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# NONLINEAR EQUATIONS OF SHELLS OF SLOWLY VARYING CURVATURES

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## 1. Introduction

A major problem in the general nonlinear theory of thin elastic shells consists in reducing the very complex general field equations to simpler, tractable forms. Extensive surveys, of efforts along these lines can be found in the works of Koiter [1] and Pietraszkiewicz [2], so we may concentrate here on those results having a direct bearing on the present work.

One early recognized possibility is to deal with only the equilibrium equations and compatibility conditions in conjunction with the constitutive relations. In this way the very involved stress-displacement relations are put aside and no restrictions as to the magnitudes of displacements and their derivatives must be adopted. These so-called "intrinsic shell equations" can be greatly simplified if the strains are small and the ratio of maximum membrane to bending strains is not very large or very small compared with unity. The resulting "lowest-order interior equations" due to John [3] and Koiter [1] permit further reduction for "quasi-shallow" shells (also called "shells of small Gaussian curvature") introduced by Koiter [1] which are characterized by the requirement of smallness of the Gaussian curvature with respect to the reciprocal of the square of the characteristic deformation wave length. Under such circumstances, the membrane forces can be represented in terms of a stress function and the bending strains through a strain function, leading to two appealingly simple differential equations in two unknowns [1].

This paper aims at extending the range of applicability of the now classic equations of quasi-shallow shells. To this end, the condition of quasi-shallowness is replaced by the weaker assumption of slow variation of curvatures over the middle surface — an assumption first proposed by Duddeck [4] in the context of linear theory and then exploited by Łukasiewicz (see [5]) in a series of papers concerning both linear and nonlinear shell problems. We borrow from Duddeck his refined expression for membrane forces in terms of a strain function which, contrary to quasi-shallow shells, takes account of the Gaussian curvature. The second of Duddeck's variables, the normal deflection of the midsurface, turns out to be unsuitable for the intended here displacement-free theory and is not used. Instead, we express the bending strains through a strain function, finding the appropriate formula from Duddeck's stress function by noting a static-geometric analogy between membrane forces and bending strains. Compared with Koiter's [1] strain function ("curvature function" in his terminology), our new formula is only slightly more complicated due to the occurence of a Gaussian curvature-related term. In the end, a relatively simple set of two governing differential equations in two unknowns — the stress and strain functions — is obtained which generalizes the equations of quasi-shallow shells and reduces to the latter upon dropping terms multiplied by the Gaussian curvature. It is a matter of course that the new equations generalize also all their predecessors involving two unknowns one of which is the normal deflection, i.e. the equations of shallow shells due to Donnell [6], Mushtari [7], and Vlasov [8], as well as equations for shells of slowly varying curvatures due to Duddeck [4] and Łukasiewicz [5].

Our work closes with a formulation of appropriate displacement-free boundary conditions to be used with the two differential equations. These include a set of static boundary conditions derived by proper simplification from Danielson's conditions [9], and a set of deformational boundary conditions which are a reduced version of those provided by Pietraszkiewicz [2].

The differential equations and boundary conditions found are truly displacement-free only for surface and edge loads whose components are known in the basis attached to the deformed shell. Consequently, dead loads are inappropriate and only pressure-like loads can be admitted.

## 2. Reduction of basic field equations

This section is devoted to reducing the general nonlinear shell equations to the so-called "lowest-order interior equations" [1, 3]. Although the outcome of this reduction is identical with [1, 3], our derivation throws new light on the subject as we: (a) make a distinction between the wave lengths corresponding to membrane and bending strains (b) introduce a wave length characterising the variation of curvatures over the midsurface; consequently, the validity criteria for the "lowest-order interior equations" become more precise than in [1, 3].

To begin with we assume, as Koiter [1] does, that the strains are small everywhere in the shell which is thin, homogeneous, and linearly elastic. The fundamental field equations now are as follows ([1], p. 34). The constitutive equations between the symmetric membrane forces $N_{\alpha\beta}$ and extensional strains $g_{\alpha\beta}$, and between the moments $M_{\alpha\beta}$ and bending strains $q_{\alpha\beta}$ read

$$g_{\alpha\beta} = \frac{1}{Eh}[(1+\nu)N_{\alpha\beta} - \nu a_{\alpha\beta}N_\lambda^\lambda], \tag{1}$$

$$M_{\alpha\beta} = D[(1-\nu)q_{\alpha\beta} + \nu a_{\alpha\beta}q_\lambda^\lambda], \tag{2}$$

where $a_{\alpha\beta}$ is the metric tensor of the undeformed middle surface, $h$ denotes the constant shell thickness, $E$ is Young's modulus, $\nu$ is Poisson's ratio, and $D = Eh^3/12(1-\nu^2)$ stands for the flexural rigidity. The force equilibrium equations are:

$$\left(N^{\beta\alpha} + \underline{\frac{1}{2}b_\lambda^\alpha M^{\beta\lambda} - \frac{1}{2}b_\lambda^\beta M^{\alpha\lambda} + q_\lambda^\alpha M^{\beta\lambda}}\right)_{|\beta} \underline{+ (b_\lambda^\alpha + q_\lambda^\alpha)M_{|\beta}^{\beta\lambda}} = -\bar{p}^\alpha, \tag{3}$$

$$M_{|\alpha\beta}^{\alpha\beta} - \left(\underline{\frac{1}{2}b_\alpha^\lambda q_{\lambda\beta} + \frac{1}{2}b_\beta^\lambda q_{\lambda\alpha} + q_\alpha^\lambda q_{\lambda\beta}}\right)M^{\alpha\beta} - (b_{\alpha\beta} + q_{\alpha\beta})N^{\alpha\beta} = \bar{p}. \tag{4}$$

Here $b_{\alpha\beta}$ is the curvature tensor of the undeformed midsurface, a vertical stroke indicates surface covariant differentiation based on the undeformed metric $a_{\alpha\beta}$, $\bar{p}^{\alpha}$ and $\bar{p}$ are surface loads tangential and normal to the deformed shell (this is indicated by the overbars). The compatibility conditions assume the form:

$$e^{\alpha\beta}e^{\lambda\mu}\left[\left(q_{\beta\lambda}+\frac{1}{2}b^{\delta}_{\beta}g_{\delta\lambda}+\frac{1}{2}b^{\delta}_{\lambda}g_{\delta\beta}\right)_{|\mu}-b^{\delta}_{\lambda}(g_{\delta\beta|\mu}+g_{\delta\mu|\beta}-g_{\beta\mu|\delta})\right]=0, \tag{5}$$

$$e^{\alpha\beta}e^{\lambda\mu}\left(g_{\alpha\mu|\beta\lambda}+b_{\alpha\mu}q_{\beta\lambda}+\frac{1}{2}q_{\alpha\mu}q_{\beta\lambda}\right)=0, \tag{6}$$

where $e_{\alpha\beta}$ is the permutation tensor based on $a^{\alpha\beta}$.

In order to compare the various terms in the above equations, we assume that the surface coordinates have the dimension of length. This makes it possible to introduce the relations:

$$g_{\alpha\beta}=0(g), \quad q_{\alpha\beta}=0(q), \quad b_{\alpha\beta}=0(1/R),$$

to define the parameters $g$, $q$ and $R$ which are the absolute maximum stretching and bending strains, and the smallest principal radius of curvature of the midsurface. The rates of change of the strains and curvatures will be characterized by means of wave lengths $L_g$, $L_q$ and $L_r$ as follows:

$$g_{\alpha\beta|\lambda}=0(g/L_g), \quad q_{\alpha\beta|\lambda}=0(q/L_q),$$
$$b_{\alpha\beta|\lambda}=0(1/RL_r), \quad K_{|\alpha}=0(K/L_r),$$

where $K$ is the Gaussian curvature of the undeformed midsurface; the same wave lengths will be used in evaluating higher-order derivatives, e.g. $g_{\alpha\beta|\lambda\eta}=0(g/L_g^2)$, etc.
These definitions and relations (1) and (2) imply that:

$$N_{\alpha\beta}=0(Ehg), \quad N_{\alpha\beta|\lambda}=0(Ehg/L_g),$$
$$M_{\alpha\beta}=0(Eh^3q), \quad M_{\alpha\beta|\lambda}=0(Eh^3q/L_q),$$

where use has been made of the fact that $a_{\alpha\beta}=0(1)$. Now the magnitudes of the individual terms in the equilibrium and compatibility equations (3) - (6) are:

(3): $Ehg/L_g$, $\underline{Eh^3q/RL_q, Eh^3q/RL_r, Eh^3q^2/L_q}$,

(4): $Eh^3q/L_q^2$, $\underline{Eh^3q^2/R, Eh^3q^3}$, $Ehg/R$, $Ehgq$,

(5): $q/L_q$, $\underline{g/RL_g, g/RL_r}$,

(6): $g/L_g^2$, $q/R$, $q^2$,

where in evaluating (5) and (6) one should remember that $e^{\alpha\beta}=0(1)$.

In order to simplify equations (3) - (6), we first take notice of the well-known fact that uncoupled constitutive equations (1) and (2) are approximate ones because of omission of terms conforming to relations [1, 9]:

$$h/R \ll hq/g \ll (R/h, 1/hq). \tag{7}$$

Now the underlined terms in (4) are seen to be negligible and can be dropped.

To proceed further, we assume that the shell curvatures vary slowly compared with the strains variation, in the sense that

$$L_g \leqslant L_r, \quad L_q \leqslant L_r. \tag{8}$$

Keeping this in mind, we find that the underscored terms in (3) and (5) may be neglected in comparison with the remaining contributions provided that:

$$h/R \ll (hq/g)\,(L_g/L_q) \ll (R/h,\, 1/hq). \tag{9}$$

Deleting all the underlined terms in (3) - (5) and transforming the first terms in (4) and (6) as in [1], we finally get reduced equilibrium and compatibility equations of the form:

$$N^{\beta\alpha}_{|\beta} = -\bar{p}^{\alpha}, \quad e^{\alpha\beta}e^{\lambda\mu}q_{\beta\lambda|\mu} = 0, \tag{10, 11}$$

$$Dq^{\alpha|\beta}_{\alpha|\beta} - (b_{\alpha\beta}+q_{\alpha\beta})N^{\alpha\beta} = \bar{p}, \tag{12}$$

$$N^{\alpha|\beta}_{\alpha|\beta} - Eh e^{\alpha\beta}e^{\lambda\mu}\left(b_{\alpha\mu} + \frac{1}{2}q_{\alpha\mu}\right)q_{\beta\lambda} = -(1+\nu)\bar{p}^{\alpha}_{|\alpha}. \tag{13}$$

These are exactly the "lowest-order interior equations" of [1, 3]. Their validity depends on the requirements (7) - (9) of which only (7) can be found in Koiter's work [1]. Practically, the curvatures vary smoothly in most shell applications so that relations (8) are true. As for (9), these conditions reduce to Koiter's assumptions (7) only for deformations characterized by equal wave lengths of the membrane and bending strains; this is very often assumed for analytical convenience, but surely unjustified, in general, physically (see [10]).

## 3. Governing equations in terms of stress and strain functions

The system of equations (10) - (13) lends itself to further simplifications resulting in two coupled equations for two unknowns — a stress and a strain function. For quasi-shallow shells such equations have been found by Koiter [1]. Here we propose two more general and more complex equations valid for shells of slowly varying curvatures.

An appropriate approximate solution to the equilibrium equations (10) has been found by Duddeck [4] in the form:

$$N^{\beta\alpha} = e^{\beta\lambda}e^{\alpha\mu}(F_{|\mu\lambda} + a_{\mu\lambda}KF) + \bar{P}^{\alpha\beta}, \tag{14}$$

where $\bar{P}^{\alpha\beta}$ is a particular solution to (10); this formula is of interest thanks to the $KF$ term absent in works on shallow and quasi-shallow shells. Introducing (14) into (10) and making use of the well-known geometric relations:

$$e^{\beta\lambda}F_{|\mu\lambda\beta} = Ke_{\mu\beta}F^{|\beta}, \quad e^{\alpha\mu}e_{\mu\beta} = -a^{\alpha}_{\beta},$$

the residual error in (10) is found to be equal to $K_{|\alpha}F$. This quantity is $0(KF/L_r)$, while the principal term in (10), $N^{\beta\alpha}_{|\beta}$, is, by (14), $0(F/L^3_g)$, so that the relative error in (10) is negligible when:

$$(L_g/L_r)\,(KL^2_g) \ll 1. \tag{15}$$

A welcome feature of the compatibility equations (11) is their similarity, in the sense of static-geometric analogy, to the equilibrium equations (10). Remembering this, we immediately find from (14) the solution to (11) in terms of a strain function $W$:

$$q_{\alpha\beta} = W_{|\alpha\beta} + a_{\alpha\beta} KW. \tag{16}$$

This expression was apparently first used by the present writer [11] in the linear theory of shells of slowly varying curvatures; neglecting the $KW$ terms, (16) assumes the form familiar from Koiter's quasi-shallow shells [1].

Introduction of (16) into (11) produces in the latter equations a residual error that is negligible when:

$$(L_q/L_r)(KL_q^2) \ll 1. \tag{17}$$

Substitution of (14) and (16) into (12) and (13) finally yields

$$\begin{aligned} D(W|_\alpha^\alpha + 2KW)|_\beta^\beta - e^{\alpha\lambda} e^{\beta\mu} (b_{\alpha\beta} + W_{|\alpha\beta} + a_{\alpha\beta} KW)(F_{|\mu\lambda} + a_{\mu\lambda} KF) = \\ = (b_{\alpha\beta} + W_{|\alpha\beta} + a_{\alpha\beta} KW)\overline{P}^{\alpha\beta} + \overline{p}, \end{aligned} \tag{18}$$

$$\begin{aligned} (F|_\alpha^\alpha + 2KF)|_\beta^\beta - Ehe^{\alpha\beta} e^{\lambda\mu} \left( b_{\alpha\mu} + \frac{1}{2} W_{|\alpha\mu} + \frac{1}{2} a_{\alpha\mu} KW \right) (W_{|\lambda\beta} + a_{\lambda\beta} KW) = \\ = (1+\nu) \overline{P}^{\alpha\beta}_{|\alpha\beta} - \overline{P}^\alpha_\alpha|_\beta^\beta. \end{aligned} \tag{19}$$

The two just derived governing differential equations in two unknowns, $F$ and $W$, are the major novel finding of this account. Recall that they are valid for small strains and under the assumptions (7) - (9), (15) and (17). For shells of slowly varying curvatures in the sense of (8), the requirements (15) and (17) are clearly less restrictive than the assumption $KL^2 \ll 1$ (here $L = L_g = L_q$) adopted in the theory of quasi-shallow shells [1]. Consequently, our equations (18) and (19) generalize those of quasi-shallow shells; the former reduce to the latter when the $K$ terms are dropped. As an example consider a spherical shell: it has constant curvatures ($1/L_r = 0$) and thus represents a shell of slowly varying curvatures for all deformations with finite wave lengths $L_g$, $L_q$, whereas it belongs to the class of quasi-shallow shells only for sufficiently small products $KL_g^2$ and $KL_q^2$ of Gaussian curvature and the wave lengths squared.

## 4. Boundary conditions

Boundary conditions suitable for our differential equations (18) and (19) must not involve displacements if they are to be of any value. This quality possess the static conditions provided that the edge load components are known in the natural basis of the deformed shell. An appropriate set of such conditions may be easily obtained from Danielson's conditions ([9], Eqs. (4.10) - (4.12)) upon neglecting small terms satisfying relations (7). The result is:

$$N^{\alpha\beta} n_\beta = \overline{N}^\alpha, \quad (1-\nu) D q^{\alpha\beta} n_\alpha n_\beta + \nu D q_\alpha^\alpha = \overline{M}_n, \tag{20, 21}$$

$$D q_\beta^\beta|^\alpha n_\alpha + D(1-\nu)(q^{\alpha\beta} n_\alpha t_\beta)_{,s} = \overline{M}_{t,s} - \overline{Q}, \tag{22}$$

where $\overline{N}^\alpha$ are the components of the membrane force, $\overline{Q}$ is the shear force, $\overline{M}_n$ is the bending moment and $\overline{M}_t$ represents the torque, all prescribed per unit length of the undeformed edge but resolved with respect to the deformed basis (its vectors have approximately the same magnitudes as the undeformed base vectors, because of small strains, but may have quite different directions, since displacements and rotations are not restricted); $n_\alpha$ is the unit normal to the undeformed edge surface, $t_\alpha$ is the unit tangent to the undeformed edge curve, $(\ )_{,s}$ indicates differentiation with respect to the undeformed arc length.

Deformational boundary conditions represent another type of displacement-free conditions. A set suitable for our purposes is readily derived from Pietraszkiewicz ([2], Eqs. (4.4.40) and (6.3.8)), after simplifications based on (7), in the form:

$$q^{\alpha\beta}t_\alpha t_\beta = k_t, \qquad q^{\alpha\beta}t_\alpha n_\beta = k_{tn}, \tag{23, 24}$$

$$t^\alpha e^{\lambda\beta}q_{\lambda\alpha|\beta} + (t^\alpha n^\beta g_{\alpha\beta})_{,s} = k_n, \qquad t^\alpha t^\beta g_{\alpha\beta} = g_t. \tag{25, 26}$$

Here $k_t$, $k_{tn}$ and $k_n$ denote the changes of the normal curvature, the geodesic torsion and the geodesic curvature of the boundary curve, and $g_t$ is its elongation. When the edge is clamped, for instance, all these quantities are zero.

With (14) and (16) the above static and deformational boundary conditions may be easily represented in terms of the stress and strain functions.

## 5. Conclusions

The two differential equations and the static and deformational boundary conditions obtained in this paper for shells of slowly varying curvatures, undergoing small strains with unrestricted displacements and rotations are fairly simple, but must be used with discretion. First, a word of caution should be said in regard to the simplifications made in deriving the equations, which were based on a qualitative rather than quantitative argument. Therefore it is imperative that each solution of our simplified equations be checked for consistency with the original, unsimplified equations. These latter equations, as Koiter [1] points out, are unsuitable for shell stability problems and so are, of course, the reduced equations presented here. Finally, there is apparently no variational formulation equivalent to our differential equations and boundary conditions; this is a serious drawback from a computational viewpoint.

In theoretical perspective, our result seems worth while, as it considerably expands the limits of validity of the various similar equations known previously.

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Резюме

НЕЛИНЕЙНЫЕ УРАВНЕНИЯ ОБОЛОЧЕК С МЕДЛЕННО ИЗМЕНЯЮЩИМИСЯ КРИВИЗНАМИ

Общие нелинейные уравнения равновесия и условия совместности деформации тонких, упругих оболочек сведены к двум уравнениям для функции напряжений и функции деформаций. Приведены соответствующие статические и деформационные граничные условия. Предположено, что деформации малы а кривизны изменяются медленно, но перемещения и обороты не ограничены. Полученные результаты обобщают известные уравнения пологих оболочек, оболочек малой Гауссовой кривизны и оболочек с медленно изменяющимися кривизнами.

Streszczenie

NIELINIOWE RÓWNANIA POWŁOK O WOLNO ZMIENIAJĄCYCH SIĘ KRZYWIZNACH

Ogólne nieliniowe równania równowagi i warunki nierozdzielności cienkich powłok sprężystych zredukowano do dwóch równań z funkcją naprężeń i funkcją odkształceń. Przedstawiono odpowiednie statyczne i deformacyjne warunki brzegowe. Założono małe odkształcenia i łagodną zmienność krzywizn powłoki, natomiast przemieszczenia i obroty nie są ograniczone. Otrzymane wyniki są ogólniejsze od znanych równań powłok o małej wyniosłości, powłok o małej krzywiźnie Gaussa oraz powłok o łagodnie zmiennych krzywiznach.

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