MECHANIKA
TEORETYCZNA
I STOSOWANA
3, 27, (1989)

MODIFIED HYPOTHESIS OF THEORY OF THIN MAGNETOELASTIC SHELLS

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1. Introduction

The magnetoelastic interaction effects are particularly significant in shell-like constructions. However, the electrodynamic forces caused by initial magnetostatic field do not violate the validity of the Kirchhoff hypothesis. In a case of a perfectly conductible shell the Kirchhoff hypothesis is a sufficient approximation for the Lorentz body force to be expressed entirely in terms of the midsurface deflection by means of the relationships of electrodynamics (cf. [1]). It should be stated that in this case, i.e., when the conductivity increases to infinity, the magnetoelasticity equations do not ensure the vanishing of the normal component of the current density vector at the boundary bordered on a vacuum. Upon the neglect of the perturbed electric field in beams of finite conductivity Dunkin, Eringen [2] and Suhubi [3] obtained the simplified equation of the bending and twisting vibration, respectively, expressed in terms of displacement. For the shells made of a real conductor Ambartsumian, Baghdasarian and Belubekian [4] supplemented the Kirchhoff hypothesis, assuming that the tangent components of the perturbed electric field vector and the normal component of the perturbed magnetic field vector are constant across the shell thickness. As a consequence, some of Maxwell's equations in scalar form are not used during the derivation of the differential governing equations of the shell theory [5] (Apart from a case of spherical shell in radial magnetostatic field the equations simplified with the aid of so-called Vlasov's engineering theory being considered). It was proved in [6] that the above electromagnetic restrictions are too strong for plates so far as a general case of initial magnetostatic field is concerned. It should be apparent that these restrictions are too strong for shells, too. Note that the electromagnetic quantities determined in the same way as in [1] are not found to be constant across the shell thickness.

In the present paper we apply the modified hypothesis of thin magnetoelastic plates and shells (see [6]) as the simplest correction removing the shortcomings caused by the former hypothesis.

2. Basic equations of linear magnetoelasticity

We shall deal with an elastic, diamagnetic or paramagnetic, homogenous and isotropic solid. The initial magnetostatic field, being described in a body region by the equations:

$$\mathrm{rot}\boldsymbol{B} = 0, \quad \mathrm{div}\boldsymbol{B} = 0, \tag{1}$$

where $\boldsymbol{B}$ is the magnetostatic induction vector, causes no body force in the state of rest. The motion of a particle with respect to initial frame may be described by the displacement vector $\boldsymbol{w}$. In the linear theory all the terms containing more than one secondary quantity, i.e., quantity due to motion of a body, ought to be omitted. Thus, in the quasi-stationary case (cf. [7]) the perturbed electromagnetic field satisfies the following equations:

$$\begin{gathered}\mathrm{rot}\,\boldsymbol{e}+\dot{\boldsymbol{b}} = \boldsymbol{0}, \quad \mathrm{rot}\,\boldsymbol{b} = \mu\boldsymbol{j},\\ \mathrm{div}[\in\boldsymbol{e}-(\in-\in_0)\boldsymbol{B}\times\boldsymbol{v}] = \varrho_e, \quad \mathrm{div}\boldsymbol{b} = 0,\end{gathered} \tag{2}$$

in which the vectors $\boldsymbol{b}$, $\boldsymbol{e}$ and $\boldsymbol{j}$ denote the perturbed magnetic field induction, the perturbed electric field intensity and the current density, respectively; μ, $\in$ and $\in_0$ the magnetic permeability of solid (equal with permeability of vacuum), the electric permittivity of solid and permittivity of vacuum, respectively; ϱ_e denotes the charge density; $\boldsymbol{v}$ the velocity vector ($\boldsymbol{v} = \dot{\boldsymbol{w}}$); dot represents differentiation with respect to time, t. The divergence of $(2)_{1,2}$ yieds, respectively,

$$\mathrm{div}\dot{\boldsymbol{b}} = 0, \quad \mathrm{div}\boldsymbol{j} = 0. \tag{3}$$

The electromagnetic constitutive equation is given by:

$$\boldsymbol{j} = \lambda(\boldsymbol{e}-\boldsymbol{B}\times\boldsymbol{v}), \tag{4}$$

where λ is the conductivity. In a case of a perfect conduction (4) ought to be replaced by:

$$\boldsymbol{e} = \boldsymbol{B}\times\boldsymbol{v}. \tag{5}$$

Making use of (5), or (4) and $(3)_2$, we simplify $(2)_3$ in view of $(1)_1$ as follows:

$$\varrho_e = -\in_0\boldsymbol{B}\,\mathrm{rot}\boldsymbol{v}. \tag{6}$$

At the bounding surface of the body considered we must fulfil the discontinuity conditions:

$$\begin{gathered}[\boldsymbol{e}-\boldsymbol{B}\times\boldsymbol{v}]\times\boldsymbol{n} = \boldsymbol{0}, \quad \left[\frac{\boldsymbol{b}}{\mu}\right]\times\boldsymbol{n} = -\boldsymbol{j}^p,\\ [\in\boldsymbol{e}-(\in-\in_0)\boldsymbol{B}\times\boldsymbol{v}]\boldsymbol{n} = \varrho_e^p, \quad [\boldsymbol{b}]\boldsymbol{n} = 0, \quad [\boldsymbol{j}]\boldsymbol{n} = 0,\end{gathered} \tag{7}$$

where $\boldsymbol{j}^p$ is the surface current density vector, ϱ_e^p the surface-bound charge density, $\boldsymbol{n}$ the unit normal vector to the external surface of the body; double square brackets mean the difference in the values outside and inside the solid considered. Surface currents are observed when one of the media is a perfect conductor (cf. [7]). If the bounding surface of the body considered is bordered on a vacuum, then $(7)_5$ reduces to:

$$\boldsymbol{j}\boldsymbol{n} = 0. \tag{8}$$

If the body made of a real conductor borders on a perfect conductor, then with the aid of (5) and (4) we simplify $(7)_1$ to:

$$\boldsymbol{j}\times\boldsymbol{n} = \boldsymbol{0}. \tag{9}$$

Under the assumption that the only mechanical effect of the electromagnetic field is the Lorentz force, the stress equation of motion takes the form:

$$\mathrm{div}\sigma+\boldsymbol{l}+\boldsymbol{q}-\varrho\ddot{\boldsymbol{w}} = 0, \tag{10}$$

where the vector product:

$$l = j \times B \tag{11}$$

is the Lorentz body force, q the mechanical body force, σ the stress tensor and ϱ the mass density. Analogously, the kinetic boundary condition may be written down as:

$$\sigma n - S = F, \tag{12}$$

where the vector:

$$S = j^p \times B, \tag{13}$$

describes the surface tractions of electromagnetic origin, the vector F represents the surface load of mechanical origin.

Obviously, the well known geometrical relations remain unchanged by the presence of the electromagnetic field. Similarly, the mechanical constitutive equation is taken to be the Hooke's law.

3. Equations of magnetoelastic shell theory

We shall consider a thin conductible (not perfectly) shell of uniform thickness, the lower and upper faces being surrounded by a vacuum, and the edges being bordered on a perfectly conductible medium. Let u^i $(i = 1, 2, 3)$ be identified with a set of normal coordinates in a shell space and its certain neighbourhood, the midsurface being determined by equation $u^3 = 0$, and u^α $(\alpha = 1, 2)$ being the curvature coordinates. Lame's coefficients corresponding to a given coordinate system are:

$$H_\alpha = (1 + u^3 K_\alpha) A_\alpha, \quad H_3 = 1, \tag{14}$$

where $A_\alpha = A_\alpha(u^\beta)$ and $K_\alpha = K_\alpha(u^\beta)$ $(\beta = 1, 2)$, K_α being the principal curvatures of the midsurface.

We put the components of the magnetostatic induction vector in the similar form:

$$B_i = (1 - u^3 \eta_i) B_{0i}, \tag{15}$$

where B_{0i} and η_i depend only on u^β, η_i being of the order K_α. Naturally, for plates $\eta_i = 0$ (cf. [4], [6]). Here and afterwards the vectors indicated by the subscript „0" are identical to their space counterparts evaluated at $u^3 = 0$. In [5] the components B_i are, in general, arbitrary functions of u^3 (for plates too), thereby the complexity of the governing equations of the shell theory becomes too great, even if shallow shells are concerned.

We adopt the classical assumptions of the Kirchhoff-Love's theory (also known as Love's first approximation). Thus, in particular, we may write:

$$w_\alpha = w_{0\alpha} + u^3 \chi_\alpha, \quad w_3 = w_{03}, \tag{16}$$

where:

$$\chi_\alpha = -\frac{w_{0,3\alpha}}{A_\alpha} + K_\alpha w_{0\alpha}, \tag{17}$$

are the angles of rotation of the straight lines perpendicular to the midsurface; comma represents differentiation with respect to the appropriate coordinate.

Furthermore, we introduce the electromagnetic approximation which corresponds to that of [6]:

$$e_\alpha = e_{0\alpha} + \frac{u^3}{h} s_\alpha, \quad b_3 = b_{03} + \frac{u^3}{h} g, \tag{18}$$

where $e_{0\alpha}$, b_{03}, s_α and g depend on u^β and t, h being the shell thickness. Above electromagnetic hypothesis, unlike the former hypothesis in which case $s_\alpha = 0$ and $g = 0$ (cf. [4]), is in agreement with (16). Note that in termoelastic study [8] the thickness distribution of the temperature field is also approximated by a linear formula.

Taking all the foregoing assumptions into account, we formulate the shell theory as an approximation of the magnetoelasticity theory, in final formulae the terms of the order hK_α being omitted. First, integration of (1) across the shell thickness, i.e., with respect to u^3 between the limits $-h/2$ to $h/2$, after using (14) and (15), yields:

$$\begin{gathered} B_{03,\alpha} - (K_\alpha - \eta_\alpha) A_\alpha B_{0\alpha} = 0, \quad (A_1 B_{01})_{,2} - (A_2 B_{02})_{,1} = 0, \\ (A_2 B_{01})_{,1} + (A_1 B_{02})_{,2} + (K_1 + K_2 - \eta_3) A_1 A_2 B_{03} = 0. \end{gathered} \tag{19}$$

From (4), and the use of (15), (16) and $(18)_1$, the tangent components of the current density vector at any point of the shell region and at any point of the midsurface read, respectively:

$$\begin{gathered} j_\alpha = j_{0\alpha} + u^3 \lambda \left\{ \frac{s_\alpha}{h} + (-1)^\nu [B_{03}(\dot{\chi}_\nu - \eta_3 \dot{w}_{0\nu}) + \eta_\nu B_{0\nu} \dot{w}_{03}] \right\}, \\ j_{0\alpha} = \lambda [e_{0\alpha} + (-1)^\alpha (B_{0\nu} \dot{w}_{03} - B_{03} \dot{w}_{0\nu})], \end{gathered} \tag{20}$$

where the notation $\nu = 3 - \alpha$ (cf. [9]) is employed. Introducing $(20)_1$ and $(18)_2$ to the first two scalar equations of $(2)_2$, and integrating these equations across the shell thickness, we obtain:

$$j_{0\alpha} = \frac{(-1)^\nu}{\mu} \left(\frac{1}{A_\nu} b_{03,\nu} - \frac{1}{h} b^r_\nu \right), \tag{21}$$

where:

$$b^r_\alpha = b^+_\alpha - b^-_\alpha, \quad b^s_\alpha = b^+_\alpha + b^-_\alpha, \tag{22}$$

the quantities indicated by a superscripts "$\pm$" being referred to the surfaces $u^3 = \pm h/2$. Applying to the same scalar equations of $(2)_2$ the integral operator:

$$\int\limits_0^{u^3} du^3 - \frac{1}{2} \left(\int\limits_0^{h/2} du^3 + \int\limits_0^{-h/2} du^3 \right),$$

we find:

$$\begin{aligned} b_\alpha = \frac{1}{2} b^s_\alpha + \frac{u^3}{h} b^r_\alpha - \frac{h}{8} \left[1 - 4 \left(\frac{u^3}{h} \right)^2 \right] \bigg\{ & \frac{1}{A_\alpha} g_{,\alpha} - (-1)^\alpha \mu (\lambda s_\nu + hK_\alpha j_{0\nu}) + \\ & - \mu \lambda h [B_{03}(\dot{\chi}_\alpha - \eta_3 \dot{w}_{0\alpha}) + \eta_\alpha B_{0\alpha} \dot{w}_{03}] \bigg\}. \end{aligned} \tag{23}$$

The third scalar equation deduced from $(2)_2$ and evaluated at $u^3 = \pm h/2$, on account of (8), i.e., $j_3^{\pm} = 0$, leads to:

$$(A_2 b_2^r)_{,1} - (A_1 b_1^r)_{,2} = (A_2 b_2^s)_{,1} - (A_1 b_1^s)_{,2} = 0. \tag{24}$$

Substituting (23) into the third scalar equation of $(2)_2$, and making use of (24), we arrive at:

$$j_3 = \left[1 - 4\left(\frac{u^3}{h}\right)^2\right] j_{03}, \tag{25}$$

where:

$$\begin{aligned} j_{03} = {} & \frac{\lambda h^2}{8A_1 A_2}\left\{A_2\left[B_{03}(\dot{\chi}_2 - \eta_3 \dot{w}_{02}) + \eta_2 B_{02}\dot{w}_{03} + K_2 \frac{j_{01}}{\lambda}\right]\right\}_{,1} + \\ & - \frac{\lambda h^2}{8A_1 A_2}\left\{A_1\left[B_{03}(\dot{\chi}_1 - \eta_3 \dot{w}_{01}) + \eta_1 B_{01}\dot{w}_{03} - K_1 \frac{j_{02}}{\lambda}\right]\right\}_{,2} + \lambda h \frac{s}{8}, \end{aligned} \tag{26}$$

the unknown s being defined by:

$$s = \frac{1}{A_1 A_2}\left[(A_2 s_1)_{,1} + (A_1 s_2)_{,2}\right]. \tag{27}$$

On comparing of $(20)_2$ and (21) we have:

$$e_{0\alpha} = (-1)^{\alpha}\left[B_{03}\dot{w}_{0\gamma} - B_{0\gamma}\dot{w}_{03} - \frac{1}{\mu\lambda}\left(\frac{b_{03,\gamma}}{A_\gamma} - \frac{b_\gamma^r}{h}\right)\right]. \tag{28}$$

Putting (4) in the inverted form and using (15), (16), and (25) we obtain:

$$e_3 = B_{01}\dot{w}_{02} - B_{02}\dot{w}_{01} + u^3(B_{01}\dot{\chi}_2 - B_{02}\dot{\chi}_1) + \left[1 - 4\left(\frac{u^3}{h}\right)^2\right]\frac{j_{03}}{\lambda}. \tag{29}$$

Thus, through (18), $(20)_1$, (23), (25), and (29) the thickness distribution of the quantities: e_i, b_i and j_i has been established. Now we proceed to deduce three electromagnetic equations expressed entirely in terms of the following unknowns: w_{0i}, j_{03}, b_{03}, g, b_α^r, b_α^s and its derivates.

Integration of $(2)_{1,4}$ across the shell thickness, after use of (14), (15), (16), (18), (23), and (29), gives, rspectively:

$$\begin{aligned} & \left(1 + \mu\lambda \frac{h^2}{12}\frac{\partial}{\partial t}\right)\left[(-1)^{\alpha}\left(\frac{s_\alpha}{h} + K_\gamma \frac{j_{0\alpha}}{\lambda} - B_{03}(\dot{\chi}_\gamma - \eta_3 \dot{w}_{0\gamma}) - \eta_\gamma B_{0\gamma}\dot{w}_{03}\right]\right. + \\ & + \frac{h}{12}\frac{\dot{g}_{,\gamma}}{A_\gamma} - \frac{\dot{b}_\gamma^s}{2} + \frac{(-1)^{\alpha}}{\lambda}\left[(K_\alpha - K_\gamma)j_{0\alpha} - \frac{2}{3}\frac{j_{03,\alpha}}{A_\alpha}\right] + \frac{1}{A_\alpha}(B_{0\alpha}\dot{w}_{0\gamma} - B_{0\gamma}\dot{w}_{0\alpha})_{,\alpha} + \\ & + B_{03}[\dot{\chi}_\gamma - (\eta_3 - K_\alpha)\dot{w}_{0\gamma}] + B_{0\gamma}(\eta_\gamma - K_\alpha)\dot{w}_{03} = 0, \\ & (A_2 e_{02})_{,1} - (A_1 e_{01})_{,2} + A_1 A_2 \dot{b}_{03} + \\ & + \frac{h}{12}(K_2 A_2 s_2)_{,1} - (K_1 A_1 s_1)_{,2} + (K_1 + K_2)A_1 A_2 \dot{g}] = 0, \end{aligned} \tag{30}$$

$$\left(1-\frac{h^2}{12}\Delta\right)g+h(K_1+K_2)b_{03}+\frac{h}{2A_1A_2}[(A_2b_1^s)_{,1}+(A_1b_2^s)_{,2}]-\mu\lambda\frac{h^3}{12}\Delta_B\dot{w}_{03}+$$
$$+\frac{\mu\lambda h^3}{12A_1A_2}\left\{A_1\left[(K_2-\eta_3)B_{03}\dot{w}_{02}+\eta_2B_{02}\dot{w}_{03}+K_2\frac{j_{01}}{\lambda}+\frac{s_1}{h}\right]\right\}_{,2}+ \qquad \begin{matrix}(30)\\ [\text{cd.}]\end{matrix}$$
$$+\frac{\mu\lambda h^3}{12A_1A_2}\left\{A_2\left[(K_1-\eta_3)B_{03}\dot{w}_{01}+\eta_1B_{01}\dot{w}_{03}-K_1\frac{j_{02}}{\lambda}-\frac{s_2}{h}\right]\right\}_{,1}=0,$$

where:

$$\Delta=\frac{1}{A_1A_2}\left[\frac{\partial}{\partial u^1}\left(\frac{A_2}{A_1}\frac{\partial}{\partial u^1}\right)+\frac{\partial}{\partial u^2}\left(\frac{A_1}{A_2}\frac{\partial}{\partial u^2}\right)\right],$$
$$\Delta_B=\frac{1}{A_1A_2}\left[\frac{\partial}{\partial u^1}\left(B_{03}\frac{A_2}{A_1}\frac{\partial}{\partial u^1}\right)+\frac{\partial}{\partial u^2}\left(B_{03}\frac{A_1}{A_2}\frac{\partial}{\partial u^2}\right)\right].$$

Multiplying $(30)_1$ by A_γ, next taking the partial derivate with respect to u^α, and combining both results, i.e., for $\alpha = 1$ and $\alpha = 2$, in view of (26) and (27) we arrive at the first desired equation:

$$\left[1-\frac{h^2}{12}\left(\Delta-\mu\lambda\frac{\partial}{\partial t}\right)\right]j_{03}+\lambda\frac{h^2}{8}\left[\Delta(B_{02}\dot{w}_{01}-B_{01}\dot{w}_{02})+(K_1-K_2)\left(\frac{B_{02}}{A_1}\frac{\partial}{\partial u^1}+\right.\right.$$
$$\left.\left.+\frac{B_{01}}{A_2}\frac{\partial}{\partial u^2}\right)\dot{w}_{03}\right]+\frac{\lambda h^2}{8A_1A_2}\{[(K_1-\eta_2)A_2B_{02}]_{,1}-[(K_2-\eta_1)A_1B_{01}]_{,2}\}\dot{w}_{03}+$$
$$+\frac{\lambda h^2}{8A_1A_2}\{[(K_1+K_2-\eta_3)A_1B_{03}\dot{w}_{01}]_{,2}-[(K_1+K_2-\eta_3)A_2B_{03}\dot{w}_{02}]_{,1}\}+ \qquad (31)$$
$$+\frac{h^2}{8\mu A_1A_2}\left\{\left[(K_1-K_2)\left(b_{03,2}-\frac{A_2}{h}b_2^r\right)\right]_{,1}-\left[(K_2-K_1)\left(b_{03,1}-\frac{A_1}{h}b_1^r\right)\right]_{,2}\right\}=0.$$

Applying approximations: $h(K_\alpha s_\alpha)_{,\gamma} \ll 12e_{0\alpha,\gamma}$ and $hK_\alpha g \ll 12b_{03}$, and replacing $e_{0\alpha}$ by (28), we simplify $(30)_2$, to:

$$\left(\Delta-\mu\lambda\frac{\partial}{\partial t}\right)b_{03}-\frac{1}{A_1A_2h}[(A_2b_1^r)_{,1}+(A_1b_2^r)_{,2}]+$$
$$+\frac{\mu\lambda}{A_1A_2}[(A_2B_{01}\dot{w}_{03})_{,1}+(A_1B_{02}\dot{w}_{03})_{,2}-(A_1B_{03}\dot{w}_{02})_{,2}-(A_2B_{03}\dot{w}_{01})_{,1}]=0. \qquad (32)$$

In order to express the terms with s_α in $(30)_3$ by the desired unknowns we return to the scalar equation $(2)_1$ corresponding to u^3—direction. Integration of this equation with the weight one across the shell thickness results in:

$$(A_2s_2)_{,1}-(A_1s_1)_{,2}=h[(A_1K_1e_{01})_{,2}-(A_2K_2e_{02})_{,1}]-A_1A_2[h(K_1+K_2)\dot{b}_{03}+\dot{g}].$$

With the aid of this above formula, by application of (28) and (21), and under the assumptions: $h^3K_\alpha\Delta b_{03} \ll 12g$ and $h^3K_{\alpha,\beta}b_{03,\beta} \ll 12A_1A_2g$ equation $(30)_3$ becomes:

$$\left[1-\frac{h^2}{12}\left(\Delta-\mu\lambda\frac{\partial}{\partial t}\right)\right]g+h(K_1+K_2)\left(1+\mu\lambda\frac{h^2}{12}\frac{\partial}{\partial t}\right)b_{03}+$$
$$+\frac{h}{2A_1A_2}[(A_2b_1^s)_{,1}+(A_1b_2^s)_{,2}-\mu\lambda\frac{h^3}{12}\Delta_B\dot{w}_{03}+ \tag{33}$$
$$+\frac{\mu\lambda h^3}{12A_1A_2}\{[(\eta_1-K_2)A_2B_{01}\dot{w}_{03}]_{,1}+[(\eta_2-K_1)A_1B_{02}\dot{w}_{03}]_{,2}+$$
$$+[(K_1+K_2-\eta_3)A_2B_{03}\dot{w}_{01}]_{,1}+[(K_1+K_2-\eta_3)A_1B_{03}\dot{w}_{02}]_{,2}\}=0.$$

The equations (31) to (33) may be futher transform with the aid of (19). According to (32) and (33) $B_3=0$ is the only case in that the unknown g is negligible in comparison with the unknown b_{03}.

After using of (14) to (17) from (6) we obtain:

$$\varrho_e=2\in_0(B_{01}\dot{\chi}_2-B_{02}\dot{\chi}_1+B_{03}\dot{\delta}),$$

where:

$$\delta=\frac{1}{2A_1A_2}[(A_1w_{01})_{,2}-(A_2w_{02})_{,1}],$$

denotes the angle of in-plane rotation of an infinitezimal midsurface element.

Keeping in mind (10), we supplement the equations of motion of a shell (cf. [9]) by the terms due to (11), namely:

$$\begin{gathered}(A_\nu N_\alpha)_{,\alpha}+(A_\alpha N_{\nu\alpha})_{,\nu}+A_{\alpha,\nu}N_{\alpha\nu}-A_{\nu,\alpha}N_\nu+A_\alpha A_\nu(K_\alpha Q_\alpha+p_\alpha-\varrho h\ddot{w}_{0\alpha}+f_\alpha)=0,\\ (A_2Q_1)_{,1}+(A_1Q_2)_{,2}-A_1A_2(K_1N_1+K_2N_2-p_3+\varrho h\ddot{w}_{03}-f_3)=0,\\ (A_\nu M_\alpha)_{,\alpha}+(A_\alpha M_{\nu\alpha})_{,\nu}+A_{\alpha,\nu}M_{\alpha\nu}-A_{\nu,\alpha}M_\nu-A_\alpha A_\nu(Q_\alpha-f_{\alpha+3})=0,\end{gathered} \tag{34}$$

where:

$$f_i=\int_{-h/2}^{h/2}l_i du^3,\quad f_{\alpha+3}=\int_{-h/2}^{h/2}l_\alpha u^3du^3, \tag{35}$$

N_α, $N_{\alpha\nu}$, Q_α, M_α and $M_{\alpha\nu}$ are the stress resultants and couple resultants, respectively, p_i the components of the mechanical load per unit area of the midsurface. Introducing (11) into (35) through (15), (20), (25) and (21), we find:

$$\begin{aligned}f_\alpha&=-\frac{B_{03}}{\mu}\left(h\frac{b_{03,\alpha}}{A_\alpha}-b_\alpha^r\right)+(-1)^\alpha\frac{2}{3}hB_{0\nu}j_{03},\\ f_3&=\frac{1}{\mu}\left[B_{01}\left(h\frac{b_{03,1}}{A_1}-b_1^r\right)+B_{02}\left(h\frac{b_{03,2}}{A_2}-b_2^r\right)\right],\\ f_{\alpha+3}&=\lambda B_{03}\frac{h^3}{12}\left[(-1)^\nu\frac{s^\nu}{h}-B_{03}(\dot{\chi}_\alpha-\eta_3\dot{w}_{0\alpha})-\eta_\alpha B_{0\alpha}\dot{w}_{03}\right].\end{aligned} \tag{36}$$

After some manipulation and simplification, using $(30)_3$, (21), $(20)_2$ and (19), we arrive at:

$$f_\alpha+K_\alpha f_{\alpha+3}\approx f_\alpha,$$
$$f_3+\frac{1}{A_1A_2}[(A_2f_4)_{,1}+(A_1f_5)_{,2}]\approx \tag{37}$$

$$\approx f_3 + \frac{B_{03}}{\mu}\left\{g + h(K_1+K_2)b_{03} + \frac{h}{2A_1A_2}[(A_2 b_1^s)_{,1} + (A_1 b_2^s)_{,2}]\right\}.$$

The mechanical constitutive equations have the form (cf. [9]):

$$\begin{aligned} N_\alpha &= C(\varepsilon_\alpha + \nu\varepsilon_\beta), & N_{\alpha\beta} &= C(1-\nu)\left(\frac{\omega}{2} + \frac{h^2}{12}K_\beta \tau\right), \\ M_\alpha &= D(\varkappa_\alpha + \nu\varkappa_\alpha), & M_{\alpha\beta} &= D(1-\nu)\tau, \end{aligned} \tag{38}$$

where:

$$C = \frac{Eh}{1-\nu^2} \quad \text{and} \quad D = C\frac{h^2}{12},$$

denote the stretching and bending stiffness, respectively; ε_α, ω, $\varkappa_\alpha$ and τ being the measures of strain. The components of the midsurface displacement vector may be introduced into the formulae (38) by using of the geometrical relations:

$$\begin{aligned} \varepsilon_\alpha &= \frac{1}{A_\alpha}\left(w_{0\alpha,\alpha} + \frac{A_{\alpha,\beta}}{A_\beta} w_{0\beta}\right) + K_\alpha w_{03}, & \omega &= \omega_1 + \omega_2, \\ \varkappa_\alpha &= \frac{1}{A_\alpha}\left(\chi_{\alpha,\alpha} + \frac{A_{\alpha,\beta}}{A_\beta}\chi_\Delta\right), & \tau &= \frac{1}{2}(\tau_1 + \tau_2 - K_1\omega_2 - K_2\omega_1), \end{aligned} \tag{39}$$

where:

$$\omega_\alpha = \frac{1}{A_\alpha}\left(w_{0\beta,\alpha} - \frac{A_{\alpha,\beta}}{A_\beta} w_{0\alpha}\right), \quad \tau_\alpha = \frac{1}{A_\alpha}\left(\chi_{\beta,\alpha} - \frac{A_{\alpha,\beta}}{A_\beta}\chi_\alpha\right),$$

χ_α being determined by (17).

After expressing the transverse shear forces Q_α by means of $(34)_3$ and making use of (36) to (39), from $(34)_{1,2}$ we obtain three equations of motion which together with electromagnetic equations (31) to (33) may be put in the form:

$$\begin{aligned} & L_k^1 w_{01} + L_k^2 w_{02} + L_k^3 w_{03} + L_k^4 j_{03} + L_k^5 b_{03} + L_k^6 g + \\ & + L_k^7 b_1^r + L_k^8 b_2^r + L_k^9 b_1^s + L_k^{10} b_2^s = p_k, \quad (k = 1, \dots, 6) \end{aligned} \tag{40}$$

where $p_4 = p_5 = p_6 = 0$ and $L_1^8 = L_2^7 = L_\alpha^6 = L_\alpha^9 = L_\alpha^{10} = L_3^4 = L_4^6 = L_4^9 = L_4^{10} = L_5^4 = L_5^6 = L_5^9 = L_5^{10} = L_6^4 = L_6^7 = L_6^8 = 0$. The nonvanishing differential operators are not recorded for the sake of brevity. Since the right-hand sides of $(36)_{1,2}$ and (37), unlike in [5], contain no displacement terms, the operators L_{ij} $(i, j = 1, 2, 3)$ are affected by none component of the magnetostatic induction vector.

On account of extra unknowns: b_α^r and b_α^s the six equations (40) must be supplemented by:

$$\operatorname{rot}\bar{\tilde{\boldsymbol{b}}} = \boldsymbol{0}, \quad \operatorname{div}\bar{\tilde{\boldsymbol{b}}} = 0, \tag{41}$$

where $\tilde{\boldsymbol{b}} = \tilde{\boldsymbol{b}}(x^i, t)$ denotes the magnetic induction vector in the upper and lower region of vacuum, coordinates x^i being referred to both regions of vacuum and the shell space. We assume that there exists one-to-one transformation $x^i = x^i(u^j)$ as far as the certain neighbourhoods of the upper and lower shell faces are concerned. The magnetic induction in vacuum regions may be found by solving (41) under the continuity conditions $(7)_4$ for $u^3 = \pm h/2$, namely:

$$b_{03} \pm \frac{1}{2} g = \tilde{b}_3|_{u^3 = \pm h/2} \tag{42}$$

As a shell is made of a real conductor, the upper and lower faces carry no currents, thereby in view of $(7)_2$ (with $j^p = 0$) the unknowns (22) are identical to its vacuum counterparts, i.e.:

$$b^r_\alpha = \tilde{b}_\alpha|_{u^3 = h/2} - \tilde{b}_\alpha|_{u^3 = -h/2},$$
$$b^s_\alpha = \tilde{b}_\alpha|_{u^3 = h/2} + \tilde{b}_\alpha|_{u^3 = -h/2}.$$

The set of equations (40) may be reduced, provided the initial magnetostatic field satisfies the restriction $B_\alpha B_3 = 0$. If $B_3 = 0$, then unknowns: g, b^s_α and equation (33) may be set aside. Correspondingly, the second term in l.h.s. of (42) is to omit, for in this case unknown g is negligible in comparison with unknown b_{03}. If $B_\alpha = 0$, then unknown j_{03} and equation (31) may be set aside.

The order of the differential equations (40) with respect to basic unknowns: w_{0i}, j_{03}, b_{03} and g is fourteen, therefore the boundary conditions at each edge are seven (four of mechanical origin and three of electromagnetic origin). In sequel we shall consider the edge $u^\alpha = \text{const}$. The geometrical conditions are the same as in the Kirchhoff-Love's theory, namely:

$$w_{0i} = \hat{w}_{0i}, \qquad \chi_\alpha = \hat{\chi}_\alpha.$$

Above and afterwards the boundary quantities known a priori are indicated by "∧" The kinetic conditions may be written down as follows:

$$N_\alpha = \hat{N}_\alpha + \int_{-h/2}^{h/2} S_\alpha du^3,$$

$$N_{\alpha\nu} + K_\alpha M_{\alpha\nu} = \hat{N}_\nu + K_\alpha \hat{M}_{\alpha\nu} + \int_{-h/2}^{h/2} S_\nu du^3,$$

$$Q_\alpha + \frac{M_{\alpha\nu,\nu}}{A_\nu} = \hat{Q}_\alpha + \frac{\hat{M}_{\alpha\nu,\nu}}{A_\nu} + \int_{-h/2}^{h/2} S_3 du^3 + \int_{-h/2}^{h/2} \frac{S_{\nu,\nu}}{A_\nu} u^3 du^3,$$

$$M_\alpha = \hat{M}_\alpha + \int_{-h/2}^{h/2} S_\alpha u^3 du^3.$$

In accord with (12), (13) and $(7)_2$, and under the assumption that the permeability is continuous, the terms due to electromagnetic surface tractions read:

$$S_\alpha = -\frac{1}{\mu}[B_{0\nu}(\hat{b}_\nu - b_\nu) + B_{03}(\hat{b}_3 - b_3)],$$

$$S_\nu = \frac{B_{0\alpha}}{\mu}(\hat{b}_\nu - b_\nu), \qquad S_3 = \frac{B_{0\alpha}}{\mu}(\hat{b}_3 - b_3).$$

In the case considered (perfectly conductible outward medium) the electromagnetic conditions follow from (9) which for the edge u^α = const. is equivalent to:

$$j_3 = 0. \quad j_\nu = 0.$$

Integration of the above equations across the shell thickness, with the aid of (20), (21) and (25), leads to:

$$\begin{gathered} j_{03} = 0, \quad \frac{h}{A_\alpha} b_{03,\alpha} - b_\alpha^r = 0, \\ (-1)^\nu \frac{s_\nu}{h} - B_{03}(\dot{\chi}_\alpha - \eta_3 \dot{w}_{0\alpha}) - \eta_\alpha B_{0\alpha} \dot{w}_{03} = 0. \end{gathered} \tag{43}$$

Making use of $(43)_{1,2}$, $(30)_1$ and (21), we change $(43)_3$ to:

$$\begin{gathered} \frac{h}{12A_\alpha} g_{,\alpha} - \frac{1}{2} b_\alpha^s + \frac{1}{A_\nu}(B_{0\nu} w_{0\alpha} - B_{0\alpha} w_{0\nu})_{,\nu} + \\ + (\eta_\alpha - K_\nu) B_{0\alpha} w_{03} + B_{03}[\chi_\alpha - (\eta_3 - K_\nu) w_{0\alpha}] = 0. \end{gathered} \tag{44}$$

Note that through (15), (16) and (23), (44) may be put in the form:

$$\int_{-h/2}^{h/2} [b_\alpha - \mathrm{rot}_\alpha(\boldsymbol{w} \times \boldsymbol{B})] du^3 = 0.$$

4. Final remarks

The modification (18) of the electromagnetic hypothesis hitherto applied in the shell theory consists in the inclusion of the antisymmetrical terms with respect to the thickness coordinate. This correction results in a full coupling of the differential governing equations (40). Namely, each one of displacement equations of motion influenced by the magnetostatic field (cf. (36) and (37)) is mutually coupled at least with one (in general with two) of the electromagnetic equations (31) to (33). Such a coupling does not appear when the former hypothesis is invoked, i.e., when $s_\alpha = 0$ and $g = 0$, and as it was proved in [6] for the problem of transverse vibrations of a plate, the resulting solutions are false for lack of this coupling. Note that in magnetoelasticity theory the equation of motion (10) is coupled with the Maxwell's equations (2) in every case of magnetostatic field. In the present paper, unlike in [5], all the Maxwell's equations (2) in scalar form have been used during the derivation of the equations of the shell theory. The modification (18) involves an increase both in the number of unknowns and equations. Nonetheless, due to (15) and the elimination of the tangent components of the perturbed electric field vector the final equations attain a relatively simple form. The termoelastic effects may be added as in [8].

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Резюме

МОДИФИЦИРОВАННАЯ ГИПОТЕЗА ТЕОРИИ ТОНКИХ МАГНИТОУПРУГИХ ОБОЛОЧЕК

Из уравнений теории магнитоупругости выведены уравнения колебаний тонких проводящих оболочек при допущении, что тангенциальные компоненты вектора напряженности возбуждаемого электрического поля и нормальная компонента вектора напряженности возбуждаемого магнитного поля изменяются линейно по толщине оболочки. Подтверждено, что гипотеза Амбарцумяна-Багдасаряна-Белубекяна слишком сильная в общем случае магнитостатического поля.

Streszczenie

ZMODYFIKOWANA HIPOTEZA TEORII CIENKICH POWŁOK MAGNETOSPRĘŻYSTYCH

Z równań teorii magnetosprężystości wyprowadzono równania drgań cienkich powłok przewodzących, zakładając że składowe styczne wektora natężenia wzbudzonego pola elektrycznego i składowa normalna wektora natężenia wzbudzonego pola magnetycznego zmieniają się liniowo na grubości powłoki. Przyjęto, że materiał powłoki ma właściwości diamagnetyczne lub paramagnetyczne. Potwierdzono, że hipoteza Ambarcumiana-Bagdasariana-Biełubiekiana jest zbyt silna w ogólnym przypadku pola magnetostatycznego.

Praca wpłynęła do Redakcji dnia 26 stycznia 1988 roku.