MECHANIKA
TEORETYCZNA
I STOSOWANA
1, 26, 1988

THE KORTEWEG-DE VRIES EQUATIONS FOR WAVES PROPAGATION IN AN INFINITE TUBE

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1. Introduction

In recent years, there has been growing interest in the partial differential equations which govern wave phenomena on the basis of the reductive Taniuti-Wei's [1], the multiple-scaling [2], the Lagrangian [3], and Shen's [4] methods. Amongst them, the number of equations appeared on linear waves in tubes [5 - 9]. It was shown that in the absence of dissipation of energy the fundamental set of nonlinear equations for the irrotational motion of waves in a liquid filled a tube can be reduced to the Korteweg-de Vries equation [10]. Also Burgers equation was obtained for dissipative systems [10 - 12]. In 1968 Johnson [13] introduced the so-called Korteweg-de Vries-Burgers equation for a wave propagation on an elastic tube containing a viscous fluid which may be regarded as a simple model of an artery. Recently the discussion of an incompressible fluid that is confined within an infinitely long circular cylinder with thin walls of elastic rings leads to the Korteweg-de Vries equation [14] which also may be obtained in this case via Lagrangian method [15].

The main purpose of this note is to broaden Lamb equations [14] to allow compressibility of fluid and to take more realistic model equation, describing motion of a tube wall, into consideration.

The organization of this note is as follows. In the next Section fundamental sets of equations are presented. Section 3 and 4 deal with derivation of the Korteweg-de Vries equation for a tube with elastic rings and the Korteweg-de Vries equation with varying coefficients. Section 5 presents construction of the same equation via the multiple-scaling method. Last Section is devoted to the short summary of this note.

2. Physial models

In this note we consider the one-dimensional irrotational fluid waves of characteristic amplitude l and characteristic length λ in an infinitely long tube with thin walls of elastic rings and a diameter 2a to take into account nonlinearity and dispersion of medium on the assumption that $l \ll 2a \ll \lambda$. The tube wall is assumed to be incompressible and we ignore axial motions of the wall and bending moments are neglected. Then we take as the set of relevant basic equations:

— equation of continuity,

$$(\varrho A)_t+(\varrho AV)_x = 0, \tag{2.1}$$

— Euler's equation,

$$V_t+VV_x+\frac{1}{\varrho}p_x = 0, \tag{2.2}$$

— Newton's equation,

$$A_{tt}+\frac{E}{a^2\varrho_m}A-\frac{2\pi a}{\varrho_m h}p+\frac{\pi(2aq-Eh)}{\varrho_m h} = 0, \tag{2.3}$$

— *equation of state,*

$$\varrho = \varrho(p) \equiv D\cdot p, D = \text{const.}, \tag{2.4}$$

where we used the following notations: ϱ — liquid density, A — area of the crossection, V — liquid velocity, a — tube radius at the undisturbed uniform state, ϱ — density of the tube material, E — Young's modulus in the circumferential direction, p — liquid pressure, q — outside pressure. The subscripts x and t imply partial diferentiation.

The other model equation governing motion of a tube wall without rings as a linear viscoelastic solid characterized by its relaxation time was that employed previously in [16] and for our aim may be written in the following form

$$\frac{Eh}{(1-\nu^2)a^2}r+\varrho h r_{tt}-\frac{\varrho h^3}{12}r_{xxtt}-p = \frac{Eh}{(1-\nu^2)a}-q, \tag{2.5}$$

where: ν is a Poisson's coefficients and r is a tube radius at the disturbed uniform state.

We define two dimensionless small parameters, namely:

$$\varepsilon = \frac{2a}{\lambda}, \qquad \delta = \frac{l}{2a}, \tag{2.6}$$

which measure the weakness of dispersion and nonlinearity, respectively. The Korteweg-de Vries equation will be derived on assumption that $\delta = \varepsilon^2$.

3. Derivation of the Korteweg-de Vries equation for tube with rings

Our primary aim is to derive an approximate single equation from the fundamenta set of equations (2.1) - (2.4). For this purpose we apply the reductive Taniuti-Wei's method [2]. Assuming that A, V, p are slowly varying functions in a reference frame moving with the speed V_0, we introduce the following coordinate-stretching:

$$\xi = \varepsilon(x-V_0 t), \qquad \tau = \varepsilon t. \tag{3.1}$$

In new coordinates ξ, τ, equations (2.1) - (2.4) may be rewritten in the form

$$\varepsilon^2(pA)_\tau-V_0(pA)_\xi+(pAV)_\xi = 0, \tag{3.2}$$

$$Dp[\varepsilon^2 V_\tau+(V-V_0)V_\xi]+p_\xi = 0, \tag{3.3}$$

$$\varepsilon^2 A_{\xi\xi}+\frac{E}{a^2\varrho_m}A-\frac{2\pi a}{\varrho_m h}p+\frac{\pi(2aq-Eh)}{\varrho_m h}=0(\varepsilon^2). \tag{3.4}$$

On the other hand, since we are concerned with weak nonlinear waves, we expand the dependent variables as power series in δ around the undisturbed uniform state:

$$\left.\begin{aligned} p &= q+\delta p_1+\dots,\\ V &= \delta V_1+\delta^2 V_2+\dots,\\ A &= A_0+\delta A_1+\dots. \end{aligned}\right\} \tag{3.5}$$

Substituting (3.5) and $\varepsilon^2=\delta$ into the above set of equations (3.2) - (3.4) and equating all the coefficients of the various powers of ε to zero, we have the equations:

$$qA_0V_{1\xi}-V_0(A_0p_{1\xi}+qA_{1\xi})=0, \tag{3.6}$$

$$p_{1\xi}-qV_0V_{1\xi}=0, \tag{3.7}$$

$$EhA_1-2\pi a^3p_1=0. \tag{3.8}$$

Hence, we obtain

$$A_1=\frac{2\pi a^3}{Eh}p_1, \tag{3.9}$$

$$V_1=\frac{V_0(EhA_0+2\pi qa^3)}{EhqA_0}p_1 \tag{3.10}$$

$$V_0^2=\frac{EhA_0}{D(A_0Eh+2\pi qa^3)}. \tag{3.11}$$

Finally, from ε^4, the second-order perturbed terms can be eliminated and the compatibility condition (3.11) gives rise to the Korteweg-de Vries equation for p_1

$$p_{1\tau}+\beta p_1p_{1\xi}+\alpha p_{1\xi\xi\xi}=0. \tag{3.12}$$

The nonlinear β and the dispersive α coefficients are described by the formulae

$$\beta=\frac{V_0[(Eh)^2+aqEh+6(aq)^2]}{Ehq(Eh+2aq)}, \tag{3.13}$$

$$\alpha=\frac{\varrho_m qa^3V_0}{E(Eh+2aq)}. \tag{3.14}$$

4. Derivation of the Korteweg-de Vries equation with varying coefficients

We consider now the fundamental set of equations (2.1), (2.2), and (2.5) which describe wave propagation in an infinite thin-walled tube without rings neglecting bending moments and axial motion of the tube wall. We assume that the undisturbed radius a is varying slowly along axial direction and rewrite the above mentioned equations for $\varrho=$ const. $=\varrho_0$ in the following form:

$$\varrho_0(V_t+VV_x)+(Br)_x+\varrho_0hr_{xtt}-\frac{\varrho_0h^3}{12}r_{xxxtt}-C_x=0, \tag{4.1}$$

$$(r^2)_t+(r^2V)_x=0, \tag{4.2}$$

where we introduce the notation:

$$B = \frac{Eh}{(1-\nu^2)a^2}, \tag{4.3}$$

$$C = \frac{Eh}{(1-\nu^2)a} - q. \tag{4.4}$$

We investigate ingoing solutions of equations (4.1) and (4.2) in the small amplitude approximation using the same reductive method. Because $a = a(x)$, we introduce the following coordinate-stretching of the reference moving frame:

$$\xi = \varepsilon\left(\int \frac{dx}{V_0} - t\right), \qquad \eta = \varepsilon^{3/2}x. \tag{4.5}$$

Now V_0 is a function of x. We take $\varepsilon^2 = \delta$ into consideration. Expansion of r, V into power series of the same parameter

$$r = a + \varepsilon r_1 + \ldots, \qquad V = \varepsilon V_1 + \varepsilon^2 V_2 + \ldots, \tag{4.6}$$

leads to the decomposition of equations (4.1) and (4.2) establishing the relationship among the first-order perturbed quantities from collecting terms by ε:

$$V_1 = \frac{2V_0}{a} r_1, \tag{4.7}$$

$$V_0^2 = \frac{aB}{2\varrho_0}. \tag{4.8}$$

From the second-order equations ε^2, the compatibility condition give rise to the Korteweg-de Vries equation with varying coefficients

$$2aBV_{1\eta} + 5\varrho_0 V_1 V_{1\xi} + \frac{a\varrho_0 h}{V_0} V_{1\xi\xi\xi} = 2\left[V_0\left(\frac{Ba}{2V_0}\right)_\eta - Ba_\eta\right] V_1. \tag{4.9}$$

5. Derivation of the Korteweg-de Vries equation via multiple-scaling method

Our next purpose is to apply the multiple-scaling method [2] to derive the Korteweg-de Vries equation which describe small amplitude and long waves. The fundamental set of equations (4.1) and (4.2) may be rewritten in the following form:

$$V_t + VV_x + r_x + r_{xtt} + r_{xxxtt} = 0, \tag{5.1}$$

$$(r^2)_t + (r^2 V)_x = 0, \tag{5.2}$$

where dimensionless variables are introduced by the transformations:

$$x \to \frac{h}{\sqrt{12}} x, \quad t \to \sqrt{\frac{\varrho_0 h}{B}}\, t, \quad r \to \frac{h}{12} r, \quad V \to \sqrt{\frac{Bh}{12\varrho_0}}\, V. \tag{5.3}$$

In equations (5.1) and (5.2) we introduce the multiple spatial and temporal scales $x_n = \varepsilon^n x$ and $t_n = \varepsilon^n t$ for $n = 1, 2, \ldots$ The dependent variables are expanded around the undisturbed uniform state into the asymptotic series in terms of the parameter δ by writing

$$V = \sum_{n=1}^{\infty} \delta^n V_n, \qquad r = \tilde{l} + \sum_{n=1}^{\infty} \delta^n r_n. \tag{5.4}$$

$\tilde{l}$ is the undisturbed dimensionless radius of tube. The derivative operators are considered to be of the form

$$\begin{aligned} \frac{\partial}{\partial t} &= \varepsilon \frac{\partial}{\partial t_1} + \varepsilon^2 \frac{\partial}{\partial t_2} + \dots, \\ \frac{\partial}{\partial x} &= \varepsilon \frac{\partial}{\partial x_1} + \varepsilon^2 \frac{\partial}{\partial x_2} + \dots. \end{aligned} \tag{5.5}$$

Substituting (5.5) and (5.4) into equations (5.1) and (5.2), we obtain a sequence of equations by equating the coefficients of like powers of ε. The first three sets of perturbation equations are as follows:

$$\varepsilon^2: \begin{cases} V_{1t_1} + r_{1x_1} = 0, & (5.6) \\ 2r_{1t_1} + \tilde{l} V_{1x_1} = 0, & (5.7) \end{cases}$$

$$\varepsilon^3: \begin{cases} V_{1t_2} + r_{1x_2} = 0, & (5.8) \\ 2r_{1t_2} + \tilde{l} V_{1x_2} = 0, & (5.9) \end{cases}$$

$$\varepsilon^4: \begin{cases} V_{2t_1} + V_{1t_3} + V_1 V_{1x_1} + r_{2x_1} + r_{1x_3} + r_{1x_1t_1^2} = 0, & (5.10) \\ 2\tilde{l} r_{2t_1} + r_{1t_3} + 2r_1 r_{1t_1} + 2\tilde{l} V_1 r_{1x_1} + \tilde{l}^2 (V_{2x_1} + V_{1x_3}) + \end{cases}$$

$$+ 2\tilde{l} r_1 V_{1x_1} = 0. \tag{5.11}$$

From equations (5.6) - (5.9), we find

$$V_1 = V_1 \left(x_1 - \sqrt{\frac{\tilde{l}}{2}}\, t_1 \right) \equiv V(\xi_1), \tag{5.12}$$

$$r_1 = r_1(\xi_1) = \sqrt{\frac{\tilde{l}}{2}}\, V_1, \tag{5.13}$$

$$V_1 = V_1 \left(x_2 - \sqrt{\frac{\tilde{l}}{2}}\, t_2 \right) \equiv V_1(\xi_2), \qquad r_1 = r_1(\xi_2). \tag{5.14}$$

The fourth-order equations (5.10) and (5.11) lead to the following equation after removing second-order terms by assuming that V_2 depends on x_1 and t_1 through ξ_1:

$$V_{1t_3} + \sqrt{\frac{\tilde{l}}{2}}\, V_{1x_3} + \frac{5}{4} V_1 V_{1\xi_1} + \left(\frac{\tilde{l}}{2} \right)^{3/2} V_{1\xi_1\xi_1\xi_1} = 0. \tag{5.15}$$

Transforming to the coordinate system moving with a phase velocity $\sqrt{\frac{\tilde{l}}{2}}$, i.e.,

$$\xi_3 = x_3 - \sqrt{\frac{\tilde{l}}{2}}\, t_3, \qquad \tau = t_3, \tag{5.16}$$

we can obtain the Korteweg-de Vries equation

$$V_{1\tau} + \frac{5}{4} V_1 V_{1\xi_1} + \left(\frac{\tilde{l}}{2}\right)^{3/2} V_{1\xi\xi_1\xi_{11}} = 0. \tag{5.17}$$

6. Summary

Basing on the rigorous developed in the reductive theory, we have derived the Korteweg-de Vries equations as a first-order of approximation of waves in an infinite thin-walled tube having taken into account the fundamental sets of equations. These equations model also impulse propagation in an a arterial system, small intensines and a nervous system. The problem of impulse propagation was considered via various methods by Scott [17] for the nervous system and by Greenwald et al [18] for the arterial stenoses and aneurysms.

The formulae (3.11), (4.8), and (5.16) may be used to determine physical parameters such as Young's modulus having measured the velocity of the moving frame [19]. Various models of the tubes may be tested against experiments.

The Korteweg-de Vries equation with constant coefficients was discussed in some details to obtain N — soliton [20] and N — periodical wave [21] solutions. These equations were reviewed for water waves by Johnson [22].

Solution of the Korteweg-de Vries equation with varying coefficients was considered in the context of a solitary wave propagation from one uniform cross section of a symmetric triangular channel into another through a transition region. Numerical results showed that the solitary wave is desintegrated into a train of solitons of decreasing amplitudes [23].

The author would like to express the sincere thanks to the referee for his valuable comments.

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Резюме

УРАВНЕНИЯ КОРТЕВЕГА-ДЕ ФРИЗА ДЛЯ РАСПРОСТРАНЕНИЯ ВОЛН В БЕСКОНЕЧНО ДЛИННОЙ ТРУБЕ

В работе применено теорию нелинейных волн, основанную на методе редукции Тапоти-Веи и методе многих параметров для получения уравнения Кортевега-де Фриза для распространения нелинейных и дисперсионных волн в трубе.

Streszczenie

RÓWNANIA KORTEWEGA-DE VRIESA DLA PROPAGACJI FAL W RURZE O NIESKOŃCZONEJ DŁUGOŚCI

W pracy zastosowano teorię fal nieliniowych opartą na metodzie redukcji Taniuti-Wei i metodzie wielu skal od otrzymania równania Kortewega-de Vriesa dla propagacji nieliniowych i dyspersyjnych fal w rurach.

Praca wpłynęła do Redakcji dnia 20 września 1985 roku.