MECHANIKA
TEORETYCZNA
I STOSOWANA
3, 26 (1988)

HOMOGENIZATION OF FIRST STRAIN-GRADIENT BODY

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The problem of the homogenization of first strain-gradient body is studied by means of the Γ-convergence method. Assuming the form of internal energy for real ε-periodic structure the homogenized internal energy of the homogeneous (effective) body is determined. The coefficients in the homogenized energy functional are effective material constants and they depend on the solutions of a so-called cell-problem.

1. Introduction

Homogenization method is applied to describe the global elastic response of the non-simple material body with periodic microstructure. As a result one obtain the closed form of effective (homogenized) internal energy function in which any quantities can be calculated explicitly for a given real structure as far as so-called „problem on a cell" is solved. Microstructure is understood here as a real heteregeneous non-simple elastic body (first strain-gradient model) whose properties vary rapidly and periodically with space. The real dimension of a single cell of periodicity is big enough to apply the concept of continuum but the number of cells is too large to apply any numerical procedure for solving the proper system of partial differential equation. For this purpose one seek the behaviour of limiting process when the numbers of cells goes to infinity and at the same time their characteristic dimension becomes infinitely small. In the problem of the first strain-gradient theory we deal with the system of partial differential equations of the 4^{th} order with rapidly varying coefficient and to get any limiting result we decide to use the concept of Γ-convergence rather than homogenization theorem based on G-convergence i.e. convergence of a sequence of the partial differential operators ([1] [2]). In mathematical description is the problem going to the limit (Γ-convergence) of a sequence whose terms are the energy functionals involving the small parameter ε. The limit is a functional with constant coefficients which we call effective material parameters. We follow the homogenization theorem [4] and we apply it to the case of the first strain-gradient model of elasticity with periodic microstructure.

2. Equations of the first strain-gradient model of elasticity

The statical equations of equilibrium are [5]:

$$\tau_{ij,i}-\mu_{ijk,ik}+X_j=0 \quad \text{in } \Omega\subset R^3,\ i,j,k=1,2,3, \tag{2.1}$$

here $\tau_{ij}=\tau_{ji}$, μ_{ijk} denotes the stress tensor and the couple stress tensor, respectively. X_j is the body force vector per unit volume.
We suppose that the internal energy per unit volume has the form:

$$\mathscr{A}(\varepsilon_{ij},\varkappa_{ijk})=\frac{1}{2}K_{ijkl}\varepsilon_{ij}\varepsilon_{kl}+\frac{1}{2}M_{ijklmn}\varkappa_{ijk}\varkappa_{lmn}+N_{ijklm}\varepsilon_{ij}\varkappa_{klm}, \tag{2.2}$$

where:

$$\varepsilon_{ij}=u_{(i,j)},\qquad \varkappa_{ijk}=\varepsilon_{jk,i}, \tag{2.3}$$

$$K_{ijkl}=K_{klij}=K_{jikl},\ M_{ijklmn}=M_{lmnijk}, \tag{2.4}$$
$$N_{ijklm}=N_{jiklm},$$

and K_{ijkl}, M_{ijklmn}, N_{ijklm} are bounded and measurable functions in $\bar{\Omega}$ (Ω is a region in R^3 occupied by the body).

Then the constitutive equations become:

$$\tau_{ij}\equiv\partial\mathscr{A}/\partial\varepsilon_{ij}=K_{ijpq}\varepsilon_{pq}+N_{ijpqr}\varkappa_{pqr},$$
$$\mu_{ijk}\equiv\partial\mathscr{A}/\partial\varkappa_{ijk}=N_{pqijk}\varepsilon_{pq}+M_{ijkpqr}\varkappa_{pqr}. \tag{2.5}$$

Moreover, we assume that the form $\mathscr{A}(\cdot,\cdot)$ is positive definite i.e. there exists such a number $c>0$ that for all $X\in\Omega$ holds:

$$\mathscr{A}(\varepsilon_{ij},\varkappa_{ijk})\geqslant c\sum_{i,j,k=1}^{3}(\varepsilon_{ij}^2+\varkappa_{ijk}^2). \tag{2.6}$$

Now, we define the microperiodic structure of the real medium.

Let:

$$Y=[0,Y_1]\times[0,Y_2]\times[0,Y_3]\subset R^3, \tag{2.7}$$

after [2] we shall call it a basic cell.

Moreover we assume that functions:

$$K_{ijkl}(y),\ M_{ijklmn}(y),\ N_{ijklm}(y)\in L^\infty(Y),\ y\in Y, \tag{2.8}$$

i.e. they are bounded and measurable functions and can be extended to the whole R^3 as Y-periodic functions. Now we define Y periodic coefficients by the following assumptions:

$$K^\varepsilon_{ijkl}(x)=K_{ijkl}(y),$$
$$M^\varepsilon_{ijklmn}(x)=M_{ijklmn}(y),\qquad x=\frac{y}{\varepsilon}, \tag{2.9}$$
$$N^\varepsilon_{ijklm}(x)=N_{ijklm}(y).$$

For a fixed ε the internal energy function per unit volume has the form:

$$\mathscr{A}(\varepsilon_{ij},\varkappa_{ijk})=\frac{1}{2}K^\varepsilon_{ijkl}(x)\varepsilon_{ij}\varepsilon_{kl}+\frac{1}{2}M^\varepsilon_{ijklmn}(x)\varkappa_{ijk}\varkappa_{lmn}+N^\varepsilon_{ijklm}(x)\varepsilon_{ij}\varkappa_{klm}. \tag{2.10}$$

3. The Concept of Γ — convergence and homogenization theorem

Let (X, τ) be a topological space and (F_h), $h \in N$, $F_h: X \to \bar{R}$ a sequence of functions ($\bar{R}$ denotes closure of R).

Following [4] we define:

$$\Gamma^-(\tau)\limsup_{\substack{h\to\infty \\ y\to x}} F_h(y) = \sup_{U\in\tau(x)}\limsup_{h\to\infty}\inf_{y\in U} F_h(y), \tag{3.1}$$

$$\Gamma^-(\tau)\liminf_{\substack{h\to\infty \\ y\to x}} F_h(y) = \sup_{U\in\tau(x)}\liminf_{h\to\infty}\inf_{y\in U} F_h(y), \tag{3.2}$$

where $\tau(x)$ is the family of open sets, for the topology τ, containing x.

When:

$$\Gamma^-(\tau)\limsup_{\substack{h\to\infty \\ y\to x}} F_h(y) = \Gamma^-(\tau)\liminf_{\substack{h\to\infty \\ y\to x}} F_h(y). \tag{3.3}$$

We shall denote their common value by:

$$\Gamma^-(\tau)\lim_{\substack{h\to\infty \\ y\to x}} F_h(y), \tag{3.4}$$

or briefly by:

$$\Gamma^-(\tau)\lim_{h\to\infty} F_h(x). \tag{3.5}$$

We shall say that $F = \Gamma^-(\tau)\lim\limits_{h\to\infty} F_h$, iff

$$\forall x \in X \quad F(x) = \Gamma^-(\tau)\lim_{h\to\infty} F_h(x). \tag{3.6}$$

In other words $\{F_h\}_{h\in N}$ converges in the sense of Γ-convergence to the limit $F(x)$. The homogenization theorem given in [4] is used to formulate appropriate theorem in the case of the first strain-gradient model of elasticity. In [4] the theorem is given for the case of scalar field (see below) $\alpha \in R^1$, but the proof of the theorem can be repeated in 3 dimensional case without any important changes. The proof is long and we decided to omit it. Now, we shall formulate theorem:

let:

$$f\colon R^3 \times R^3 \times R^{3^2} \times R^{3^3} \to R^+,$$
$$(x, \alpha, \beta, \xi) \to f(x, \alpha, \beta, \xi),$$

be an integrand satisfying:

$$\begin{array}{ll} \text{(i)} & x \to f(x, \alpha, \beta, \xi) \text{ is } Y\text{— periodic}, \\ & (Y \text{ is a basic cell in } R^3), \\ \text{(ii)} & \xi \to f(x, \alpha, \beta, \xi) \text{ is convex}, \\ \text{(iii)} & \lambda|\xi|^2 \leqslant f(x, \alpha, \beta, \xi) \leqslant \Lambda(1+|\alpha|^2+|\beta|^2+|\xi|^2), \\ & (\lambda, \Lambda \text{— constant}), \\ \text{(iv)} & |f^{1/2}(x, \alpha, \beta, \xi)-f^{1/2}(x, \alpha', \beta', \xi)| \leqslant s(|\alpha-\alpha'|+|\beta-\beta'|), \\ & (s \text{ — constant}). \end{array} \tag{3.7}$$

Let:

$$F_\varepsilon(u, \Omega) = \int_\Omega f\left(\frac{x}{\varepsilon}, u(x), Du(x), D^2u(x)\right) dx, \tag{3.8}$$

then,

$$\forall \Omega \subset \Theta_n,$$

$\forall u \in [W^{2,2}_{loc}(R^3)]^3$ (Θ_n-family of open bounded sets in R^3),

$$\Gamma^-(s^- W^{1,2}(\Omega)) \lim_{\varepsilon \to 0} F_\varepsilon(u, \Omega) = F_0(u, \Omega), \tag{3.9}$$

with

$$F_0(u, \Omega) = \int_\Omega f_0(u(x), Du(x), D^2u(x)) dx, \tag{3.10}$$

and

$$f_0(\alpha, \beta, \xi) = \min_{u \in W_Y} \frac{1}{|Y|} \int_Y f(x, \alpha, \beta, D^2u(x) + \xi) dy, \tag{3.11}$$

where: $W_Y = \{u \in [W^{2,2}_{loc}(R^3), \; u \text{ is } Y\text{-periodic}\}$,
s — topology in $W^{1,2}(\Omega)$.
The spaces $W^{2,2}_{loc}(R^3)$, $W^{1,2}(\Omega)$ are the proper Sobolev spaces. Using (2.10) we define:

$$f(x, \alpha, \beta, \xi) = \mathscr{A}^\varepsilon(\beta, \xi) = \frac{1}{2} K(y)\beta\beta + \frac{1}{2} M(y)\xi\xi + N(y)\beta\xi \tag{3.12}$$

with $y = \dfrac{x}{\varepsilon}$.
Now, we verify the assumptions (i)—(iv).
The (i) follows from (2.8) and (2.9) i.e. from the assumption about periodic structure of the body.
The (ii) follows from the square form of f as a function of ξ.
The (iii) is fulfilled becouse of (2.6) and the fact that all quantities $K_{ijkl}(y)$, $M_{ijklmn}(y)$, $N_{ijklm}(y)$ are bounded.
The (iv) is proved by using the average value theorem. The energy density of the homogenized body writes:

$$f_0(\beta, \xi) = \min_{\Theta \in W_Y} \frac{1}{|Y|} \int_Y \left[\frac{1}{2} K\beta\beta + \frac{1}{2} M(D^2\Theta + \xi)(D^2\Theta + \xi) + N\beta(D^2\Theta + \xi)\right] dy, \tag{3.13}$$

(we use abbreviate notation).
To find Θ which minimizes the functional we shall calculate the variations of integrand with respect to Θ:

$$\delta_\Theta \left[\frac{1}{2} K\beta\beta + \frac{1}{2} M(D^2\Theta + \xi)(D^2\Theta + \xi) + N\beta(D^2\Theta + \xi)\right]. \tag{3.14}$$

Assuming:

$$\delta_\Theta(\cdot) = 0, \tag{3.15}$$

we take:

$$\Theta = \chi(y)\xi(x) + \overline{\chi}(y)\beta(x) \tag{3.16}$$

and as a final result we get the set of equations which should be fullfiled on the basic cell Y:

$$\begin{aligned} \nabla\nabla(M\nabla\nabla\chi) &= -\nabla\nabla M, \\ \nabla\nabla(M\nabla\nabla\overline{\chi}) &= -\nabla\nabla N. \end{aligned} \tag{3.17}$$

The above system of equations determines the „cell-problem". One can see that the coefficients $K_{ijkl}(y)$ are not taken into account. The solution $(\chi, \overline{\chi})$ exists and is unique (with accuracy up to polinomial of the 1st order) iff:

$$(\chi, \overline{\chi}) \text{ is } Y \text{ periodic and } (\chi, \overline{\chi}) \in W^{4,2}(Y).$$

Substituting (3.16) into (3.13) we get:

$$\begin{aligned} f_0(\beta, \xi) = & \left(\frac{1}{|Y|}\int_Y \frac{1}{2} K dy\right)\beta\beta + \left[\frac{1}{|Y|}\int_Y \frac{1}{2} M(D^2\chi+I)(D^2\chi+I)\right]\xi\xi + \\ & + \left[\frac{1}{|Y|}\int_Y \frac{1}{2} MD^2\overline{\chi}D^2\overline{\chi}dy\right]\beta\beta + \left[\frac{1}{|Y|}\int_Y ND^2\overline{\chi}dy\right]\beta\beta + \\ & + \left[\frac{1}{|Y|}\int_Y M(D^2\chi+I)D^2\overline{\chi}dy\right]\beta\xi + \left[\frac{1}{|Y|}\int_Y N(D^2\chi+I)dy\right]\beta\xi, \end{aligned} \tag{3.18}$$

or:

$$f_0(\beta, \xi) = \frac{1}{2} K^{ef}\beta\beta + \frac{1}{2} M^{ef}\xi\xi + N^{ef}\beta\xi, \tag{3.19}$$

where:

$$\begin{aligned} K^{ef} &= \frac{1}{|Y|}\int_Y K dy + \frac{1}{|Y|}\int_Y MD^2\overline{\chi}D^2\overline{\chi}dy + \frac{1}{|Y|}\int_Y 2ND^2\overline{\chi}dy, \\ M^{ef} &= \frac{1}{|Y|}\int_Y M(D^2\chi+I)(D^2\chi+I)dy, \\ N^{ef} &= \frac{1}{|Y|}\int_Y M(D^2\chi+I)D^2\overline{\chi}dy + \frac{1}{|Y|}\int_Y N(D^2\chi+I)dy. \end{aligned} \tag{3.20}$$

The effective functional $F_0(u, \Omega)$ has the simple form:

$$F_0(u, \Omega) = \int_\Omega \left(\frac{1}{2} K^{ef}_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + \frac{1}{2} M^{ef}_{ijklmn}\varkappa_{ijk}\varkappa_{lmn} + N^{ef}_{ijklm}\varepsilon_{ij}\varkappa_{klm}\right)dx. \tag{3.21}$$

It is clear that fields χ and $\overline{\chi}$ are not need to be known.
We use only the second derivatives of them.

4. One-dimensional example

Assuming that all quantities depend on one variable only we deal with reduced problem of (2.1):

$$\frac{d\tau}{dx}-\frac{d^2}{dx^2}\mu+X=0 \quad \text{in} \quad \Omega\subset R^1, \tag{4.1}$$

where:

$$\begin{aligned}\tau &= K^\varepsilon(x)\frac{du}{dx}+N^\varepsilon\frac{d^2u}{dx^2},\\ \mu &= M^\varepsilon(x)\frac{d^2u}{dx^2}+N^\varepsilon\frac{du}{dx}.\end{aligned} \tag{4.2}$$

If, independently, we formally reduced the cell problem to a one-dimensional we get:

$$\begin{aligned}\frac{d^2}{dy^2}M(y)\frac{d^2}{dy^2}\chi(y) &= -\frac{d^2}{dy^2}M(y),\\ \frac{d^2}{dy^2}M(y)\frac{d^2}{dy^2}\bar{\chi}(y) &= -\frac{d^2}{dy^2}N(y).\end{aligned} \quad y\in Y \tag{4.3}$$

The solutions (second derivatives of χ, $\bar{\chi}$) have the form:

$$\begin{aligned}\frac{d^2\chi(y)}{dy^2} &= -1+\frac{C_1}{M(y)},\\ \frac{d^2\bar{\chi}(y)}{dy^2} &= \frac{C_2}{M(y)}-\frac{N(y)}{M(y)},\end{aligned} \tag{4.4}$$

where:

$$C_1=\left\langle\frac{1}{M(y)}\right\rangle^{-1},$$

$$C_2=\left\langle\frac{N(y)}{M(y)}\right\rangle C_1, \quad \langle\cdot\rangle\equiv\frac{1}{|Y|}\int_Y(\cdot)dy.$$

Using formulae (3.20) we get:

$$\begin{aligned}K^{ef} &= \langle K\rangle+\left\langle\frac{N}{M}\right\rangle^2\left\langle\frac{1}{M}\right\rangle^{-1}-\left\langle\frac{N^2}{M}\right\rangle,\\ M^{ef} &= \left\langle\frac{1}{M}\right\rangle^{-1},\\ N^{ef} &= \left\langle\frac{1}{M}\right\rangle^{-1}\left\langle\frac{N}{M}\right\rangle.\end{aligned} \tag{4.5}$$

If one assumed the stronger conditions of continuity for the functions $K^\varepsilon_{ijkl}(x)$, $M^\varepsilon_{ijklmn}(x)$, $N^\varepsilon_{ijklm}(x)$ one can use the G — convergence method (compare [3], [6], [7]) to obtain the effective properties of the medium under consideration. But in the case this method provides to a long calculations (4^{th} order differential operator) and the proof of the proper homogenization theorem is not trivial one. In contrast, the Γ-convergence concept applied in this paper gives the results (i.e. effective material parameters) almost immediately and in a very elegant manner.

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Резюме

ГОМОГЕНИЗАЦИЯ ГРАДИЕНТНОЙ СРЕДЫ ПЕРВОГО ПОРЯДКА

Проведена гомогенизация градиентной среды первого порядка методом Γ-сходимости. Предлагая вид внутренней энергии для действительной ε-периодической структуры получено гомогенизированную внутреннюю энергию для однородной эффективной среды. Коэффициенты в гомогенизированном функционале энергии являются эффективными материальными постоянными. Они зависят от решения т. наз. задачи на ячейке.

Streszczenie

HOMOGENIZACJA OŚRODKA GRADIENTOWEGO PIERWSZEGO RZĘDU

Przeprowadzono homogenizację ośrodka gradientowego 1-go rzędu metodą Γ-zbieżności. Zakładając postać energii wewnętrznej dla rzeczywistej ε-periodycznej struktury wyznaczono zhomogenizowaną energię wewnętrzną dla jednorodnego ciała efektywnego. Współczynniki w zhomogenizowanym funkcjonale energii są efektywnymi stałymi materiałowymi. Zależą one od rozwiązania tzw. problemu na komórce.

Praca wplynęła do Redakcji dnia 28 września 1987 roku.