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THERMAL STRESSES IN A LONG IN-HOMOGENEOUS AELOTROPIC CYLINDER SUBJECTED TO ν-RAY HEATING

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1. Introduction

In recent years, an intensive attention has been paid to the determination of thermal stresses in isotropic cylinders subjected to internal heat generation due to axisymmetric radiation. The aim of this paper is to find the thermal stresses in the case of an inhomogeneous transversely isotropic long hollow cylinder, the outer curved surface of which is perfectly insulated and the source of generation of heat being due to ν-ray radiation. For the non-homoneity of the material, it is assumed that the elastic constants, co-efficients of thermal expansion and thermal conductivity vary linearly with the radial distance. Finally, the author has shown numerically, for the material magnesium, that the hoop — stress on the inner boundary gradually increases as the thickness of the cylinder increases for arbitrary values of absorption constant.

2. Formulation and solution of the problem, distribution of temperature

We use the cylindrical co-ordinates and take the z-axis coinciding with the axis of the cylinder. Let the temperature be symmetrical about the axis of the cylinder and be independent of axial co-ordinate. If H denotes the rate at which heat is generated in the vessel, we have the following law [1]:

$$H = H_i e^{-\mu(r-a)}, \tag{1}$$

where H_i = heat generation rate on the inside wall of the cylinder, a = inside radius and μ = the absorption coefficient for ν-ray energy.

For the present problem, the temperature satisfies the conductivity equation [4]:

$$K\left(\frac{d^2T}{dr^2}+\frac{1}{r}\frac{dT}{dr}\right)+\frac{dK}{dr}\cdot\frac{dT}{dr} = H_i e^{-\mu(r-a)}, \tag{2}$$

where K = thermal conductivity of the material.

For inhomogeneity of the material we assume:

$$K = K_0 r, \tag{3}$$

where K_0 is a non-zero positive constant.

Using (3) in (2) we obtain:

$$r\frac{d^2T}{dr^2}+2\frac{dT}{dr} = -\frac{H_i}{K_0}\,e^{-\mu(r-a)}. \tag{4}$$

The outer wall being insulated and the inner wall kept at a constant temperature, the boundary conditions are:

$$\left.\begin{aligned} T &= T_i \quad \text{on} \quad r = a \\ \text{and} \quad \frac{dT}{dr} &= 0 \quad \text{on} \quad r = b \end{aligned}\right\}. \tag{5}$$

The solution of equation (4) is:

$$T = -\frac{H_i}{K_0\mu^2}\cdot\frac{e^{-\mu(r-a)}}{r}-\frac{A}{r}+B, \tag{6}$$

where A and B are constants.
Using (5) in (6) we get:

$$T = T_i+p\left[\frac{1-\lambda}{a}+\frac{\lambda}{r}-\frac{1}{r}\,e^{-\mu(r-a)}\right], \tag{7}$$

where

$$\lambda = (1+\mu b)e^{-\mu(b-a)} \quad \text{and} \quad p = \frac{H_i}{K_0\mu^2}. \tag{8}$$

3. Stress distribution

We assume that the axial displacement is zero throughout so that considering the axially symmetric character of the problem, the nonvanishing components of stress tensors are σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz} and σ_{rz}.

Thus the stress-strain relation for transversely isotropic material are given by [5]:

$$\begin{aligned} \sigma_{rr} &= c'_{11}e_{rr}+c'_{12}e_{\theta\theta}+c'_{13}e_{zz}-b'_1T, \\ \sigma_{\theta\theta} &= c'_{12}e_{rr}+c'_{11}e_{\theta\theta}+c'_{13}e_{zz}-b'_1T, \\ \sigma_{zz} &= c'_{13}e_{rr}+c'_{13}e_{\theta\theta}+c'_{33}e_{zz}-b'_2T, \\ \sigma_{rz} &= c'_{44}e_{rz}, \end{aligned} \tag{9}$$

where $b'_1 = (c'_{11}-c'_{12})\alpha'_1+c'_{13}\alpha'_2$ and $b'_2 = 2c'_{13}\alpha'_1+c'_{33}\alpha'_2$, c'_{ij} are elastic constants and functions of r. T is the temperature at a point (r, θ, z) and α'_1 and α'_2 are the coefficients of linear expansion along and perpendicular to the z-axis, respectively.

Considering the axisymmetric character of the problem, the strain components are given by:

$$e_{rr} = \frac{\partial u}{\partial r},\quad e_{\theta\theta} = \frac{u}{r},\quad e_{zz} = \frac{\partial w}{\partial z},\quad e_{rz} = \frac{\partial u}{\partial z}+\frac{\partial w}{\partial r},$$

where $u_r = u$, $u_\theta = 0$, $u_z = w$.

Assuming u to be dependent on r along and $w = 0$ the above components reduce to:

$$e_{rr} = \frac{du}{dr}, \quad e_{\theta\theta} = \frac{u}{r}, \quad e_{zz} = 0, \quad e_{rz} = 0. \tag{10}$$

For nonhomogeneity of the material we assume:

$$c'_{ij} = c_{ij} r, \quad \alpha'_i = \alpha_i r, \tag{11}$$

where c_{ij} and α_i are non-zero positive constants.

The relations (9) with (10) and (11) reduce to:

$$\begin{aligned} \sigma_{rr} &= c_{11} r \frac{du}{dr} + c_{13} u - b_1 r^2 T, \\ \sigma_{\theta\theta} &= c_{12} r \frac{du}{dr} + c_{11} u - b_1 r^2 T, \\ \sigma_{zz} &= c_{13} r \frac{du}{dr} + c_{13} u - b_2 r^2 T, \\ \sigma_{rz} &= 0, \end{aligned} \tag{12}$$

where:

$$b_1 = (c_{11} + c_{12})\alpha_1 + c_{13}\alpha_2, \quad b_2 = 2c_{13}\alpha_1 + c_{33}\alpha_2. \tag{13}$$

The stress-equations of equilibrium in absence of the body forces are (Timoshenko and Goodier):

$$\begin{aligned} \frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} &= 0, \\ \frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} &= 0. \end{aligned} \tag{14}$$

The second equation of (14) authomatically holds and the first, by (12) and (7) becomes:

$$r^2 \frac{d^2 u}{dr^2} + 2r \frac{du}{dr} + \left(\frac{c_{12}}{c_{11}} - 1\right) u = \frac{b_1}{c_{11}} \left[2\left\{T_i + \frac{1-\lambda}{a} p\right\} r^2 + \lambda p r - p(1-\mu r) r e^{-\mu(r-a)} \right]. \tag{15}$$

The particular integral of equation (15) [3] is:

$$u = \frac{b_1}{c_{11}} \left[\frac{2\left(T_i + \frac{1-\lambda}{a} p\right)}{s+6} r^2 + \frac{\lambda p}{s+2} r + p \frac{e^{\mu a}}{\beta_2 - \beta_1} \left\{ r^{\beta_1} \int_a^r e^{-\beta_1} (1-\mu r) e^{-\mu r} dr - r^{\beta_2} \int_a^r r^{-\beta_2} (1-\mu r) e^{-\mu r} dr \right\} \right],$$

where $\beta_1, \beta_2 = \frac{-1 \pm \sqrt{1-4s}}{2}$ and $s = \frac{c_{12}}{c_{11}} - 1$,

so that $\beta_1 + \beta_2 = -1$.

Thus the general solution of (15) is:

$$u = A_1 r^{\beta_1} + A_2 r^{\beta_2} + \frac{b_1}{c_{11}} \left[\frac{2\left(T_i + \frac{1-\lambda}{a} p\right)}{s+6} r^2 + \frac{\lambda p}{s+2} r + \right.$$

$$\left. + p \frac{e^{\mu a}}{\beta_2 - \beta_1} \left\{ r^{\beta_1} \int_a^r r^{-\beta_1}(1-\mu r) e^{-\mu r} dr - r^{\beta_2} \int_a^r r^{-\beta_2}(1-\mu r) e^{-\mu r} dr \right\} \right], \tag{16}$$

where A_1 and A_2 are constants.

Thus the stresses as calculated from (12) are:

$$\sigma_{rr} = c_{11}(m_1 A_1 r^{\beta_1} + m_2 A_2 r^{\beta_2}) + b_1 \left[\frac{s}{s+6} \left(T_i + \frac{1-\lambda}{a} p \right) r^2 + \right.$$

$$\left. + p \frac{e^{\mu a}}{\beta_2 - \beta_1} \{ m_1 F_1(r) - m_2 F_2(r) \} \right],$$

$$\sigma_{\theta\theta} = c_{11}(n_1 A_1 r^{\beta_1} + n_2 A_2 r^{\beta_2}) + b_1 \left[\frac{3s}{s+6} \left(T_i + \frac{1-\lambda}{a} p \right) r^2 + rpe^{-\mu(r-a)} + \right.$$

$$\left. + p \frac{e^{\mu a}}{\beta_2 - \beta_1} \{ n_1 F_1(r) - n_2 F_2(r) \} \right], \tag{17}$$

$$\sigma_{zz} = c_{13}(l_1 A_1 r^{\beta_1} + l_2 A_2 r^{\beta_2}) + T_i(6l - b_2) r^2 + p \left[(6l - b_2) \frac{1-\lambda}{a} r^2 + \right.$$

$$\left. + \{(2l - b_2)\lambda + b_2 e^{-\mu(b-a)}\} r + b_1 \frac{c_{12}}{c_{11}} \cdot \frac{e^{\mu a}}{\beta_2 - \beta_1} \{ l_1 F_1(r) - l_2 F_2(r) \} \right],$$

where: $m_i = s + 1 + \beta_i, \quad n_i = (s+1)\beta_i + 1,$

$$l_i = \beta_i + 1, \qquad l = \frac{b_1}{s+6} \frac{c_{13}}{c_{11}}, \quad (i = 1, 2),$$

and

$$F_i(r) = r^{\beta_i} \int_a^r r^{-\beta_i}(1-\mu r) e^{-\mu r} dr, \quad (i = 1, 2).$$

So that $F_i(a) = 0$.

A distribution of normal force according to (17) is required to be applied at the ends of the cylinder just to maintain $w = 0$ throughout. Let us suppose axial stress $\sigma_{zz} = c_1$ (constant) on the system such that choosing c_1 properly, we can make the resultant forces on the ends zero. According to Saint-Venant's Principle, such a distribution produces local effect only at the ends.

Due to superposition of the uniform axial stress c_1, σ_{rr} and $\sigma_{\theta\theta}$ will be undisturbed in value, while u is effected. A term c_1/c_{13} should be added to the expression of u in (16). The question of displacements being set aside, we set the boundary conditions to determine the constants A_1 and A_2 for our problem. In this case:

$$\sigma_{rr} = 0 \quad \text{on } r = a \quad \text{and } r = b. \tag{18}$$

Using te boundary condition (18) we get:

$$A_1 = \frac{b_1}{R}\cdot\frac{L_2 a^{\beta_2} - L_1 b^{\beta_2}}{c_{11} m_1}, \qquad A_2 = \frac{b_1}{R}\cdot\frac{L_1 b^{\beta_1} - L_2 a^{\beta_1}}{c_{11} m_2}, \tag{19}$$

with

$$R = \frac{1}{a^{\beta_1} b^{\beta_2} - a^{\beta_2} b^{\beta_1}},$$

where:

$$L_1 = \frac{s}{s+6} a^2 T_i + \left\{(1-\lambda)\frac{s}{s+6} + 1\right\} ap,$$

$$L_2 = \frac{s}{s+6} b^2 T_i + \left[\frac{1-\lambda}{a}\cdot\frac{s}{s+6} b^2 + b e^{-\mu(b-a)} + f_1(b)\right] p,$$

and

$$f_1(r) = \frac{e^{\mu a}}{\beta_2 - \beta_1}[m_1 F_1(r) - m_2 F_2(r)].$$

Substituting the values of A_1 and A_2 we get the stress components as follows:

$$\frac{\sigma_{rr}}{b_1} = \frac{s}{s+6} T_i \Phi_1(r) + p\Phi_2(r), \tag{20}$$

$$\frac{\sigma_{\theta\theta}}{b_1} = \frac{s}{s+6} T_i \Psi_1(r) + p\Psi_2(r), \tag{21}$$

$$\frac{c_{11}}{c_{13}}\cdot\frac{\sigma_{zz}}{b_1} = \frac{s}{s+6} T_i \chi_1(r) + p\chi_2(r), \tag{22}$$

where:

$$\Phi_1(r) = R[R_1 a^2 + R_2 b^2] + r^2,$$

$$\Phi_2(r) = R\left[R_1\left\{(1-\lambda)\frac{s}{s+6} + 1\right\} a + R_2 S\right] + \frac{s}{s+6}\,\frac{1-\lambda}{a} r^2 + r e^{-\mu(r-a)} + f_1(r),$$

$$\Psi_1(r) = R[P_1 a^2 + P_2 b^2] + 3r^2,$$

$$\Psi_2(r) = R\left[P_1\left\{(1-\lambda)\frac{s}{s+6} + 1\right\} a + P_2 S\right] + \frac{1-\lambda}{a}\cdot\frac{3s}{s+6} r^2 + r e^{-\mu(r-a)} + f_2(r),$$

$$\chi_1(r) = R[Q_1 a^2 + Q_2 b^2] + \left(1 - R\frac{s+6}{s}\right) r^2,$$

$$\chi_2(r) = R\left[Q_1\left\{(1-\lambda)\frac{s}{s+6} + 1\right\} a + Q_2 S\right] + \left(\frac{6}{s+6} - R\right)\frac{1-\lambda}{a} r^2 +$$
$$-\left(\frac{2}{s+2} - R\right)\lambda r + R r e^{-\mu(r-a)} + f_3(r),$$

$$f_2(r) = \frac{e^{\mu a}}{\beta_2 - \beta_1}[n_1 F_1(r) - n_2 F_2(r)], \qquad f_3(r) = \frac{e^{\mu a}}{\beta_2 - \beta_1}[l_1 F_1(r) - l_2 F_2(r)],$$

$$R_1 = b_1^{\beta_1} r^{\beta_2} - b^{\beta_2} r^{\beta_1}, \qquad R_2 = a^{\beta_2} r^{\beta_1} - a^{\beta_1} r^{\beta_2},$$

$$P_1 = \frac{n_2}{m_2} b^{\beta_1} r^{\beta_2} - \frac{n_1}{m_1} b^{\beta_2} r^{\beta_1}, \quad P_2 = \frac{n_1}{m_1} a^{\beta_2} r^{\beta_1} - \frac{n_2}{m_2} a^{\beta_1} r^{\beta_2},$$

$$Q_1 = \frac{l_2}{m_2} b^{\beta_1} r^{\beta_2} - \frac{l_1}{m_1} b^{\beta_2} r^{\beta_1}, \quad Q_2 = \frac{l_1}{m_1} a^{\beta_2} r^{\beta_1} - \frac{l_2}{m_2} a^{\beta_1} r^{\beta_2},$$

$$S = \frac{1-\lambda}{a} \frac{s}{s+6} b^2 + be^{-\mu(b-a)} + f_1(b).$$

4. Numerical results and discussions

We calculate our numerical results for the following range of parameters: $10 \leqslant \mu \leqslant 30$ $1.5 < b < 6.0$ and $a = 1$.

We consider the material to be made of magnesium, for which the elastic constants on the inner boundary $r = a = 1$ are given by [2]:

$$c_{11}^1 = c_{11} = 0.565\times 10^{12} \text{ dynes/cm}^2,$$
$$c_{12}^1 = c_{12} = 0.232\times 10^{12} \quad ,, \quad ,, ,$$
$$c_{13}^1 = c_{13} = 0.181\times 10^{12} \quad ,, \quad ,, ,$$
$$c_{33}^1 = c_{33} = 0.587\times 10^{12} \quad ,, \quad ,, ,$$
$$c_{44}^1 = c_{44} = 0.168\times 10^{12} \quad ,, \quad ,, .$$

The coefficients of linear thermal expansion of the said material on the inner boundary $r = a = 1$ are:

$$\alpha_1^1 = \alpha_1 = 27.7\times 10^{-6} \text{ cms/c},$$
$$\alpha_2^1 = \alpha_2 = 26.6\times 10^{-6} \text{ cms/c}.$$

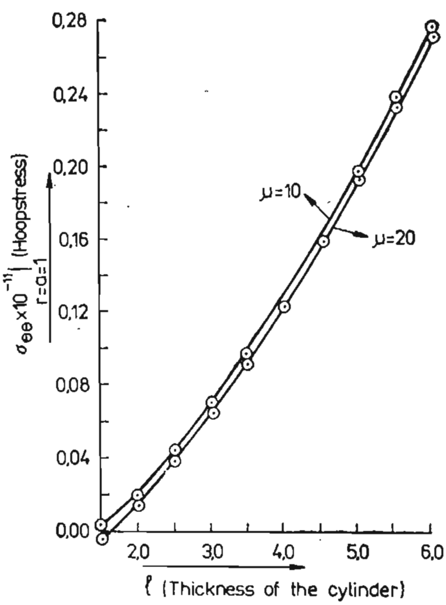

Fig. 1. Variation of the hoopstress on the inner wall with the thickness of the cylinder when $\mu = 10$, $\mu = 20$

Further we choose arbitrarily:

$$T_i = 500°\text{C} \quad \text{and} \quad H_i = 1.$$

With this data, we calculate $\times [= 10^{-11}\sigma_{\theta\theta}|_{r=a=1}]$ and show graphically for different values of b and μ.

From fig. 1, corresponding to $\mu = 10$ and $\mu = 20$ it is clear that the values of hoop stresses gradually increase and tend to coincide as the thickness of hollow cylinder increases. In fig. 2, corresponding to $\mu = 30$, the curve is totally different from that in fig. 1.

From fig. 1 and 2, we can conclude that the hoop stress increases as the thickness of the cylinder increases.

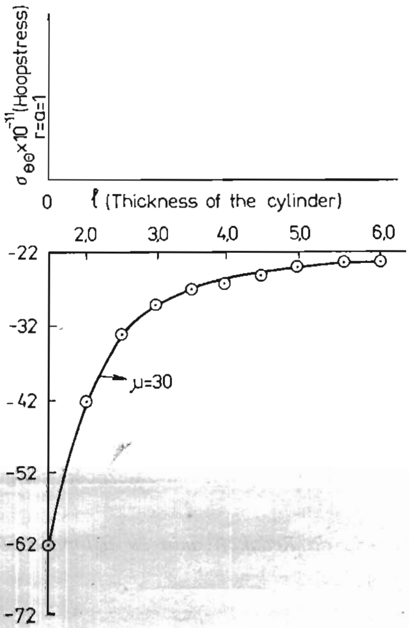

Fig. 2. Variation of hoopstress on the inner wall with the thickness of the cylinder when $\mu = 30$

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Резюме

ТЕОРЕТИЧЕСКИЕ НАПРЯЖЕНИЯ В ДОЛГОМ НЕОДНОРОДНОМ АНИЗОТРОПНОМ ЦИЛИНДРЕ ПОД ДЕЙСТВИЕМ РАДИАЦИИ

В работе рассуждается решение задачи термических напряжений в случае неоднородных, поперечно изотропных, долгих, полых цилиндров. Внешная цилиндрическая поверхность идеально изолирована, а источник тепла возникает от действия радиации. Упругие постоянные, коэффициенты термического линейного разширения и термопроводности изменяются линейно с радиусом. Примером таких поперечно изотропных тел являются металлы магнезии и цинка, а также натуральное дерево. В случае некоторых технологических процессов поперечно изотропические металлы могут стать неоднородными, а дерево в процессе натурального роста может стать изотропным и неоднородным. Численный пример касается тангенциальных напряжений для материала типа магнезии, как функции толщины цилиндра.

Streszczenie

NAPRĘŻENIA CIEPLNE W DŁUGIM, NIEJEDNORODNYM WALCU ANIZOTROPOWYM PODDANYM DZIAŁANIU PROMIENIOWANIA

W pracy podajemy rozwiązanie zagadnienia naprężeń cieplnych w przypadku niejednorodnych, poprzecznie izotropowych długich i wydrążonych walców. Zewnętrzna pobocznica walca jest idealnie izolowana, a źródłem ciepła jest promieniowanie. Współczynniki sprężystości, rozszerzalności cieplnej i przewodnictwa cieplnego zmieniają się liniowo wraz z promieniem. Jako przykłady metali mogą posłużyć magnezyt i cynk, a z materiałów naturalnych drewno. Obróbka technologiczna metali poprzecznie izotropowych może spowodować niejednorodność, a drewno na skutek uprzedniego naturalnego wzrostu może się czasami zachowywać jak materiał izotropowy i niejednorodny. Obliczenia wykonano dla magnezytu, pokazano na wykresie rozkład naprężeń obwodowych w zależności od grubości walca.

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