

NEW EFFICIENT APPROXIMATION OF WEIGHTING FUNCTIONS FOR SIMULATIONS OF UNSTEADY FRICTION LOSSES IN LIQUID PIPE FLOW

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Most papers dealing with calculations and simulations of the unsteady liquid pipe flow are based on the assumption that the formula for quasi-steady friction (Darcy-Weisbach formula) can be applied. In the case of fast changes, like fast transients e.g. water hammer, it fails. In this work, the wall shear stress is presented as a sum of quasi-steady and unsteady component. The unsteady component of the wall shear stress is modeled as a convolution of local fluid acceleration and a weighting function. The original weighting function has usually a very complicated structure, and what is more, makes impossible to carry out an efficient simulation of dynamical runs. In this paper, in order to enable efficient calculation of the unsteady component of the wall shear stress, new weighting functions are presented as sums of exponential components.

Key words: transient flow, weighting function, friction losses

1. Introduction

Most paper dealing with numerical calculations and simulations of the unsteady liquid pipe flow are based on the assumption that the formula for quasi-steady friction can be applied. However it is only correct in the case of slow changes in the fluid velocity field in the pipe cross-section, i.e. either for small accelerations or for low frequencies. It fails in the case of simulation of a fast-changing flow, e.g. in the case of water hammer simulation, because received results significantly differ from the results of experimental studies.

Earlier models of unsteady friction losses were based on instantaneous values of velocity and acceleration (Daily *et al.*, 1956; Cartens and Roller, 1959;

Safwat and Polder, 1973; Shuy and Apelt, 1987; Brunone *et al.*, 1991; Vitkovsky *et al.*, 2000; Bughazem and Anderson, 2000; Bergant *et al.*, 2001). Currently, models based on the history of the flow are commonly used. The forerunner in this group of models was Zielke (1968), who presented the instantaneous wall shear stress τ_u in form of an integral convolution of the mean local acceleration of the liquid and a weighting function $w(t)$

$$\tau_u = \frac{2\mu}{R} \int_0^t \frac{\partial v}{\partial t}(u)w(t-u) du \quad (1.1)$$

where: μ – dynamic viscosity, R – inner radius of pipe, v – instantaneous mean flow velocity, t – time, u – time used in the convolution integral, $w(t)$ – weighting function.

This dependence is correct for any changes in the average velocity of flow in the pipe cross-section and relates to laminar flow. This model requires time-consuming numerical calculations due to continuous referring to the history of the flow velocity. Therefore, it has been simplified by the introduction of the so-called effective weighting function. Trikha (1975) was first to present the effective expression of the Zielke weighting model, but this relationship has a limited range of applications. Then, new computing models, based on the approximation of the Zielke weighting function were created by Kagawa *et al.* (1983), Suzuki *et al.* (1991) and Schohl (1993).

In the case of transient turbulent flow, in most of the scientific papers, the models of unsteady friction losses are based on the two-dimensional Reynolds equation and the Boussinesqu hypothesis. In addition, the experimental data for the turbulent viscosity coefficient distribution in the pipe cross section (in its various layers) are used. In the literature one can find so-called two region models (Vardy *et al.*, 1993; Vardy and Brown, 1995, 1996; Popov, 1982; Brown *et al.*, 1969), three-region models (Brown, 1969) and four-region models (Ohmi *et al.*, 1985; Zarzycki, 1994, 2000). Similarly, as it was in the case of laminar flow, the relation describing the instantaneous wall shear stress in form of Eq. (1.1) can also be used for unsteady turbulent flow, but then weighting functions $w(t)$ are determined on the basis of the above mentioned multi-region models. In this case, expressions which describe the weighting functions effectively are represented by the following models: Vardy and Brown (2003, 2004), Zarzycki and Kudźma (2004). In the turbulent flow, weighting functions depend not only on dimensionless time (as in the case of laminar flow), but also on the Reynolds number and relative roughness height. Domains of dimensionless time and Reynolds numbers of presented weighting functions do not always correspond to practical applications. Accordingly, this work extended the scope

of practical applications of new weighting functions. In the case of laminar flow, the most strict model (Zielke, 1968) was used as a base for the process of approximation (a new function is valid in the range $10^{-9} \leq \hat{t} \leq \infty$). In the case of turbulent flow, the commonly used weighting functions: by Zarzycki (2000) and by Vardy and Brown (2003) were modified to the range of Reynolds numbers $2320 \leq \text{Re} \leq 10^8$ and dimensionless time $10^{-9} \leq \hat{t} \leq \infty$.

Using the method of characteristics (MOC) for solution of the equations of motion and continuity, there are a few examples of the application of the new weighting function to the waterhammer phenomenon. The results of numerical calculations are compared with the results of experimental studies using the experimental data by Holmboe and Rouleau (1967) and by Adamkowski and Lewandowski (2004). The comparison confirmed a good agreement.

2. Governing equations

The unsteady flow, accompanying the water hammer effect, may be described by a set of the following partial differential equations (Ohmi *et al.*, 1985; Zarzycki, 1994, 2000):

— equation of continuity

$$\frac{\partial p}{\partial t} + c^2 \rho \frac{\partial v}{\partial x} = 0 \quad (2.1)$$

— equation of motion

$$\frac{\partial v}{\partial t} + \frac{1}{\rho} \frac{\partial p}{\partial x} + g \sin \gamma + \frac{2\tau}{\rho R} = 0 \quad (2.2)$$

where: $v = v(x, t)$ – average velocity of the liquid in the pipe cross-section, $p = p(x, t)$ – average pressure in the pipe cross-section, τ – wall shear stress, ρ – density of the fluid, g – gravitational acceleration, γ – inclination angle of the hydraulic line, c – velocity of the pressure wave propagation, t – time, x – axial location along the pipe.

Among methods, which enable one to resolve the system of the above equations, particular attention should be paid to the method of characteristics (MOC), which perfectly interprets the essence of natural phenomena of transient flow, and at the same time is characterized by fast convergence, taking easily to take into account various boundary conditions and high accuracy of calculation results. With its help, one can easily solve a system of partial differential equations of the quasi-linear hyperbolic type, Eqs. (2.1) and (2.2). The solution is to find the equivalent to four ordinary differential equations,

which are then solved using the method of finite differences. Approximation to first-order differential schemes gives satisfactory results. Most computer programs used in computational simulations of transients in pipe systems have implemented simple computational algorithms, which adopt a quasi-steady hydraulic resistance. Novelty of this work is to include description of the unsteady hydraulic resistance of the transient flow in a pipeline, for which it is easy to conduct an efficient simulation for both laminar and turbulent flow.

3. Equations of unsteady hydraulic resistance

Zielke analyzed the relationship presented in the work of Brown describing the impedance of a hydraulic line versus frequency. Application of the inverse Laplace transform brought him to the following expression (Zielke, 1968)

$$\tau(t) = \underbrace{\frac{\rho v |v|}{8} \lambda}_{\tau_q} + \underbrace{\frac{2\mu}{R} \int_0^t w(t-u) \frac{\partial v}{\partial t}(u) du}_{\tau_n} \quad (3.1)$$

where: $w(t-u)$ – weighting function; t – actual time in numerical simulation; u – integral variable having dimension of time.

The first component τ_q of equation (3.1) represents the quasi-steady amount of wall shear stress. The other one, τ_n , describes the impact of the unsteady effect of flow on the wall shear stress.

3.1. The laminar flow

Zielke (1968) presented the weighting functions for laminar flow in the following form

$$w(\hat{t}) = \begin{cases} \sum_{i=1}^6 m_i \hat{t}^{(i-2)/2} & \text{for } \hat{t} \leq 0.02 \\ \sum_{i=1}^5 e^{-n_i \hat{t}} & \text{for } \hat{t} > 0.02 \end{cases} \quad (3.2)$$

where: $\hat{t} = \nu t / R^2$ – dimensionless time, coefficients m_i and n_i successively take the following values: $m_i = 0.282095, -1.25, 1.057855, 0.9375, 0.396696, -0.351563$; $n_i = 26.3744, 70.8493, 135.0198, 218.9216, 322.5544$.

Numerical calculation of the time-dependent component of the wall shear stress τ_u (second component in Eq. (3.1)) can be made using the first-order differential approximation (Zarzycki and Kudźma, 2004; Zielke, 1968)

$$\begin{aligned}\tau_n &= \frac{2\mu}{R} \sum_{j=1}^{k-1} (v_{i,j+1} - v_{i,j}) w \left((k-j)\Delta\hat{t} - \frac{\Delta\hat{t}}{2} \right) \\ &= \frac{2\mu}{R} \sum_{j=1}^{k-1} (v_{i,k-j+1} - v_{i,k-j}) w \left(j\Delta\hat{t} - \frac{\Delta\hat{t}}{2} \right)\end{aligned}\quad (3.3)$$

where: j – number of computational time step changing from 1 to k for $k \geq 2$, $\Delta\hat{t} = \nu\Delta t/R^2$.

Determination of the wall shear stress using above formula (3.3) is very time consuming. Trikha (1975) was first to develop an effective method of solving the integral convolution (later Kagawa *et al.* (1983), Suzuki *et al.* (1991) and Schohl (1993) had improved that method). In this paper, the effective solution of Kagawa *et al.* (1983) is used

$$\tau_u = \frac{2\mu}{R} \sum_{i=1}^k \underbrace{\left(y_i(t)e^{-n_i\Delta\hat{t}} + m_i e^{-n_i\Delta\hat{t}/2} [v(t+\Delta t) - v(t)] \right)}_{y_i(t+\Delta t)} \quad (3.4)$$

This method, however, requires that the weighting function has to be written as a finite sum of exponential expressions

$$w(\hat{t}) = \sum_{i=1}^k m_i e^{-n_i\hat{t}} \quad (3.5)$$

Number of exponential terms that make up the final form of the effective weighting function, affects the range of its applicability as well as its degree of fit to the original function (according to Zielke). Over the past 35 years, many authors have presented their effective weighting functions for the case of laminar flow (Kagawa *et al.*, 1983; Schohl, 1993; Trikha, 1975; Vardy and Brown, 2004; Vitkovský *et al.*, 2004). For the ranges of their applicability and their estimated coefficients – see Tables A1 and A2 in Appendix A.

The course of the laminar weighting function is shown in Fig. 1.

3.2. Turbulent flow

Both Vardy and Brown (1995, 1996, 2003, 2004), and Zarzycki (1994, 2000) suggest that the relation for the unsteady part of the wall shear stress presented

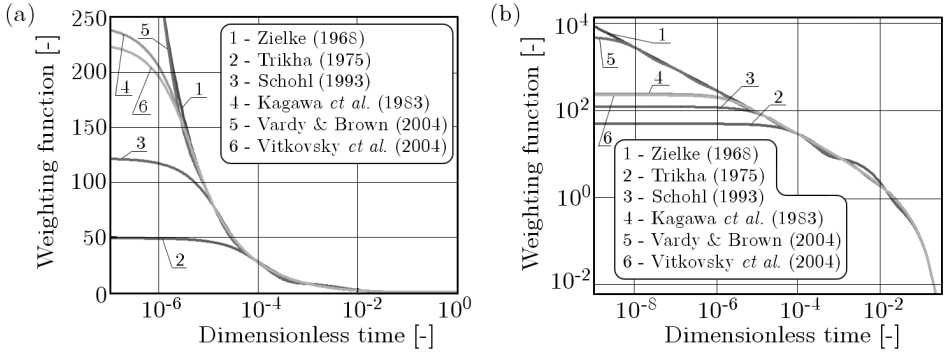


Fig. 1. Comparison of laminar function runs; (a) log-linear scale, (b) log-log scale

by Zielke, Eq. (3.1), is also true for the turbulent flow. However, the weighting function in this case has different shape due to its dependence on the Reynolds number. The original turbulent weighting functions are:

— Zarzycki's weighting function (Zarzycki, 2000)

$$w(\hat{t}, \text{Re}) = C \frac{1}{\sqrt{\hat{t}}} \text{Re}^n \quad (3.6)$$

where: $C = 0.299635$, $n = -0.005535$.

— Vardy's and Brown's weighting function (Vardy and Brown, 2003, 2004)

$$w(\hat{t}, \text{Re}) = \frac{A^* e^{-B^* \hat{t}}}{\sqrt{\hat{t}}} \quad (3.7)$$

where: $A^* = \sqrt{1/(4\pi)}$ and $B^* = \text{Re}^\kappa / 12.86$, $\kappa = \log(15.29/\text{Re}^{0.0567})$ for smooth pipes and $A^* = 0.0103\sqrt{\text{Re}}(\varepsilon/D)^{0.39}$ and $B^* = 0.352\text{Re}(\varepsilon/D)^{0.41}$ for flows in rough pipes. The ratio ε/D is called the relative roughness (in one of the recent works by Vardy and Brown, there is a proposition to set the parameters A^* and B^* from more complicated equations, shown in Vardy and Brown (2007)).

But they were not suitable for efficient simulations using expression (3.3) for wall shear stress calculations. From computational point of view it is very important to be able to conduct effective (in terms of high accuracy, fast working numerical scheme and less memory usage) simulations of transients both in laminar and turbulent flow. Therefore, in literature, one can find, so-called, *effective expressions* which are approximations of original models of turbulent weighting functions. For ranges of their applicability and their estimated coefficients – see Tables A3 and A4 in Appendix A.

4. Developing effective weighting functions

4.1. Weighting function for laminar flow

In view of the need to extend the applicability of the effective function of weighting which was noted by, among others, Vardy and Brown (2004, 2007), a new model which is the sum of exponential expressions will be presented further.

The new model is consistent with the original function by Zielke in the following range of applicability $10^{-9} \leq \hat{t} < \infty$. The final form of the new function consists of 26 exponential expressions. Since the value of Zielke weighting function for the dimensionless time $\hat{t} > 0.02$ must be determined from the following formula (3.2)

$$w(\hat{t}) = \sum_{i=1}^5 e^{-n_i \hat{t}} \quad (4.1)$$

where: $n_1 = 26.3744$, $n_2 = 70.8493$, $n_3 = 135.0198$, $n_4 = 218.9216$, $n_5 = 322.5544$.

In this work these first five exponential expressions were kept unchanged (in order to receive perfect fit of the new weighting function in this interval of dimensionless time). For the mapping of the interval $10^{-9} \leq \hat{t} < 0.02$ it was decided to add extra similar components. It was assumed that a very good accuracy would be received by describing each dimensionless time interval $10^{n-1} \leq \hat{t} < 10^n$ with three exponential expressions (except for the range $10^{-3} \leq \hat{t} < 0.02$, which also describes the three formulas – so the worst match of the new weighting function was expected). In each of those intervals, matching was carried out to uniformly distributed 1000 points (in log scale), which were the results obtained by using the original function of weighting according to Zielke.

The effective weighting function coefficients were determined by using the LSQNONLIN function, which is a module of MATLAB. In this function the Levenberg-Marquardt algorithm is implemented, considered as one of the most effective among the minimization algorithms. It combines the linear approximation away from the minimum and square approximation near the minimum, so that it is specialized to problems of the method of least squares.

Following the procedure outlined above, all coefficients of the *new effective weighting function for laminar flow* were determined

$$w_{apr}(\hat{t}) = \sum_{i=1}^{26} m_i e^{-n_i \hat{t}} \quad (4.2)$$

where: $m_1 = 1, m_2 = 1, m_3 = 1, m_4 = 1, m_5 = 1, m_6 = 2.141, m_7 = 4.544, m_8 = 7.566, m_9 = 11.299, m_{10} = 16.531, m_{11} = 24.794, m_{12} = 36.229, m_{13} = 52.576, m_{14} = 78.150, m_{15} = 113.873, m_{16} = 165.353, m_{17} = 247.915, m_{18} = 369.561, m_{19} = 546.456, m_{20} = 818.871, m_{21} = 1209.771, m_{22} = 1770.756, m_{23} = 2651.257, m_{24} = 3968.686, m_{25} = 5789.566, m_{26} = 8949.468, n_1 = 26.3744, n_2 = 70.8493, n_3 = 135.0198, n_4 = 218.9216, n_5 = 322.5544, n_6 = 499.148, n_7 = 1072.543, n_8 = 2663.013, n_9 = 6566.001, n_{10} = 15410.459, n_{11} = 35414.779, n_{12} = 80188.189, n_{13} = 177078.960, n_{14} = 388697.936, n_{15} = 850530.325, n_{16} = 1835847.582, n_{17} = 3977177.832, n_{18} = 8721494.927, n_{19} = 19120835.527, n_{20} = 42098544.558, n_{21} = 92940512.285, n_{22} = 203458923.000, n_{23} = 445270063.893, n_{24} = 985067938.878, n_{25} = 2166385706.058, n_{26} = 4766167206.672.$

4.2. Weightig functions for turbulent flow

The new effective weighting function for turbulent flow will be based on the Zarzycki original model ((Zarzycki, 1994, 2000; Zarzycki and Kudźma, 2004). The form of a new function is selected so that it is easy to scale it. Scaling will depend in detail on multiplying the estimated coefficients (assuming a constant value of the Reynolds number Re during the transient flow) by a function of the Reynolds number (which will be fixed prior to simulation) (Vitkovský *et al.*, 2004). It will ensure a good fit for the standard functions (the original weighting function) of the effective function throughout the range of applicability.

The weighting function by Zarzycki is a function of dimensionless time and the Reynolds number Re (3.6). Following the way presented by Vitkovský *et al.* (2004), it can be written

$$w_{apr}(\hat{t}, Re) = \frac{CRe^n}{\sqrt{\hat{t}}} \approx \sum_{i=1}^k m_i e^{-n_i \hat{t}} \quad (4.3)$$

Then, dividing the above dependence by CRe^n , the weighting function has form

$$\begin{aligned} w_{apr}^* &= \frac{1}{\sqrt{\hat{t}}} \approx \sum_{i=1}^k \frac{m_i}{CRe^n} e^{-n_i \hat{t}} & w_{apr}^* &\approx \frac{Re^{-n}}{C} \sum_{i=1}^k m_i e^{-n_i \hat{t}} \\ w_{apr}^* &\approx \frac{Re^{0.005535}}{0.299635} \sum_{i=1}^k m_i e^{-n_i \hat{t}} & w_{apr}^* &\approx \sum_{i=1}^k m_i^* e^{-n_i \hat{t}} \end{aligned} \quad (4.4)$$

where: m_i^* – universal coefficients for the effective weighting function of turbulent flow.

This procedure shows how the universal coefficients of the effective weighting function for the turbulent flow are determined. The current form of the effective weighting function is determined with their help as a result of the reverse scaling. In this paper, the universal coefficients m_i^* were determined by adjusting the effective weighting function to the original by Zarzycki for the constant Reynolds number $\text{Re} = 10000$

$$m_i^* = \frac{m_i(\text{Re} = 10000)}{C} \text{Re}^{0.005535} = \frac{m_i(\text{Re} = 10000)}{0.299635} 10000^{0.005535} \quad (4.5)$$

The making use of universal coefficients must ensure that the number of $\text{Re} = 10000$ are derived values, which were initially estimated, i.e. $m_i(\text{Re} = 10000)$. Therefore, the desired effective weighting function for turbulent flow must have the following form

$$w_{apr}(\hat{t}, \text{Re}) \approx \frac{C}{\text{Re}^{-n}} \sum_{i=1}^k m_i^* e^{-n_i \hat{t}} = \frac{0.299635}{\text{Re}^{0.005535}} \sum_{i=1}^k m_i^* e^{-n_i \hat{t}} \quad (4.6)$$

Assume that the scope of the new effective weighting function for turbulent flow must be as follows: $10^{n-1} \leq \hat{t} < 10^n$ to be fully suited for the use in present technical issues. Then it was found that for each interval $10^{n-1} \leq \hat{t} < 10^n$ (for $n = -9$ to $n = 3$) for a good estimate of the coefficients characterizing this function, just two power terms ($m_i e^{-n_i \hat{t}}$) are efficient. Hence, for the entire range of applicability in the field of dimensionless time $10^{-9} \leq \hat{t} < 10^3$ 24 exponential expressions were finally received. The sum of these expressions is the new effective weighting function for turbulent flow. In order to determine such a large number of factors (in total 48 coefficients) it was assumed that the successive coefficients of determination in steps must be followed. For each step of estimation, there were two new exponential terms added (4 consecutive coefficients) to the searched function (power series). The searching procedure was started for the dimensionless time period $10^2 \leq \hat{t} < 10^3$.

In order to accurately estimate the coefficients in each interval of analysis $10^{n-1} \leq \hat{t} < 10^n$, the estimation was based on equally distributed 1000 points (similar like when setting the new effective function for laminar flow), which were the results obtained from using the original weighting function by Zarzycki.

Following the above procedure, all the estimated coefficients of the new effective weighting function for turbulent flow were determined

$$w_{apr}(\hat{t}, \text{Re}) = \frac{C}{\text{Re}^{-n}} \sum_{i=1}^{24} m_i^* e^{-n_i \hat{t}} = \frac{0.299635}{\text{Re}^{0.005535}} \sum_{i=1}^{24} m_i^* e^{-n_i \hat{t}} \quad (4.7)$$

where: $m_1^* = 0.06054$, $m_2^* = 0.09698$, $m_3^* = 0.17971$, $m_4^* = 0.31240$, $m_5^* = 0.56562$, $m_6^* = 0.98348$, $m_7^* = 1.77243$, $m_8^* = 3.08626$, $m_9^* = 5.57348$, $m_{10}^* = 9.7254$, $m_{11}^* = 17.591$, $m_{12}^* = 30.723$, $m_{13}^* = 55.603$, $m_{14}^* = 97.138$, $m_{15}^* = 175.825$, $m_{16}^* = 307.176$, $m_{17}^* = 551.342$, $m_{18}^* = 954.362$, $m_{19}^* = 1727.71$, $m_{20}^* = 3171.2$, $m_{21}^* = 5899.4$, $m_{22}^* = 11013$, $m_{23}^* = 19923$, $m_{24}^* = 37929$, $n_1 = 0.000671$, $n_2 = 0.00838$, $n_3 = 0.04504$, $n_4 = 0.1790$, $n_5 = 0.6457$, $n_6 = 2.159$, $n_7 = 7.088$, $n_8 = 22.563$, $n_9 = 72.215$, $n_{10} = 227.12$, $n_{11} = 723.19$, $n_{12} = 2270.23$, $n_{13} = 7226.1$, $n_{14} = 22686.2$, $n_{15} = 72226.7$, $n_{16} = 226796$, $n_{17} = 720015$, $n_{18} = 2234661$, $n_{19} = 7050737$, $n_{20} = 22553627$, $n_{21} = 74840660$, $n_{22} = 253286747$, $n_{23} = 856109205$, $n_{24} = 2893640000$.

The range of applicability of the new effective weighting functions, presented in the last two subsections, (laminar flow $10^{-9} \leq \hat{t} < \infty$, turbulent flow $10^{-9} \leq \hat{t} < 10^3$ and $2300 \leq \text{Re} \leq 10^8$) virtually guarantees a very accurate assessment of hydraulic resistance which is useful in simulation of the unsteady cavitating flow in pipelines. However, one can move beyond this range, when it is necessary to substantially thicken the grid of characteristics (tracking changes of flow parameters in a number of cross-sections in the analyzed pressure line). Then, in the numerical analysis, the changes in a very short time $\hat{t} < 10^{-9}$ can be taken into account (see example in Appendix B).

Therefore, there are no decisive experimental results supporting one of the two (considered in this paper) original weighting functions for turbulent flow (by Vardy and Brown and by Zarzycki). There is already an effective weighting function presented by Vitkovský *et al.* (2004) (for details see Appendix A) based on the model by Vardy and Brown (3.7), which is characterized by a good fit, but a small range of applicability. There is therefore no need to present a completely new function. It is simply enough to extend the applicability of the existing one by finding new extra exponential terms.

To minimize the impact of the last term of the Vitkovský *et al.* (2004) function on quality of mapping of the expanded function to the original function, it was also replaced by a new one (new estimated values of the coefficients: m_{10}^* and n_{10}^*). Moreover, in the estimation process it was assumed that the lower range of applicability of the new function must be $\hat{t} = 10^{-9}$. For this purpose, it was necessary to find seven new exponential expressions. The procedure used to achieve this goal was almost identical to that used in the estimating of the weighting function for the turbulent flow based on the original weighting function by Zarzycki. Thus, below is only presented the final form of this function

$$w_{apr}(\hat{t}, \text{Re}) = \sum_{i=1}^{16} A^* m_i^* e^{-(n_i^* + B^*)\hat{t}} \quad (4.8)$$

where: A^* and B^* are parameters known from Vardy's and Brown's equation (3.7) and $m_1^* = 5.03362$, $m_2^* = 6.4876$, $m_3^* = 10.7735$, $m_4^* = 19.904$, $m_5^* = 37.4754$, $m_6^* = 70.7117$, $m_7^* = 133.460$, $m_8^* = 251.933$, $m_9^* = 476.597$, $m_{10}^* = 902.22$, $m_{11}^* = 1602.04$, $m_{12}^* = 2894.84$, $m_{13}^* = 5085.55$, $m_{14}^* = 9190.11$, $m_{15}^* = 16118.6$, $m_{16}^* = 29117.3$, $n_1^* = 4.78793$, $n_2^* = 51.0897$, $n_3^* = 210.868$, $n_4^* = 765.03$, $n_5^* = 2731.01$, $n_6^* = 9731.44$, $n_7^* = 34668.5$, $n_8^* = 123511$, $n_9^* = 440374$, $n_{10}^* = 1578229$, $n_{11}^* = 5481659$, $n_{12}^* = 18255921$, $n_{13}^* = 59753474$, $n_{14}^* = 192067361$, $n_{15}^* = 616415963$, $n_{16}^* = 1945566788$.

The scope of applicability of all new weighting functions presented above (4.2), (4.7) and (4.8) can be any further extended by adding the new exponential expressions.

4.3. Comparison of the new weighting functions with their original counterparts and the best known effective features from the literature

The following figures are presented in order to show a comparison of the new, proposed in the previous section, weighting functions with their inefficient counterparts and the most precise effective functions known from the literature. In addition, in order to demonstrate the degree of matching obtained by the weighting functions with their prototypes, results of quantitative analysis are graphically presented.

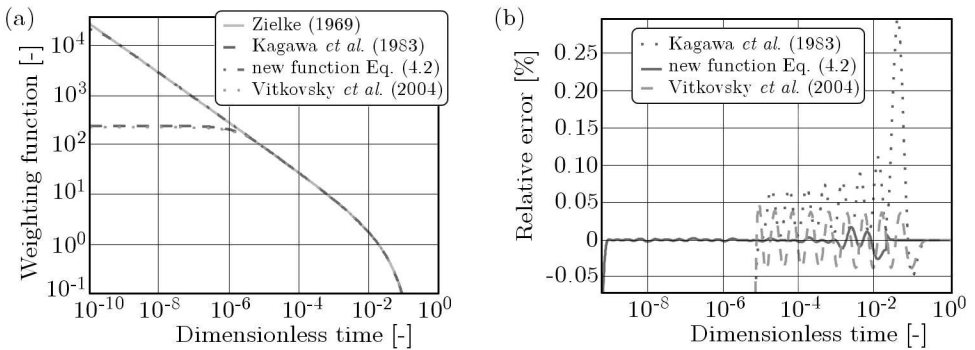


Fig. 2. Comparison of weighting functions for laminar flow: (a) log-log graph, (b) relative percentage error

As a parameter determining quantitatively the degree of matching the new efficient weighting function with its original counterpart, the relative percentage error determined from the following relationship was incorporated

$$R_{error} = \frac{w_{apr} - w}{w} 100\% \tag{4.9}$$

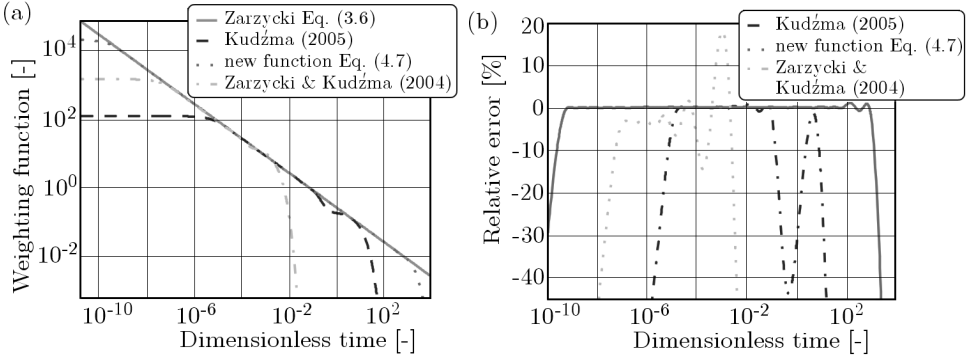


Fig. 3. Comparison of weighting functions (original Zarzycki’s and its known effective counterparts) for turbulent flow ($Re = 10^4$): (a) log-log graph, (b) relative percentage error

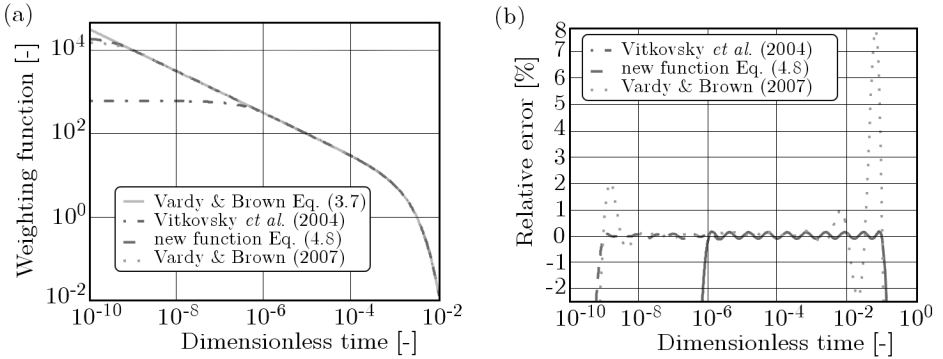


Fig. 4. Comparison of weighting functions (original Vardy and Brown and its known effective counterparts) for turbulent flow ($Re = 10^4$): (a) log-log graph, (b) relative percentage error

Comparisons of the weighting functions for turbulent flow for a wider range of the Reynolds numbers are not shown in this work. Indeed, the use of scaling procedures presented by Vitkovský *et al.* (2004) means that for other Reynolds numbers (from the range of applicability) the errors remain the same.

5. Numerical results

In order to compare the accuracy of unsteady (with the use of original and universal weighting functions) and quasi-steady models of friction in relation

to experimental data, simulations of a simple waterhammer case (tank – long liquid line and cut-off valve) were conducted.

The computed results (the method of characteristics based on a rectangular grid was used; hydraulic line was divided on $N = 16$ elements) were compared with the experimental data reported by Holmboe and Rouleau (1967), and Adamkowski and Lewandowski (2004).

5.1. Holmboe and Rouleau experiment

Holmboe and Rouleau (1967) ran their tests on a copper tube with radius 0.0127 m and length 36.09 m connected upstream to a tank which was maintained at a constant pressure by the compressed air. The liquid used in the experiment was an oil having viscosity 39.6710^{-6} m²/s. The measured speed of the pressure wave was 1324.36 m/s, and the initial flow velocity 0.128 m/s ($Re = 82$). The downstream valve was rapidly closed in the pipe line during flow. Pressure fluctuation was measured at the endpoint of the line (near the valve). From the above parameters, it followed that it was a case of laminar flow. It was determined in numerical calculations, in which the models of Zielke (3.2) and the new effective laminar model (4.2) were used. In addition, the calculation with the usage of quasi-steady model was conducted. The results of simulations and experimental data are shown in Fig. 5.

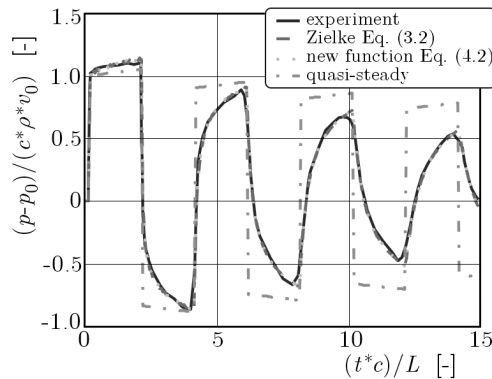


Fig. 5. Fluctuations of pressure at the endpoint of the line ($Re = 82$) – Holmboe and Rouleau experiment

It is clearly seen that the calculation using unsteady friction models is much closer to the experimental data. Therefore, in further calculations, the unsteady friction models were used instead of the quasi-steady formula. Additionally, it is significant that the differences between the simulation results

according to the classical Zielke model (3.2) and the new proposed function (4.2) (efficient way of unsteady wall shear stress computation) are hardly distinguishable.

5.2. Adamkowski and Lewandowski experiment

Adamkowski and Lewandowski (2004) conducted experiments at a test rig specially designed in order to investigate the unsteady pipe flows. Its main component was a 98.11 m long copper pipe with internal diameter of 0.016 m and wall thickness of 0.001 m. The pipe was rigidly mounted to the ground using bearings in order to minimize its vibrations. A quick-closing spring driven ball valve was installed at one end of the pipe. The initial conditions were defined by the high pressure reservoir pressure head and the initial flow velocity in the pipeline. During the tests, temperature of water was 22.6°C , and the kinematic viscosity coefficient for these conditions was $9.493 \cdot 10^{-7} \text{ m}^2/\text{s}$. Three runs were selected for the purpose of verifying new unsteady friction models. The initial flow velocities were 0.066 m/s ($\text{Re} = 1100$), 0.631 m/s ($\text{Re} = 10600$) and 0.927 m/s ($\text{Re} = 15600$). The results of simulations and experimental data are shown in Figs. 6, 7 and 8.

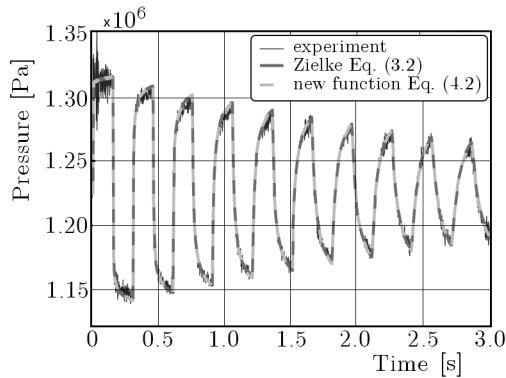


Fig. 6. Fluctuations of pressure at the endpoint of the line ($\text{Re} = 1100$) – Adamkowski and Lewandowski experiment

From the above graphs, it is clear that the use of the new efficient weighting functions (4.2), (4.7) and (4.8) ensures compatibility of results with those obtained from the original models by Zielke (3.2), Vardy and Brown (3.7) and Zarzycki (3.6).

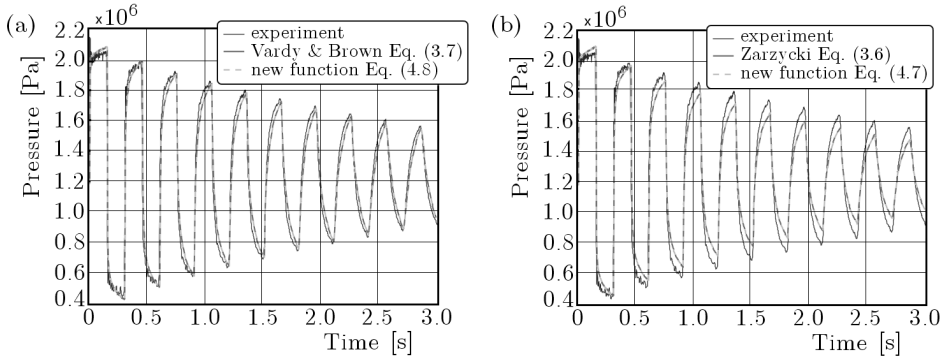


Fig. 7. Fluctuations of pressure at the endpoint of the line ($Re = 10600$) – Adamkowski and Lewandowski experiment

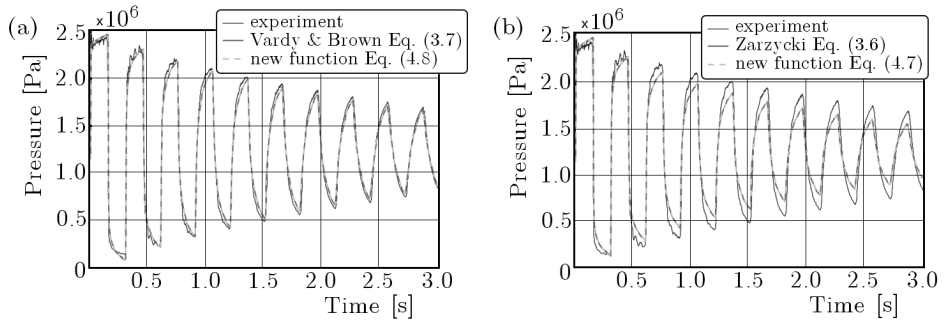


Fig. 8. Fluctuations of pressure at the endpoint of the line ($Re = 15600$) – Adamkowski and Lewandowski experiment

6. Summary

The main drawback of the original models (according to Zielke, Vardy and Brown, and Zarzycki) describing the unsteady hydraulic resistance is their inefficiency! In each successive time step they require more resources of the computer memory (this is due to augmentations in quantity of elements that make up the sum creating the solution to the integral convolutions (3.3), as well as due to enlargement of the matrix which stores information about past velocity changes $v(t)$). Often, after the time when the entire memory of the computer is used, it comes to forced interruption of performance simulation. Hence the original weighting functions that make the obtained results coincide with the results of experimental studies (Vardy and Brown, 1995; Zarzycki, 1994), even today, with tremendous development of computer technology, fit for simulations of very short transients only.

The known efficient solution of the integral convolution (3.4) for both laminar and turbulent flow (Kagawa *et al.*, 1983; Schohl, 1993; Zarzycki and Kudźma, 2004) allowed, in the case of using the weighting function as a finite sum of exponential terms $\sum_{i=1}^k m_i e^{-n_i \hat{t}}$, a significant reduction in computation time and reduced demand for operational memory. Importantly, the time of calculation with the increasing number of time steps, increased almost linearly, as shown in Fig. 9 (calculations were carried out on a standard desktop computer – Fujitsu Siemens-Intel Core 2 Quad CPU Q6600, 2.4 GHz, 2048 MB RAM). The impact of number of weighting function terms on time of calculation was also investigated. The two cases are presented in Fig. 9: 10 terms (Eq. (3.4) and Vitkovský *et al.* laminar function (2004)) and 26 terms (Eqs. (3.4) and (4.2)). The increase of time consumption for efficient cases is very small in relation to the original model (Eqs. (3.2) and (3.3)).

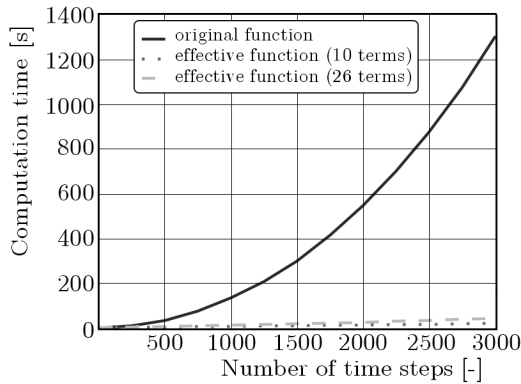


Fig. 9. Time of numerical calculations depending on the number of time steps

Numerous simulation tests carried out by the authors revealed that the numerical results using models with small ranges of applicability (e.g., by Trikha) often deviate from the experimental results (as a result of going beyond the scope of applicability of the used weighting function). Professional software, which may in future be based on weighting functions presented in this work, must prevent crossing outside the range of applicability of these functions – by informing the user about the need to change numerical parameters (e.g. the number of simultaneously observed pipe cross sections).

The presented new weighting functions characterized by the extended ranges of applicability in the domain of dimensionless time are suitable for the efficient modeling of transients. The studies show that the results of numerical simulations, in which a new weighting functions were used, overlap with those that use the original models.

In the next stage of research, the usefulness of the new weighting functions presented in this paper in simulation of transients with cavitation should be explored.

Appendix A

Table A1. Ranges of applicability of the effective laminar weighting functions

Model by	Trikha (1975)	Schohl (1993)	Kagawa <i>et al.</i> (1983)	Vitkovský <i>et al.</i> (2004)	Vardy and Brown (2004)
range of applicab.	$7.41 \cdot 10^{-5} \leq \hat{t} \leq 10$	$1.26 \cdot 10^{-5} \leq \hat{t} \leq 1$	$6.31 \cdot 10^{-6} \leq \hat{t} \leq \infty$		$10^{-8} \leq \hat{t} \leq \infty$

Table A2. Estimated coefficients of the effective laminar weighting functions

i	Trikha (1975)		Schohl (1993)		Kagawa <i>et al.</i> (1983)		Vitkovský <i>et al.</i> (2004)		Vardy and Brown (2004)	
	m_i	n_i	m_i	n_i	m_i	n_i	m_i	n_i	m_i	n_i
1	1	26.4	1.051	26.65	1	26.3744	1	26.3744	1	A
2	8.1	200	2.358	100	1.16725	72.8033	1.09301	72.044	2.1830	10^2
3	40	8000	9.021	669.6	2.20064	187.424	1.82206	166.931	2.714	$10^{2.5}$
4			29.47	6497	3.92861	536.626	3.34085	435.932	7.5455	10^3
5			79.55	57990	6.78788	1570.60	5.89377	1229.74	39.0066	10^4
6					11.6761	4618.13	10.2835	3584.84	106.8075	10^5
7					20.0612	13601.1	17.9006	10621.7	359.0847	10^6
8					34.4541	40082.5	31.1516	31757	1107.9295	10^7
9					59.1642	118153	54.4168	95563.7	3540.683	10^8
10					101.59	348316	99.4360	293268		

$$A = 26.3744$$

Table A3. Ranges of applicability of the effective turbulent weighting functions

Model by	Zarzycki and Kudźma (2004)	Kudźma (2005)	Vitkovský <i>et al.</i> (2004)	Vardy and Brown (2004)
range of applicab.	$10^{-7} \leq \hat{t} \leq 10^{-3}$	$10^{-5} \leq \hat{t} \leq 10$	$10^{-6} \leq \hat{t} \leq 10^{-1}$	$10^{-9} \leq \hat{t} \leq 10^{-1}$

Table A4. Estimated coefficients of the effective turbulent weighting functions

i	Based on Zarzycki's original expression (3.6)				Based on Vardy and Brown original expression (3.7)			
	Zarzycki and Kudźma (1975) ¹		Kudźma (2005) ²		Vitkovský <i>et al.</i> (2004)		Vardy and Brown (2004)	
	$w_{appr}(\hat{t}, Re) = \sum_{i=1}^k C^* m_i^* e^{-n_i \hat{t}}$				$w_{appr}(\hat{t}, Re) = \sum_{i=1}^k A^* m_i^* e^{-(n_i^* + B^*) \hat{t}}$			
	m_i^*	n_i	m_i^*	n_i	m_i^*	n_i^*	m_i^*	n_i^*
1	17.10735	477.887	0.2137	0.09834	5.03362	4.78793	9.06	10
2	58.51351	17790.69	1.568	8.44	6.4876	51.0897	-4.05	10 ^{1.5}
3	152.3936	207569.7	2.799	88.02	10.7735	210.868	12	10 ²
4	328.2	1464649	5.527	480.5	19.904	765.03	8.05	10 ^{2.5}
5	414.8145	6316096	10.76	2162	37.4754	2731.01	22.7	10 ³
6	640.2165	15512625	18.99	8425	70.7117	9731.44	35.2	10 ^{3.5}
7			33.26	29250	133.46	34668.5	65.9	10 ⁴
8			60.73	96940	251.933	123511	115	10 ^{4.5}
9					476.597	440374	206	10 ⁵
10					932.86	1590300	365	10 ^{5.5}
11							651	10 ⁶
12							1150	10 ^{6.5}
13							2060	10 ⁷
14							3630	10 ^{7.5}
15							6640	10 ⁸
16							10700	10 ^{8.5}
17							26200	10 ⁹

¹ $C^* = (-1.5125Re^{0.003264} + 2.55888)$; ² $C^* = (-13.27813Re^{0.000391} + 14.27658)$
 A^* and B^* are parameters known from Vardy's and Brown's equation (3.7)

Appendix B

In the numerical analysis of transients, one can sometimes go beyond the scope of applicability of the weighting functions (thus committing a serious error in determining the hydraulic resistance). This has not been previously discussed in the literature on the research of such states.

Below, there are two theoretical cases explaining the importance of extending the scope of applicability of the weighting functions (by adding new exponential expressions) to the lower range of dimensionless time $\hat{t} = 10^{-9}$.

Case I

$L = 25$ m, $\nu = 10^{-6}$ m²/s, $N = 100$, $c = 1225$ m/s, $R = 0.13$ m – we will have

$$\Delta\hat{t} = \frac{L\nu}{NcR^2} = \frac{25 \cdot 10^{-6}}{100 \cdot 1225 \cdot 0.13^2} = 1.2076 \cdot 10^{-8}$$

In the case of a rectangular grid for the first computational step ($j = 1$) using classical formula (3.3)

$$\tau_{u(i,k)} = \frac{2\mu}{R} \sum_{j=1}^{k-1} (v_{(i,k-j+1)} - v_{(i,k-j)}) w\left(j\Delta\hat{t} - \frac{\Delta\hat{t}}{2}\right)$$

as well as using effective formula (Vitkovský *et al.*, 2004)

$$\tau_{u(i,k)} = \frac{2\mu}{R} \sum_{j=1}^z \underbrace{\left[y_{j(i,k-1)} e^{-n_j \Delta\hat{t}} + m_j e^{-n_j \frac{\Delta\hat{t}}{2}} (v_{(i,k)} - v_{(i,k-1)}) \right]}_{y_{j(i,k)}}$$

the value of the weighting function is needed for the argument equal to $\Delta\hat{t}/2$. It means that for this case, the function have to be in the range of applicability beginning at $\Delta\hat{t}/2 = 6.038 \cdot 10^{-9}$. The values of different weighting functions for this argument are shown in Table B1.

Table B1. Values of the laminar weighting function for the dimensionless time $\Delta\hat{t}/2 = 6.038 \cdot 10^{-9}$

Function	Zielke (3.2)	New function (4.2)	Vardy and Brown (2004)	Vitkovský <i>et al.</i> (2004)	Kagawa <i>et al.</i> (1983)
Value of weighting function [-]	3629.103	3629.157	3494.923	226.123	241.764
Relative error [%]	0	0.0015	-3.6973	-93.7692	-93.3382

Case II

If: $L = 1000 \text{ m}$, $\nu = 10^{-5} \text{ m}^2/\text{s}$, $N = 16$, $c = 1280 \text{ m/s}$, $R = 0.008 \text{ m}$, then

$$\Delta\hat{t} = \frac{L\nu}{NcR^2} = \frac{1000 \cdot 10^{-5}}{16 \cdot 1280 \cdot 0.008^2} = 7.6 \cdot 10^{-3}$$

and $\Delta\hat{t}/2 = 3.8 \cdot 10^{-3}$.

In this case, there is no need to use many exponential expressions – just 5 is enough (for both turbulent and laminar flow) in order to properly simulate the hydraulic resistance.

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**Nowe efektywne funkcje wagowe umożliwiające symulację
niestacjonarnych oporów hydraulicznych podczas przepływów cieczy
w przewodach ciśnieniowych**

Streszczenie

W dużej części prac, dotyczących obliczania oraz symulacji niestacjonarnego przepływu cieczy w przewodach ciśnieniowych, opory wyznacza się jedynie z dobrze znanej formuły Darcy-Weisbacha. Niestety w szybkozmiennych przepływach, takich jak np. uderzenie hydrauliczne, założenie quasi-stacjonarności oporów jest poważnym błędem. W poniższej pracy naprężenie styczne na ścianie przewodu wyznacza się z sumy quasi-stacjonarnego składnika oraz niestacjonarnego. Niestacjonarny składnik naprężenia stycznego modelowany jest jako splot lokalnego przyśpieszenia cieczy oraz funkcji wagi. Klasyczne funkcje wagi mają skomplikowaną postać, co uniemożliwia efektywne wyznaczanie oporów. W tej pracy przedstawione zostały nowe efektywne postacie funkcji wagowych o rozszerzonych zakresach stosowności.

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