly as in the mentioned book by ŻYCZKOWSKI we introduce the following levels:

Point level $\mathcal{P}$ is the basic level, related to any or selected points of material continuum or to a model of structure. Tensorial notation and calculus are preferred on the level $\mathcal{P}$ in order to describe objects and their relations in the spaces, well known from the continuum mechanics.

Cross-section level $\mathcal{S}$ corresponds to such structures as bars, plates and shells in which one dimension , e.g. thickness, is much smaller than other dimensions. *Generalized variables,* e.g integral quantities, are used on this level and they are related to each other through energy or work functionals (generalized displacements versus g.loads, g.strains versus g.stresses). On the level $\mathcal{S}$ both the tensors (e.g. in shell equations) and matrices are used.

Element level $\mathcal{E}$ is introduced for a separate part of the structure (members, substructures) or for an individual finite element. In order to analyse different fields approximated functions used to be applied (e.g.

shape or basic functions). The matrices are commonly used on the level $\mathcal{E}$.

Body (structure) level $\mathcal{B}$ is also called global level contrary to local, lower levels $\mathcal{P}$, $\mathcal{S}$, $\mathcal{E}$. On the level $\mathcal{B}$ algebraic relations are preferred because of the use of computers, numerical methods of algebra and matrix calculus.

Methods of the analysis have to correspond to characteristic features of the levels. In plasticity the level $\mathcal{P}$ is especially difficult for analysis because of nonlinearity and time-type dependence of relations. That is why the transition from one to another level is not straightforward, especially with the transformations $\mathcal{P} \rightleftarrows \mathcal{S}$. In general the analysis of elastic-plastic problems needs more operations and additional computer memory than elastic analysis.

In order to describe more precisely the deformation process the definitions of active and passive processes are introduced against loading and unloading. The *active process* is related to the increase of plastic strains on the level $\mathcal{P}$ or to the development of yielding zones on the level $\mathcal{S}$. From the viewpoint of such a definition the *passive process* is related to the lack of increment of plastic strains or to a fixed zone of yielding as well as to the elastic behaviour of material. As a counterpart to the active and passive processes *loading* and *unloading* can be considered, associated with the increase or decrease of a load-type parameter. It is quite possible that for a loading of structures the passive processes can take place on the levels $\mathcal{P}$ and vice versa.

3. Incremental equations

3.1. Constitutive relations on the level $\mathcal{P}$. The strains are assumed to be small so their increments can be split into the elastic and plastic parts:

$$d\underline{\varepsilon} = d\underline{\varepsilon}^{e} + d\underline{\varepsilon}^{p}, \tag{3.1}$$

and instead of increments $d\underline{\varepsilon}$ the rates $\dot{\underline{\varepsilon}}$ can be used, calculated with respect to a conventional time of plasticity τ (any, but monotonically increasing parameter of the problem under analysis):

$$\dot{\underline{\varepsilon}} = \frac{d\underline{\varepsilon}}{d\tau} = \dot{\underline{\varepsilon}}^{e} + \dot{\underline{\varepsilon}}^{p} \ . \tag{3.2}$$

In the above formulae and in what follows the one-column matrices (vectors) are used instead of appropriate tensors, e.g. the strain and stress vectors $\underline{\varepsilon}$, $\underline{\sigma}$ are introduced.

A subsequent yield surface is defined by the following equation:

$$F\,(\underline{\sigma},\ \underline{\alpha},\ k) = 0 \ , \tag{3.3}$$

which also contains the *initial yield surface*

$$F_0 \equiv F\,(\underline{\sigma},\ \underline{0},\ k_0) = 0 \ . \tag{3.4}$$

The *associated flow rule* and the *hardening rule* are postulated according to the following relations:

$$\dot{\underline{\varepsilon}}^{p} = \frac{\partial F}{\partial \underline{\sigma}}\,\dot{\lambda} \equiv \underline{n}_F\dot{\lambda} \ , \tag{3.5}$$

$$\dot{\underline{\alpha}} = \underline{G}\,(\underline{\sigma},\ \underline{\alpha},\ \underline{\varepsilon}^{p},\ \dot{\underline{\sigma}},\ \dot{\underline{\varepsilon}}^{p}), \tag{3.6}$$

$$k = H(\varepsilon_p) \qquad \text{where} \qquad \dot{\varepsilon}_p = \left[\frac{2}{3}(\dot{\underline{\varepsilon}}^{p})^{T}\dot{\underline{\varepsilon}}^{p}\right]^{1/2} \ . \tag{3.7}$$

Using the above relations and satisfying the consistency condition, i.e. $\dot{F}=0$ for the active deformation process, the plastic parameter can be determined:

$$\dot{\lambda} = \frac{1}{g}\,\underline{n}_F^{T}\,\underline{E}^{e}\,\dot{\underline{\varepsilon}}, \tag{3.8}$$

where the hardening function g is in the form:

$$g = \underline{n}_F^{T}\,\underline{E}^{e}\,\underline{n}_F + h\ , \quad h = -\left[\frac{\partial \underline{\alpha}}{\partial \underline{\varepsilon}^{p}}\,\frac{\partial F}{\partial \underline{\alpha}} + \frac{\partial F}{\partial k}\,\frac{\partial k}{\partial \underline{\varepsilon}^{p}}\right]^{T}\underline{n}_F \ , \tag{3.9}$$

corresponding to a special case of the evolution law $\underline{\alpha} = \underline{f}(\underline{\varepsilon}^{p})$. Owing to the above relations the increment of stress vector $d\underline{\sigma}$ can be expresed as:

$$\dot{\underline{\sigma}} = (\underline{E}^{e} - \underline{E}^{P})\dot{\underline{\varepsilon}} = \underline{E}^{eP}\dot{\underline{\varepsilon}}, \tag{3.10}$$

where the *modular, plastic stiffness matrix* is:

$$E^{P} = \frac{\beta}{g}\,\underline{E}^{e}\,\underline{n}_{F}\,\underline{n}_{F}^{T}\underline{E}^{e}. \tag{3.11}$$

The zero-one parameter β depends on the type of the deformation process:

$$\begin{aligned} &\beta = 1 \quad \text{for} \quad F=0 \ \wedge \ \underline{n}_{F}^{T}\,\underline{E}^{e}\,\dot{\underline{\varepsilon}} \geq 0\,, && \text{active or neutral process,} \\ &\beta = 0 \quad \text{for} \quad \begin{cases} F=0 \\ F<0 \end{cases} \wedge \ \underline{n}_{F}^{T}\,\underline{E}^{e}\,\dot{\underline{\varepsilon}} < 0\,, && \text{passive processes.} \end{aligned} \tag{3.12}$$

In the case of passive processes the constitutive relation (3.10) is simplified to the form:

$$\dot{\underline{\sigma}} = \underline{E}^{e}\,\dot{\underline{\varepsilon}}\ , \tag{3.13}$$

where the symmetric, *modular elastic stiffness matrix* $\underline{E}^{e}$ isused.

3.2. Generalized variables on the level $\mathcal{S}$. Applying the appropriate hypotheses and definitions for integral quantities the following transition from the point level $\mathcal{P}$ to the cross-section level $\mathcal{S}$ can be deduced:

$$\mathcal{P}^{k}_{m} \longrightarrow \mathcal{S}^{l}_{n}. \tag{3.14}$$

Such a transition was considered in the book by ŻYCZKOWSKI (1981). Numbers m and n correspond there to the number of basic exertion factors on the $\mathcal{P}$ and $\mathcal{S}$ levels respectively. In comparison with ŻYCZKOWSKI additional members k and l define the number of independent displacements. The above numbers are used to define the size of vectors (one-column matrices) on the levels $\mathcal{P}$ and $\mathcal{S}$. All vectors are listed below;

the names of variables for $\mathcal{P}$ and *generalized variables* for $\mathcal{S}$ are the same:

Variables	Levels $\mathcal{P}$	$\mathcal{S}$
displacement vector	$\underline{u}_{(k\times1)}$	$\underline{d}_{(l\times1)}$
load vector	$\underline{f}_{(k\times1)}$	$\underline{p}_{(l\times1)}$
strain vector	$\underline{\varepsilon}_{(m\times1)}$	$\underline{e}_{(n\times1)}$
stress vector	$\underline{\sigma}_{(m\times1)}$	$\underline{s}_{(n\times1)}$
stiffness matrix	$\underline{E}_{(m\times m)}$	$\underline{D}_{(n\times n)}$

(3.15)

As an example we can consider a bar model under Bernoulli-Euler hypotheses which lead to the transition $\mathcal{P}^2_1 \rightarrow \mathcal{S}^2_2$ for the plane bending/tension and $\mathcal{P}^3_1 \rightarrow \mathcal{S}^4_4$ in the spatial state. In the case of Kirchhoff-Love theory of thin shells the transition is $\mathcal{P}^3_3 \rightarrow \mathcal{S}^3_6$, and $\mathcal{P}^3_5 \rightarrow \mathcal{S}^5_8$ for the Reissner - -Mindlin theory. Let us consider the Kirchhoff-Love theory of thin plates. In such a case the g. strain and stress vectors are:

$$\underset{(6\times1)}{\underline{e}} = \{ \varepsilon^0_1, \varepsilon^0_2, \gamma^0_{12}, \kappa_1, \kappa_2, \chi_{12} \} ,$$
$$\underset{(6\times1)}{\underline{s}} = \{ n_1, n_2, n_{12}, m_1, m_2, m_{12} \} . \qquad (3.16)$$

The g. stiffnes matrix can be written in the following form:

$$\underset{(6\times6)}{\underline{D}^{ep}} = \begin{bmatrix} \int_{-h/2}^{h/2} \underline{E}^{ep} dz & \int_{-h/2}^{h/2} \underline{E}^{ep}\, z dz \\ \int_{-h/2}^{h/2} \underline{E}^{ep} z dz & \int_{-h/2}^{h/2} \underline{E}^{ep} z^2 dz \end{bmatrix} . \qquad (3.17)$$

The modular stiffness matrix $\underline{E}^{ep}$ of size (3x3) depends both on constitutive equation that was used as well as on the type of deformation process. If the passive process takes place then the elastic plane strain matrix $\underline{E}^e$ should be substituted into (3.17) since $\beta=0$ in (3.11).

In general case of nonlinear stress-strain relation and local unloading the integrals over the cross section (along the plate thickness) can be computed numerically:

$$I = \int_{(A)} f(z)dA \approx \sum_{j=1}^{J} A_j \, f(z_j) \; . \tag{3.18}$$

The number of integration points J, their coordinates z_j and the weight parameters A_j influence the accuracy of the approximation (3.18).

Multilayer substitutive cross-sections are commonly used for computing the integrals (3.18) but also trapezoidal rule, Simpson, Gauss or Lobatto's formulae are used as well - cf. WASZCZYSZYN (1989), pp.77-79.

The application of quadrature formulae is time consuming since the analysis on the level $\mathcal{P}$ has to be carried out at every integration point j where appropriate information is stored and modified. That is why constitutive relations are attempted to be formulated directly on the level $\mathcal{S}$. Plastic interaction surfaces for g. stresses are used in such an approach, combined with the g. associated flow rule and other relations similar to those as given in par.3.1 at the level $\mathcal{P}$ - cf. WASZCZYSZYN (1989), pp.79-87.

The sketched approach seems to be very attractive one since the stress analysis at substitutive layers can be overcome - cf. CRISFIELD (1981), SIMO et al. (1989). But it should be emphasized that the integral (area) approach corresponds to the elastic, perfect plastic model of material and the strain -hardening or locally passive process analysis is practicallyimpossible.

The incremental relations on level the $\mathcal{S}$ are assumed in the following form:

$$\begin{aligned} \Delta\underline{e} &= \underline{L}_1(\Delta\underline{d}) + \underline{L}_{11}(\underline{d},\Delta\underline{d}) + \frac{1}{2}\underline{L}_2(\Delta\underline{d}) \; , \\ \Delta\underline{s} &= \underline{D}^{ep}\Delta\underline{e} \approx \underline{D}^{ep}[\underline{L}_1(\Delta\underline{d}) + \underline{L}_{11}(\underline{d},\Delta\underline{d})] \; , \end{aligned} \tag{3.19}$$

where $\underline{L}_1$ and $\underline{L}_2$ are linear and quadratic differential operators respectively and $\underline{L}_2(\Delta\underline{d}) = \underline{L}_{11}(\Delta\underline{d}, \Delta\underline{d})$. The constitutive equation $(3.19)_2$ is assumed to be linearized with respect to the g. displacement increment $\Delta\underline{d}$.

3.3. FE incremental equations. Let us assume the FE approximation of the g. displacement field $\Delta\underline{d}(\underline{\xi})$ on the finite element level $\mathcal{E}$:

$$\Delta\underline{d}^{(e)}(\underline{\xi}) = \underline{N}(\underline{\xi})\ \Delta\underline{q}^{(e)}, \tag{3.20}$$

where $\underline{\xi}$ is a vector of independent variables, $\underline{N}(\underline{\xi})$ is the matrix of shape functions and $\Delta\underline{q}^{(e)}$ is the vector of increments of nodal (generalized) displacements in the coordinate system of the finite element (e). The index (e) is omitted in the relation which results from (3.19) on the base of the approximation (3.20) :

$$\begin{aligned} \Delta\underline{e} &= [\underline{B}_0 + \underline{B}_1(\underline{d}) + \tfrac{1}{2}\,\underline{B}_2(\Delta\underline{q})]\ \Delta\underline{q}, \\ \Delta\underline{s} &= \underline{D}^{ep}\ [\underline{B}_0 + \underline{B}_1(\underline{d})]\ \Delta\underline{q} \end{aligned} \tag{3.21}$$

where $\underline{B}_0$ is the linear matrix, the matrices $\underline{B}_1$ and $\underline{B}_2$ depend linearly on the dispacements $\underline{d}$ and $\Delta\underline{q}$ respectively.

The above relations are used in the principle of virtual work :

$$\sum_{(e)} \int_{\Omega_{(e)}} (\delta\Delta\underline{e})^T(\underline{s} + \Delta\underline{s})\ d\Omega = \sum_{(e)} \int_{\Omega_{(e)}} (\delta\Delta\underline{d})^T(\underline{p} + \Delta\underline{p})\ d\Omega\ , \tag{3.22}$$

which is valid also for the displacement-dependent load :

$$\underline{p}(\underline{d}, \underline{\xi}) \approx \underline{p}_0(\underline{\xi}) + \frac{\partial\underline{p}}{\partial\underline{d}}\ \Delta\underline{d}\ . \tag{3.23}$$

The transformation into a global system of g. nodal displacements $\Delta\underline{q} \longrightarrow \underline{Q}$ and of the FE stiffness matrix $\underline{k}^{(e)} \longrightarrow \underline{K}^{(e)}$ is not considered in detail. Comparing the coefficients at the variation $\delta\Delta\underline{Q}$ the following FE equation, linearised with respect to the increments $\Delta\underline{Q}$, can be formulated :

$$\underline{K}_T\ \Delta\underline{Q} = \Delta\underline{P} + \underline{R}, \tag{3.24}$$

where $\underline{K}_T$ is the *tangent stiffness matrix* :

$$\underline{K}_T = \underline{K}_0 + \underline{K}_\sigma(\underline{s}) + \underline{K}_u(\underline{d}) + \underline{K}_p(\underline{p}), \tag{3.25}$$

and $\underline{R}$ is the *vector of residual forces* :

$$\underline{R} = \underline{P} - \underline{F}(\underline{s}) \ . \tag{3.26}$$

The stiffness matrix $\underline{K}_T$, vectors $\Delta\underline{P}$ and $\underline{R}$ are assembled of the following FE matrices and vectors :

$$\underline{k}_0^{(e)} = \int_{\Omega_{(e)}} \underline{B}_0^T \underline{D}^{ep} \underline{B}_0 \, d\Omega \quad - \text{ small displacement matrix,}$$

$$\underline{k}_\sigma^{(e)} = \int_{\Omega_{(e)}} \underline{B}_2^T \underline{\underline{SB}}_2 \, d\Omega \quad - \text{ initial stress matrix,}$$

$$\underline{k}_u^{(e)} = \int_{\Omega_{(e)}} (\underline{B}_0 + \underline{B}_1)^T \, \underline{D}^{ep} (\underline{B}_0 + \underline{B}_1) d\Omega \quad - \text{ initial displacement matrix}$$

$$\underline{k}_p^{(e)} = - \int_{\Omega_{(e)}} \underline{N}^T \frac{\partial \underline{p}}{\partial \underline{d}} \, \underline{N} \, d\Omega \quad - \text{ initial load matrix,} \tag{3.27}$$

$$\Delta\underline{p}^{(e)} = \int_{\Omega_{(e)}} \underline{N}^T \Delta \underline{p}_0 \, d\Omega \ , \quad \underline{p}^{(e)} = \int_{\Omega_{(e)}} \underline{N}^T \underline{p}_0 \, d\Omega \quad - \text{ incremental and total load vectors}$$

$$\underline{f}^{(e)} = \int_{\Omega_{(e)}} (\underline{B}_0 + \underline{B}_1)^T \underline{s} \, d\Omega \quad - \text{ internal force vector .}$$

The components of the matrices (3.27) can be computed by means of numerical integration over the FE domain $\Omega_{(e)}$. The integrands of the matrices depend on the cross-section stiffness $\underline{D}^{ep}$ or on the stress matrix $\underline{S}$ which are computed at every integration point by means of the quadrature formula (3.18). In such a way the transformation $\mathcal{P} \rightarrow \mathcal{S} \rightarrow \mathcal{E}$ is carried out.

The *initial load matrix* $\underline{k}_p^{(e)}$ reflects the dependence of external loads on displacements. In case of non-conservative loads the global matrix $\underline{K}_p$ depends also on boundary conditions and can be, in general, non-symmetric - cf.e.g. HIBBITT (1979), SCHWEIZERHOF and RAMM (1984).

4. Algorithms on the $\mathcal{B}$ and $\mathcal{P}$ levels

The *discrete continuation method* (step-by-step method), combined with an iteration procedure, is commonly applied to compute the displacement vector $\underline{Q}$. In Fig. 1 two possible iteration schemes are shown to pass from one equilibrium configuration ${}^{m}C$ to the other equilibrium state ${}^{m+1}C$. The intermediate configurations ${}^{i}C$, corresponding to the iteration steps i, are not in equilibrium, i.e. ${}^{i}\underline{R} \neq \underline{O}$.

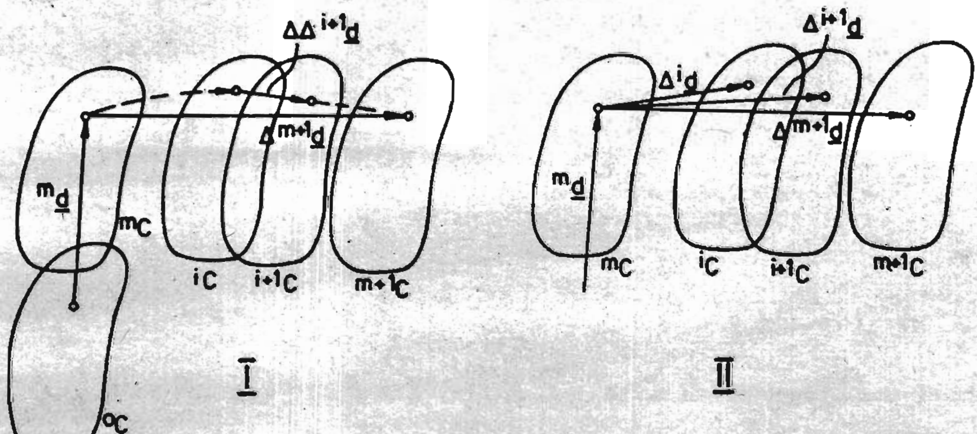

Fig. 1. Schemes for incremental procedures.

Scheme I is commonly applied to the elastic analysis and can be efficiently used also on the structure level $\mathcal{B}$ in the elastoplastic analysis. Scheme II should be preferred on the $\mathcal{P}$ level in case of elastic-plastic material in order to ensure partially the path-independence of the deformation process during the iteration process.

4.1. Extended set of equations and the Newton-Raphson method. Let us consider the *single parameter load*

$$\underline{P} = \lambda \underline{P}^{*} , \tag{4.1}$$

where λ is the load parameter and $\underline{P}^{*}$ is the load refference vector on the $\mathcal{B}$ level. In such a case and for the iteration scheme I the *incremental set of FE equations* (3.24) takes the form:

$${}^{i}\underline{K}_{T}\ \Delta\Delta^{i+1}\underline{Q} = \Delta\Delta^{i+1}\lambda\underline{P}^{*} + {}^{i}\underline{R} . \tag{4.2}$$

The displacement and load increments $\Delta\Delta^{i+1}\underline{Q}$ and $\Delta\Delta^{i+1}\lambda$ can be treated equivalently if Eqs (4.2) are completed by a *constrain equation*. From among various constrain equations - cf.e.g. WASZCZYSZYN (1983), SCHWEIZERHOF and WRIGGERS (1986), the RIKS-WEMPNER equation can be written in the following form:

$${}^{i}\underline{t}^{T}\ \Delta\Delta^{i+1}\underline{Q} + {}^{i}t_{\lambda}\ \Delta\Delta^{i+1}\lambda = {}^{i}\alpha\ \Delta^{m+1}\tau , \tag{4.3}$$

where $\{{}^{i}\underline{t},\ {}^{i}t_{\lambda}\}$ is the control vector and $\Delta^{n+1}\tau$ is the increment of the control parameter (time-type parameter). The zero - one parameter ${}^{i}\alpha$ equals 1 for i=1 and 0 for i>1. It corresponds to the predictor-corrector iteration procedure.

Eqs (4.2) and (4.3) can be written in the form of *extended set of equations*:

$${}^{i}\underline{\tilde{K}}\ \Delta\Delta^{i}\underline{\tilde{Q}} = {}^{i}\underline{\tilde{R}} , \tag{4.4}$$

where the structure of extended matrix ${}^{i}\underline{\tilde{K}}$ and the vectors $\Delta\Delta^{i}\underline{\tilde{Q}}$ and $\underline{\tilde{R}}$ are shown in Fig.2.

Specification of the control vector ${}^{i}\underline{t}$ enables us to continue the computational process in the load-displacement space under load, a selected displacement or path-parameter control. Properties of the matrix ${}^{i}\underline{\tilde{K}}$ and the Newton-Raphson method, applied to the analysis of Eq. (4.4) were discussed by WASZCZYSZYN (1983), WASZCZYSZYN and CICHOŃ (1987).

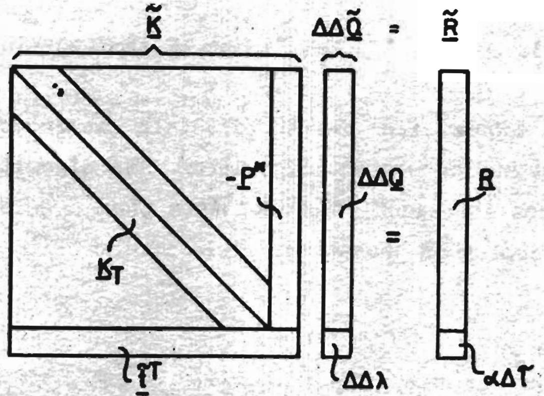

Fig.2. Structure of matrices in Eq.(4.4).

The residual force vector depends on the g. stress field ${}^1\underline{s}(\underline{\xi})$:

$$ {}^1\underline{R} = \underline{R}({}^1\underline{s}) \ . \tag{4.5}$$

In case of the elastoplastic analysis the computation of the vector ${}^1\underline{R}$ requires the coming-back to the finite elements and to substitutive layers of the cross-section at the integration points in order to compute the g. stresses there.

The tangent stiffness matrix ${}^1\underline{K}_T$ can be modified in a similar way at every iteration step if the classical Newton-Raphson method is explored. Other methods, like the modified Newton - Raphson or quasi - Newtonian methods (BFGS, DFP, Broyden's, Davidon's) can also be used — cf. WASZCZYSZYN and CICHOŃ (1987)).

4.2. Computation of the stress vector and consistent modular matrix on the level *P*. In the elastoplastic analysis the finite increment of the stress vector has to be computed

$$\Delta\underline{\sigma} = \int_{\tau}^{\tau+\Delta\tau} \dot{\underline{\sigma}}\, d\tau = \int_{\underline{\sigma}}^{\underline{\sigma}+\Delta\underline{\sigma}} d\underline{\sigma} = \int_{\underline{\varepsilon}}^{\underline{\varepsilon}+\Delta\underline{\varepsilon}} \underline{E}^{ep} d\underline{\varepsilon} = \underline{E}^{ep}(\tau+\alpha\Delta\tau)\Delta\underline{\varepsilon} \ . \tag{4.6}$$

In case of the explicit scheme $\alpha=0$ and after the yield surface is crossed a deviation from this surface can occur (cf. Fig. 3a). In order to minimize errors of such a scheme the *subincremental technique* was developed - cf. OWEN and HINTON (1980), pp.253-257. Main ideas of the technique are shown in Fig.3a.

In recent years the *implicit scheme*, i.e. $\alpha=1$ in (4.6), is rather used since it leads to more efficient and consistent algorithms. The algorithm for the computation of the stress increment $\Delta^i\underline{\sigma}$ consists then of elastic prediction and orthogonal mapping on the actual yield surface ^iF -

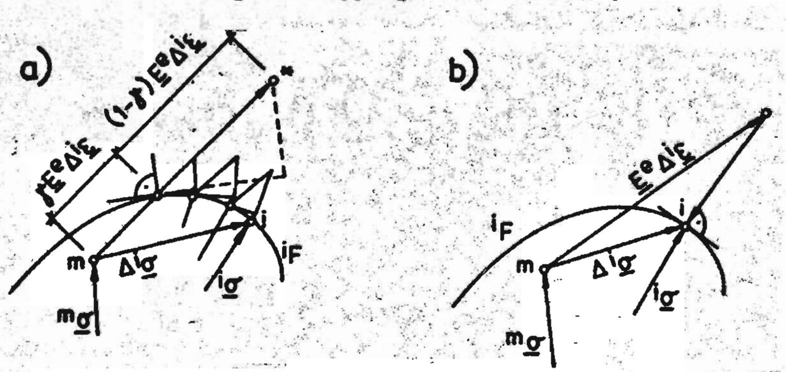

Fig.3. Subincremental and implicit scheme techniques.

cf.Fig.3b. The consistent modular matrix is obtained during the computational process.

In order to illustrate the above algorithm the HUBER-MISES-HENCKY yield function with isotropic strain hardening is assumed:

$$F = \frac{1}{2}\,\underline{\sigma}^T\underline{A}\,\underline{\sigma} - k^2(\varepsilon_p) = 0\,, \tag{4.7}$$

where $\underline{A}$ is a numerical matrix associated with deviatoric stresses.

The finite increment of strain vector is related to the equilibrium state m :

$$\Delta^i\underline{\varepsilon} = {}^i\underline{\varepsilon} - {}^m\underline{\varepsilon} = \Delta^i\underline{\varepsilon}^e + \Delta^i\underline{\varepsilon}^p\,. \tag{4.8}$$

The *elastic prediction*

$${}^i\underline{\sigma}^* = {}^m\underline{\sigma} + \underline{E}^e\Delta^i\underline{\varepsilon}\,, \tag{4.9}$$

enables us to compute the following relation:

$$ {}^{i}\underline{\sigma} = {}^{m}\underline{\sigma} + \underline{E}^{e}\Delta^{i}\underline{\varepsilon}^{e} = {}^{i}\underline{\sigma}^{*} - \underline{E}^{e}\Delta^{i}\lambda\, \underline{\Delta}\, {}^{i}\underline{\sigma}\,, \qquad (4.10) $$

where $\Delta^{i}\underline{\varepsilon}^{p} = \Delta^{i}\lambda\ \underline{\Delta}\ {}^{i}\underline{\sigma}$ has been used. Eq. (4.10) can be solved with respect to ${}^{i}\underline{\sigma}$:

$$ {}^{i}\underline{\sigma} = (\underline{I} + \Delta^{i}\lambda\ \underline{E}\ \underline{\Delta})^{-1}\ {}^{i}\underline{\sigma}^{*}\ . \qquad (4.11) $$

Using the equalit y ${}^{i}\underline{\sigma}^{T}\Delta^{i}\underline{\varepsilon}^{p} = \sqrt{3}\ {}^{i}k\cdot({}^{i}\varepsilon_{p} - {}^{m}\varepsilon_{p})$ the folowing formula for the effective plastic strain can be derived:

$$ {}^{i}\varepsilon_{p} = {}^{m}\varepsilon_{p} + \left[\frac{2}{3}\ {}^{i}\underline{\sigma}^{T}\underline{\Delta}\ {}^{i}\underline{\sigma}\right]^{1/2}\Delta^{i}\lambda\ . \qquad (4.12) $$

After substitution of (4.11) and (4.12) into (4.7) the nonlinear equation

$$ F(\Delta^{i}\lambda) = 0\ , \qquad (4.13) $$

determines the increment of plastic parameter $\Delta^{i}\lambda$. Eq. (4.13) can be solved by means of various numerical methods, e.g. RAMM and MATZENMILLER (1987) used the Newton method, PABISEK and WASZCZYSZYN (1989) combined bisection and 'regula falsi' methods.

After the value of $\Delta^{i}\lambda$ is known the stress ${}^{i}\sigma$ can be computed from (4.11) and performing the transition $\mathcal{P}\to\mathcal{G}\to\mathcal{E}$ the FE internal force vector $\underline{f}^{(e)}$ is computed according to $(3.27)_6$. The assembling process gives the vector of residual forces ${}^{i}\underline{R}$ defined by (4.5) and (3.26).

The differentiation of Eq. (4.8) with respect to the time-type parameter τ gives the equation which can be solved with respect to ${}^{i}\dot{\sigma}$:

$$ {}^{i}\dot{\underline{\sigma}} = {}^{i}\hat{\underline{E}}\ {}^{i}(\dot{\underline{\varepsilon}} - (\Delta\lambda)^{\cdot}\,\underline{\Delta}\ \underline{\sigma}\ )\ , \qquad (4.14) $$

where the following equivalent elastic matrix is

$$ {}^{i}\hat{\underline{E}} = [(\underline{E}^{e})^{-1} + \Delta^{i}\lambda\ \underline{\Delta}\ ]^{-1}\ . \qquad (4.15) $$

From the consistency condition ${}^{i}\dot{F} = 0$ the formula for $(\Delta\lambda)^{\cdot}$, similar to (3.8), can be obtained (the superscript iis omitted in what follows):

$$(\Delta\lambda)^{\cdot} = \frac{1}{\hat{g}} \underline{n}_F^{\,T} \hat{\underline{E}} \dot{\underline{\varepsilon}} , \tag{4.16}$$

where the following quantities are used :

$$\hat{g} = \underline{n}_F^{\,T} \hat{\underline{E}} \underline{n}_F + \hat{h} , \quad \underline{n}_F = \underline{A} \underline{\sigma} , \quad \hat{h} = \frac{4k^2 H'}{3-2H'\Delta\lambda} , \tag{4.17}$$

and $H'(\varepsilon_p) = \sqrt{3}\, dk/d\varepsilon_p$. Coming back to (4.14) the constitutive relation becomes

$$\dot{\underline{\sigma}} = (\hat{\underline{E}} - \hat{\underline{E}}^P)\dot{\underline{\varepsilon}} = \hat{\underline{E}}^{ep} \dot{\underline{\varepsilon}} , \tag{4.18}$$

where the *consistent modular matrix* is :

$$\hat{\underline{E}}^{ep} = \hat{\underline{E}} - \frac{1}{\hat{g}} \hat{\underline{E}} \underline{n}_F \underline{n}_F^{\,T} \hat{\underline{E}} . \tag{4.19}$$

This matrix is similar to the classical modular matrix $\underline{E}^{ep}$ in (3.10) which results from the explicit scheme in the relation (4.6).

The consistent matrix $\hat{\underline{E}}^{ep}$ is valid for the active processes, defined by the conditions $(3.12)_1$. In case of passive processes $\underline{E}^e$ is used in (4.18) instead of $\hat{\underline{E}}^{ep}$. In the consistent approach the matrix $\hat{\underline{E}}^{ep}$ is substituted in relations of the type (3.17) and using the numerical integration the consistent, cross-sectional modular matrix $\hat{\underline{D}}^{ep}$ can be computed.

The above formulation has been based on the paper by RAMM and MATZENMILLER (1987). Other yield functions were considered by MITCHELL and OWEN (1988).

The coupling of the iteration scheme I on the level $\mathcal{B}$ (Fig.1a) and the implicit scheme of integration of constitutive relation on the level $\mathcal{P}$ (Fig.3b) is called the *consistent approach*.

Such an approach was originated by R.D.KRIEG and D.B.KRIEG (1977) but in fact it was well formulated by SIMO and TAYLOR (1985). During the recent four years this approach has been introduced to majority of computer codes - cf. Proceedings of the COMPLAS-II Conference, Eds. OWEN et al. (1989).

5. Numerical example

Many examples have been devoted to the comparison of efficiency of the consistent Newton-Raphson (NR) method with the classical NR and other methods in which the subincremental technique on the level $\mathcal{P}$ has been used, e.g. papers by RAMM and MATZENMILLER (1987), MITCHELL and OWEN (1988).One of suchexamples, the perforated tension strip is shown in Fig. 4a.

One quarter of the strip has been analysed by PABISEK and WASZCZYSZYN (1989) using 28 isoparametric, 8-node quadrilateral finite elements and 4 Gauss integration points in each. The boundary of yielded zone is shown in Fig.4b for subsequent load parameters $\lambda = 0.6,\dots,1.1$. The converqence criterion has been related to the norm of residual force $\varepsilon_R = (\underline{R}^T\underline{R})^{1/2}/N < 10^{-4}$ where N = 200 is the number of degreesof freedom.

The yielding zones are close to those from ZIENKIEWICZ (1978), pp.469-471. In the frame of subincremental technique (classical NR) and $\lambda = 1.0$ the convergence has been achievied in 5 iterations and $\varepsilon_R \cdot 10^4 = 2.01$, 2.92, 4.59, 1.56, 0.152 . The consistent approach (consistent NR) needed only 2 iterations for $\varepsilon_R \cdot 10^4 = 2.38$, 0.623. For the load $\lambda = 1.1$ the classical NR has been divergent and for the consistent NR the equilibrium has been obtained after 4 iterations for $\varepsilon_R \cdot 10^4 = 5.37$, 4.38, 1.03, 0.162.

In the paper by RAMM and MATZENMILLER (1987) the same example was analysed for 132 bilinear finite elements. Large load steps were used to test various methods. Two convergence criteria were used for the Eucledian norms: $|\Delta\Delta^i\underline{Q}| / |\Delta^i\underline{Q}| < 10^{-3}$ and $|^i\underline{R}| / |^{m+1}\underline{P} - {}^m\underline{P}| < 10^{-1}$. At the first load increment $\lambda = 1.1$ about 25% of the strip area was yielded (Fig.4c). The number of iteration and average CPU time are shown in Fig.4d for the Newtonian and quasi-Newtonian updated methods.

The smallest number of iterations and lowest CPU time was obtained for the consistent NR. The modified NR and DFP fail-ed at the first load step. The classical NR and Broyden method were not convergent in the same step.

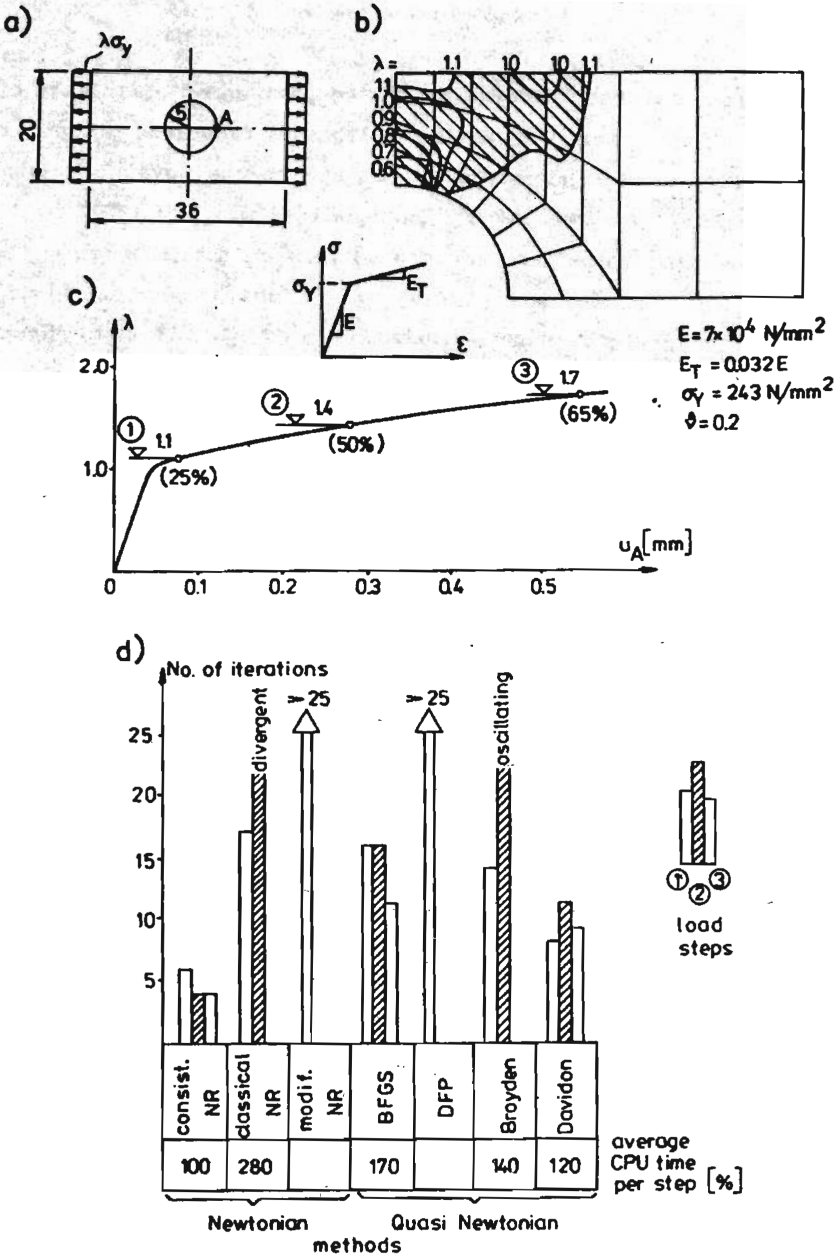

Fig.4. Perforated tension strip.

6. Conclusions

In the paper attention has been focused on basic differences between the elastic and elasto-plastic FE analysis of structures. If plastic properties of material are taken into account then the analysis on the point level $\mathcal{P}$ is of primary importance. Another difficulty is associated with transition into the cross-section level $\mathcal{S}$ where quadrature formulae (multilayer cross-sections) have to be applied to compute the generalized stresses (cross-sectional stiffnesses).

The consistent approach is shortly discussed as a combination of the implicit scheme of integration on the $\mathcal{P}$ level combined with the standard Newton-Raphson method on the structural level $\mathcal{B}$.

On example of the perforated tension strip the advantages of consistent approach over the classical Newton-Raphson and quasi-Newtonian methods have been proved.

The consistent Newton-Raphson method preserves its merits also in the large displacement analysis of elastoplastic plates and shells - cf. RAMM and MATZENMILLER (1987), SIMO and KENNEDY (1989), and for the nonlinear stability analysis of elastoplastic arches under follower loads - cf. RECZEK (1989).

The application of computational methods enables us to analyse successfully more complicated problems of thermoelastoplasticity, plastic buckling and viscoplasticity, as well as metal forming and other engineering applications of the theory of plasticity. These problems are partially reviewed in the lecture-notes by WASZCZYSZYN (1989) and recent achievements are discussed in procceedings of conferences quoted in the references of the paper. Such problems are often out of the classical assumptions of the theory of plasti-city which limited the scope of the present paper.

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Summary

PODSTAWOWE PROBLEMY ANALIZY KONSTRUKCJI SPRĘŻYSTO--PLASTYCZNYCH ZA POMOCĄ METODY ELEMENTÓW SKOŃCZONYCH

Praca ma charakter przeglądowy. Zwrócono uwagę na podstawowe różnice między analizą skończenie elementową konstrukcji sprężystych i sprężysto-plastycznych. Zasadnicze znaczenie ma analiza na poziomie punktu $\mathcal{P}$, gdzie korzysta się z równań konstytutywnych materiału sprężysto-plastycznego. Na poziomie przekroju $\mathcal{S}$ zachodzi konieczność posługiwania się wzorami kwadraturowymi celem obliczenia uogólnionych sił przekrojowych. Wskazano na zalety posługiwania się różnymi schematami procedur przyrostowych na poziomie $\mathcal{P}$ i całego układu $\mathcal{B}$. Przykład liczbowy potwierdza korzyści wynikające ze stosowania niejawnego schematu całkowania na poziomie $\mathcal{P}$ i metody Newtona-Raphsona na poziomie $\mathcal{B}$.