MECHANIKA
TEORETYCZNA
I STOSOWANA
4, 26 (1988)

YU — PL'87

THREE-DIMENSIONAL MIXED FINITE ELEMENTS

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1. Introduction

The basis for the development of three-dimensional elements are the two-dimensional plane stress elements. By introduction of the third dimension for instance the triangular elements become tetrahedron elements, the rectangular become hexahedron etc.

The development of the three-dimensional elements is similar to the development of the two-dimensional elements and doesn't represent any particular problem. However, the number of degrees of freedom (d.o.f.) of the three-dimensional elements, and consequently the number of equations which have to be solved for a particular three-dimensional problem, is large. The solutions of such problems sometimes become impossible even by computers of large capacity. Therefore the development of three-dimensional elements reducing the total number of equations for a solution of any three-dimensional problem, is very important.

The most of the developed three-dimensional elements are of stiffness type. The primary nodal unknowns are the deformations. The elements are developed by application of the minimum potential energy variational principle. The problem in the development of the elements represents the so called "parasitic stresses". In such a case the parasitic stresses are the shear stresses — the functions of the nodal axial deformations. Usually the problem is solved by application of the "reduced selective integration". However, such an approach to solution of the problem is not appropriate and reliable. As a result of the approach for instance some three-dimensional elements are good for analysis of thin systems like plate bending analysis (20 nodes, 60 d.o.f. element), the others for analysis of thick systems, like plane stress analysis (8 nodes, 24 d.o.f. element) [1].

In the early development of the mixed elements it was assumed independence of stresses and deformations. The correctness of such assumption was not verified for a long time. Therefore the mixed finite element method (FEM) was doubtful and was not widely applied. Here we develope mixed elements by application of a direct method [6]. The deformation shape function (DShF) is complete, depends on the nodal stresses and deformations. The boundary forces and deformations are derived directly from DShF, without application of any variational principle. From the distribution of these values at the nodes we derive equivalent nodal forces and nodal deformations, which represent the element matrix. This concept of development of the elements was applied in the problems of plane stress and plate bending elements [2, 3].

The three-dimensional element which we consider in this paper, is based on the plane stress element (Fig. 1a). This plane stress element gives very good results. For instance

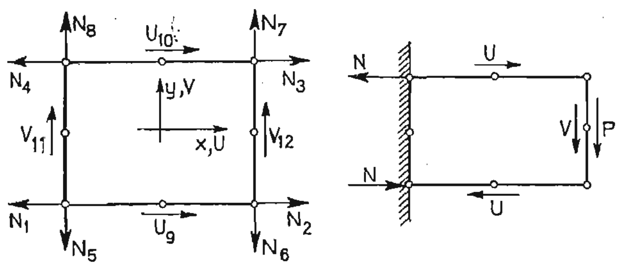

Fig. 1. Mixed plane stress element (a), cantilever represented by one element (b)

in the analysis of a cantilever (Fig. 1b) even one element, with totaly 5 equations, gives exactly the beam solution! Such results were obtained primarily due to particular distribution of the unknowns (degrees of freedom). Note that the stresses and the displacements, were taken at the midside nodes. In such cases the degrees of freedom, the displacements and the stresses, are not independent of each other. For instance by variation of the stresses we can develope the equations of the compatibility of displacements, which are already satisfied due to the presence of the displacements as the degrees of freedom at the same nodes.

The element was developed by application of the direct method of development of finite elements [6]. In this way the parasitic shear stresses $S_{xy} = f(N_{xi}, N_{yi})$, functions of the axial stresses, were automatically excluded. The exclusion of the stresses was another reason for the very good behaviour of the element. The element developed in this way is the same as the element developed by the assumption of independent stresses and deformations. It means that, in the case of energetic approach, the stresses and the displacements have to be independently assumed! The correctness of such assumption for the first time was proved in Ref. [4].

The element which we discuss is a simple prismatic three-dimensional element. The element is our first step in the development of mixed three-dimensional elements. The development of the element has to show the way of development of the mixed elements and the accuracy which could be expected. The element presented here gives very good accuracy and is very promising for further development of the mixed FEM.

2. Prismatic element with 36 d.o.f.

The element is presented on Fig. 2. At the corner nodes the unknowns are the stress components (8 × 3), and at the edge nodes are the displacements in the direction of the edges (12 × 1), it makes 36 d.o.f. in total.

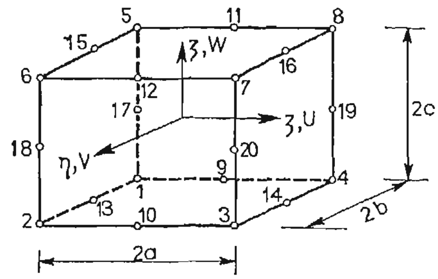

Fig. 2. Prismatic mixed element with 36 d.o.f.

2.1. The deformation shape function. The number of parameters defining the DShF is equal to the number of the degrees of freedom — 36. It means that the deformation shape function should be defined by 3 polynomials with 12 parameters each, for the 3 deformation components u, v and w separately. However, an assumption of 12 terms polynomial, as the standard procedure in the FEM, can be misleading. We shall discuss this below. Here somewhat different procedure will be considered.

If an edge of the element, for instance the edge 1 - 4, Fig. 2, is considered as an axially loaded rod, the deformations in the rod will be defined by the following expression:

$$U_{14} = U_9 + \frac{1}{2}[\varepsilon_{1x} a\xi(1-\xi/2) + \varepsilon_{4x} a\xi(1+\xi/2)], \quad (\xi = x/a). \tag{1}$$

These deformations are translated in the xy plane, in such a way that the deformations at the edge 2 - 3 vanishe, and next translated along z axis so that the side $5 \div 8$ the deformations vanish. The contribution of the nodal parameters U_9, ε_{1x} and ε_{4x} to the deformation U in the element is defined as follows:

$$U = \frac{1}{8}[2U_9 + \varepsilon_{1x} a\xi(1-\xi/2) + \varepsilon_{4x} a\xi(1+\xi/2)](1-\eta)(1-\zeta). \tag{2}$$

In this way the complete DShF can be defined and represented as follows:

$$\begin{bmatrix} U \\ V \\ W \end{bmatrix} = \begin{bmatrix} \Phi_x & \Phi_u & 0 & 0 & 0 & 0 \\ 0 & 0 & \Phi_y & \Phi_v & 0 & 0 \\ 0 & 0 & 0 & 0 & \Phi_z' & \Phi_w \end{bmatrix} \begin{bmatrix} \varepsilon_{xi} \\ U_i \\ \varepsilon_{yi} \\ V_i \\ \varepsilon_{zi} \\ W_i \end{bmatrix}, \tag{3}$$

where we have:

$$\Phi_{xi} = \frac{a}{8}(1+\xi_0/2)\xi(1+\eta_0)(1+\zeta_0),$$

$$\Phi_{yi} = \frac{b}{8}(1+\xi_0)(1+\eta_0/2)\eta(1+\zeta_0), \quad i = 1,8 \tag{4}$$

$$\Phi_{zi} = \frac{c}{8}(1+\xi_0)(1+\eta_0)(1+\zeta_0/2)\zeta,$$

$$\Phi_{ui} = \frac{1}{4}(1+\eta_0)(1+\zeta_0), \quad i = 9, 12$$

$$\Phi_{vi} = \frac{1}{4}(1+\xi_0)(1+\zeta_0), \quad i = 13, 16 \tag{5}$$

$$\Phi_{wi} = \frac{1}{4}(1+\xi_0)(1+\zeta_0), \quad i = 17, 20$$

$$\xi_0 = \xi\xi_i, \quad \eta_0 = \eta\eta_i, \quad \zeta = \zeta\zeta_i$$

It is interesting to note that the polynomial by which for instance deformations U are defined is of the following type:

$$U = a_1 + a_2 x + a_3 y + a_4 z + a_5 x^2 + a_6 xy + a_7 xz + a_8 yz + a_9 x^2 y + a_{10} x^2 z + + a_{11} xyz + a_{12} x^2 yz. \tag{6}$$

The polynomial is not a "complete" one, as usually is required. It is evident that such a polynomial is very difficult to assume in advance. The polynomial corresponds to a lline-ar variation of strains (stresses), similarly to the stresses at the element in Fig. 2.

2.2. The element matrix. The element matrix can be represented as follows:

$$\mathsf{Fk} = \begin{bmatrix} \mathsf{F}_n & \mathsf{F}_{uvw} \\ \mathsf{F}^t_{uvw} & \mathsf{K} \end{bmatrix}. \tag{7}$$

The vector of nodal parameters is the following:

$$d^t = [N_{1x} \ldots N_{8x} \ldots N_{1y} \ldots N_{8y} \ldots N_{1z} \ldots N_{8z} \ldots U_9 \ldots U_{12} V_{13} \ldots V_{16} W_1 \ldots W_{20}]. \tag{8}$$

The first row submatrices in the element matrix represents the flexibility submatrices. The submatrix F_n gives nodal displacements in direction of the axial stresses due to the same stresses. Submatrix F_{uvw} affords the displacements due to the midedge node displacements. The second row represents the stiffness submatrices. Submatrix F^t_{uvw} gives the nodal forces in the direction of the edge nodes due to the axial stresses, and submatrix K-nodal forces in the same direction due to the edge node displacements.

2.2.1. Flexibility submatrices F_n and F_{uvw}. From the DShF (3 - 5) for $\xi = -1$ the deformations on the surface 1,2, 6,5 are derived. These deformations, for instance due to ε_{1x} are as follows:

$$U = \varepsilon_{x1}\Phi_{x1} = \frac{-3a}{16}\varepsilon_{x1}(1-\eta)(1-\zeta). \tag{9}$$

The volume of these deformations is:

$$\iint U dy dz = \frac{-3}{4} abc\varepsilon_{x1}.$$

The volume is equal to the nodal deformations 1, 2, 6, 5 due to the same strain. The distribution due to the particular node is as follows:

$$F_{nij} = \iint U\delta U(\varepsilon_i) dy dz,$$

where U is as Exp. (9) and $\delta U(\varepsilon_i)$ is the variation of the displacement on the particular strains. In this way the derived nodal displacements due to ε_{x1} take the following form:

$$[U_1 U_2 U_6 U_5]^t = \frac{-abc}{12}[4 \ 2 \ 1 \ 2].$$

From ε_{x1} we obtain deformations also on the surface 3, 4, 8, 7. The nodal deformations derived in the same way for these nodes are as follows:

$$[U_3\,U_4\,U_8\,U]^t = \frac{-abc}{36}\,[2\ \ 4\ \ 2\ \ 1].$$

These two submatrices define the first column of submatrix F_n. In this way the complete submatrix F_n can be derived. The submatrix can be derived by an energetic way as follows:

$$F_{nij} = \iint \varepsilon_i\,\delta\varepsilon_i\,dx\,dy\,dz.$$

The values of the same nodal displacements due to ε_{x1} derived in this way are as follows:

$$[U_1\,U_2\,U_6\,U_5\,U_3\,U_4\,U_8\,U_7]^t = \frac{-abc}{27}\,[8\ \ 4\ \ 2\ \ 4\ \ 2\ \ 4\ \ 2\ \ 1].$$

The coefficients derived in both ways are not the same, but their sum in one row or column is the same, equal to 1. Therefore submatrix F_n defined in both ways finaly would give the same results. Here it is convenient to use F_n derived by the energetic way. Submatrix F_n derived in this way is the following:

$$\mathsf{F}_n = \begin{bmatrix} \mathsf{F}_{nx} & 0 & 0 \\ 0 & \mathsf{F}_{ny} & 0 \\ 0 & 0 & \mathsf{F}_{nz} \end{bmatrix}, \tag{10}$$

$$\mathsf{F}_n^0 = \mathsf{F}_{nx} = \frac{-abc}{27}\begin{bmatrix} 8 & 4 & 2 & 4 & 4 & 2 & 1 & 2 \\ & 8 & 4 & 2 & 2 & 4 & 2 & 1 \\ & & 8 & 4 & 1 & 2 & 4 & 2 \\ & & & 8 & 2 & 1 & 2 & 4 \\ & & & & 8 & 4 & 2 & 4 \\ & & Sym. & & & 8 & 4 & 2 \\ & & & & & & 8 & 4 \\ & & & & & & & 8 \end{bmatrix} = \mathsf{F}_{ny} = \mathsf{F}_{nz}. \tag{11}$$

If instead of strains ε we introduce stresses N, for instance for strains ε_x:

$$\varepsilon_x = \frac{1}{E}\,[N_x - \nu(N_y + N_z)],$$

submatrix F_n becomes:

$$\mathsf{F}_n = \frac{1}{E}\begin{bmatrix} \mathsf{F}_n^0 & -\nu\mathsf{F}_n^0 & -\nu\mathsf{F}_n^0 \\ -\nu\mathsf{F}_n^0 & \mathsf{F}_n^0 & -\nu\mathsf{F}_n^0 \\ -\nu\mathsf{F}_n^0 & -\nu\mathsf{F}_n^0 & \mathsf{F}_n^0 \end{bmatrix}. \tag{12}$$

In the same way submatrix F_{uvw} is derived. This submatrix can be defined as follows:

$$\mathsf{F}_{uvw} = \begin{bmatrix} \mathsf{F}_u & 0 & 0 \\ 0 & \mathsf{F}_v & 0 \\ 0 & 0 & \mathsf{F}_w \end{bmatrix} \tag{13}$$

The values of the submatrices take the form:

$$\mathsf{F}_u^t = \frac{bc}{9}\begin{bmatrix} 4 & 2 & -2 & -4 & 2 & 1 & -1 & -2 \\ 2 & 4 & -4 & -2 & 1 & 2 & -2 & -1 \\ 2 & 1 & -1 & -2 & 4 & 2 & -2 & -4 \\ 1 & 2 & -2 & -1 & 2 & 4 & -4 & -2 \end{bmatrix}, \tag{14}$$

$$\mathsf{F}_v^t = \frac{ac}{9}\begin{bmatrix} 4 & -4 & -2 & 2 & 2 & -2 & -1 & 1 \\ 2 & -2 & -4 & 4 & 1 & -1 & -2 & 2 \\ 2 & -2 & -1 & 1 & 4 & -4 & -2 & 2 \\ 1 & -1 & -2 & 2 & 2 & -2 & -4 & 4 \end{bmatrix}, \tag{15}$$

$$\mathsf{F}_w^t = \frac{ab}{9}\begin{bmatrix} 4 & 2 & 1 & 2 & -4 & -2 & -2 & -1 \\ 2 & 4 & 2 & 1 & -2 & -4 & -1 & -2 \\ 1 & 2 & 4 & 2 & -1 & -2 & -2 & -4 \\ 2 & 1 & 2 & 4 & -2 & -1 & -4 & -2 \end{bmatrix}. \tag{16}$$

2.2.2. Stiffness submatrix K. The stiffness submatrix K gives the nodal shear forces in the direction of the edge node displacements, due to the displacements of the nodes. For instance the forces due to $U_9 = 1$ are as follows:

$$N_{xy} = \frac{-G}{4b}(1-\zeta),$$

$$N_{xz} = \frac{-G}{4c}(1-\eta)$$

$$N_{yz} = 0$$

The shear force due to the shear stresses acting on the surface 1, 4, 8, 5 takes the form:

$$S_{xy} = \frac{2a2c}{2}, \quad N_{xy} = G\frac{ac}{b}.$$

Since the distribution of the shear stresses from node 9 to 11 is triangular, 2/3 of this shear force is applied to node 9 and 1/3 to node 11. In this way the defined nodal forces give the coefficients of the stiffness submatrix K. The submatrix can be represented as follows:

$$\mathsf{K} = \frac{G}{3}\begin{bmatrix} \mathsf{K}_1 & \mathsf{K}_{12} & \mathsf{K}_{13} \\ & \mathsf{K}_2 & \mathsf{K}_{23} \\ Sym. & & \mathsf{K}_3 \end{bmatrix}. \tag{17}$$

The values of submatrices in this expression are the following:

$$\mathsf{K}_1 = \begin{bmatrix} 2(\alpha c+\beta b) & -2\alpha c+\beta b & \alpha c-2\beta b & -\alpha c-\beta b \\ & 2(\alpha c+\beta b) & -\alpha c-\beta b & \alpha c-2\beta b \\ & & 2(\alpha c+\beta b) & -2\alpha c+\beta b \\ \text{Sym.} & & & 2(\alpha c+\beta b) \end{bmatrix}, \tag{18}$$

$\alpha = a/b$, $\beta = a/c$,

$$K_{12} = \begin{bmatrix} 2 & -2 & 1 & -1 \\ & 2 & -1 & -2 \\ & & 2 & -2 \\ Sym. & & & 2 \end{bmatrix} c, \tag{19}$$

$$K_{13} = \begin{bmatrix} 2 & 1 & -2 & -1 \\ & 2 & -1 & -2 \\ & & 2 & 1 \\ Sym. & & & 2 \end{bmatrix} b. \tag{20}$$

Submatrix K_2 is obtained if instead of α in K_1 we substitute α^{-1}.

$$K_{23} = \begin{bmatrix} 2 & -2 & 1 & -1 \\ 1 & -1 & 2 & -2 \\ -2 & 2 & -1 & 1 \\ -1 & 1 & -2 & 2 \end{bmatrix} a, \tag{21}$$

$$K_3 = c \begin{bmatrix} 2(\alpha^{-1}+\alpha) & \alpha^{-1}-2\alpha & -\alpha^{-1}-\alpha & -2\alpha^{-1}+\alpha \\ & 2(\alpha^{-1}+\alpha) & -2\alpha^{-1}+\alpha & -\alpha^{-1}-\alpha \\ & & 2(\alpha^{-1}+\alpha) & \alpha^{-1}-2\alpha \\ Sym. & & & 2(\alpha^{-1}+\alpha) \end{bmatrix}. \tag{22}$$

2.3. Numerical examples.

2.3.1. Simply supported square plate. The plate is subdivided in 4 elements (Fig. 3a). Since there is the symmetry, only a quarter of the plate is analyzed, (Fig. 3b). There are only 3 unknowns: N_6, U_{10} and W_{18}. The analysis gives the following results:

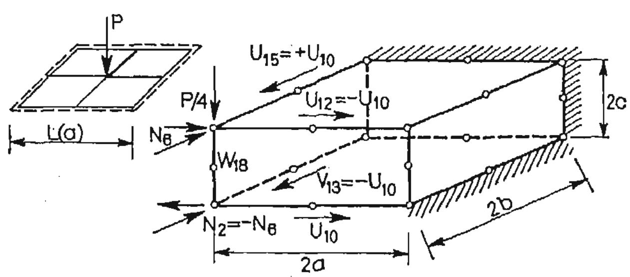

Fig. 3. Analysis of simply supported square plate (a), the quarter of the plate represented by one element (b)

$$N_6 = -\frac{9}{32c^2} P(1+\nu),$$

$$U_{10} = \frac{3aP}{16Ec^2}(1-\nu^2),$$

$$W_{18} = \frac{-3PL^2}{192D}\left[1+\frac{16c^2}{L^2(1-\nu)}\right] = -0{,}01562\frac{PL^2}{D}\left[1+\frac{16c^2}{L^2(1-\nu)}\right],$$

(Thin Plate Th.0,0116PL²/D).

The results for the case of constant distributed load q can be derived by the substitution of qa^2 instead of $P/4$. The results are as follows:

$$N_6 = \frac{-9qL^2}{128c^2}(1+\nu),$$

$$U_{10} = \frac{3qa^3}{4Ec^2}(1-\nu^2),$$

$$W_{18} = \frac{-3qL^4}{768D}\left[1+\frac{16c^2}{L^2(1-\nu)}\right] = 0{,}00391\frac{qL^4}{D}\left[1+\frac{16c^2}{L^2(1-\nu)}\right],$$

(Thin plate theory: 0,00406 qL^4/D).

The following bending moment corresponds to stress N_6:

$$M_6 = N_6\frac{(2c)^2}{6} = \frac{3qL^2}{64}(1+\nu) = 0{,}0609qL^2, \ (\nu = 0{,}3),$$

(Thin plate theory: 0,0479 qL^2)

The results for the displacement W_{18} contain two components: the thin plate theory component and the contribution of the shear forces component. The fist component is the same as is the result obtained by our very first rectangular mixed bending element [7], with the assumption that the moments and displacements are independent. The second component will be analyzed in the next chapter.

2.3.2. Cantilever. The analysis of the cantilever in Fig. 4 loaded by concentrated edge force gives the following results:

$$N_6 = \frac{3aP}{2bc^2},$$

$$W_{20} = \frac{P}{2Ec}\left[4{,}0\frac{a^3}{c^3}+2\frac{a}{c}(1+\nu)\right].$$

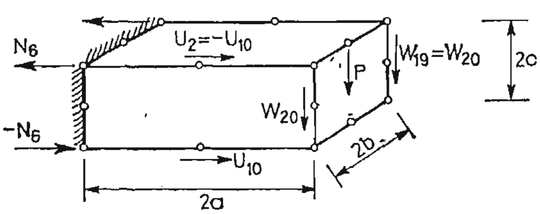

Fig. 4. Analysis of a cantilever as one finite element

These results are exactly the beam solution, with cross section shape coefficient $k = 1$. As could be exepected, the three-dimensional element gives exactly the same results as the plane stress element from Fig. 1 [2]. In the case of a cantilever loaded by a moment, the element also gives also the beam solution.

3. Reduced three-dimensional element for plate bending analysis

3.1. Element matrix. The reduced three-dimensional element is presented in Fig. 5. The primary unknowns for the element are the following:

$$d^t = [M_{1x} \ldots M_{4x} \ldots M_{1y} \ldots M_{4y} \ldots \Theta_9 \ldots \Theta_{12}, W_1 \ldots W_4]. \tag{23}$$

It means that the element is reduced to 16 d.o.f.

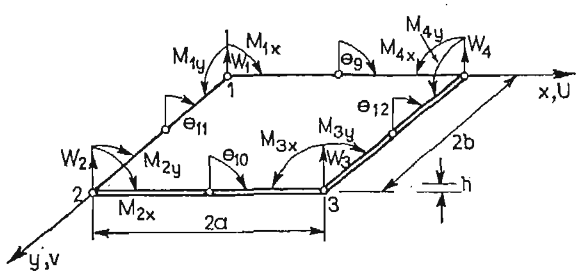

Fig. 5. Reduced plate bending element

The strains for the three-dimensional element can be substituted by the bending moments as follows:

$$\varepsilon_x = \frac{c}{D(1-\nu^2)}(M_x - \nu M_y),$$

$$\varepsilon_y = \frac{c}{D(1-\nu^2)}(-\nu M_x + M_y).$$

The displacements u and v can be substituted by the rotations, for instance according to to the following relation:

$$\Theta_9 = (U_9 - U_{11})/2c.$$

The upper stresses and displacements, and the lower stresses and displacements, in the case of plate bending are of the same intensity, but with different signes, for instance:

$$N_1 = -N_5$$
$$U_9 = -U_{11}$$

The element matrix can be represented as follows:

$$\mathsf{Fk} = \begin{bmatrix} \mathsf{F}_m & -\nu\mathsf{F}_m & \mathsf{F}_{\theta x} & 0 & 0 \\ & \mathsf{F}_m & 0 & \mathsf{F}_{\theta y} & 0 \\ & & \mathsf{K}_{\theta x} & \mathsf{K}_{\theta xy} & \mathsf{K}_{\theta xw} \\ & & & \mathsf{K}_{\theta y} & \mathsf{K}_{\theta yw} \\ Sym. & & & & \mathsf{K}_w \end{bmatrix}. \tag{24}$$

Submatrix F_m derived in this way is the same as for the element with the assumption that the moments and displacements are independent. The derivation of the other submatrices is very simple and will not be given.

3.2. Analysis of thick plates. As an illustration of the accuracy of the reduced three-dimensional element, and consequently of the original element, the simply supported square plate in Fig. 3a will be analyzed. The results of the analysis are presented in the table below and in Figs. 6, 7.

Table 1. Results of the analysis of thick simply supported square plate

Thickness / Span (h/L)	Constant distributed load		Concent. Force
	W/W^0_{ex}	M/M^0_{ex}	W/W^0_{ex}
0.0125	1.0059	1.054	1.1055
0.075	1.031	1.054	1.160
0.125	1.077	1.054	1.256
0.200	1.187	1.054	1.492
0.250	1.290	1.054	1.709

The results presented in the table are the ratio of the computed deflection W at the center of the plate versus the theoretical thin plate deflection, and the similar ratio of the computed M and theoretical M^0_{ex} moments. In the case of very thin plates ($h/L = 0.0125$) the element gives the same results as the plate bending element with the assumption that the moments and deflections are independent. The moments do not depend on the thickness of the plate. The results are derived by subdivision of the quarter of the plate on 2×2 elements.

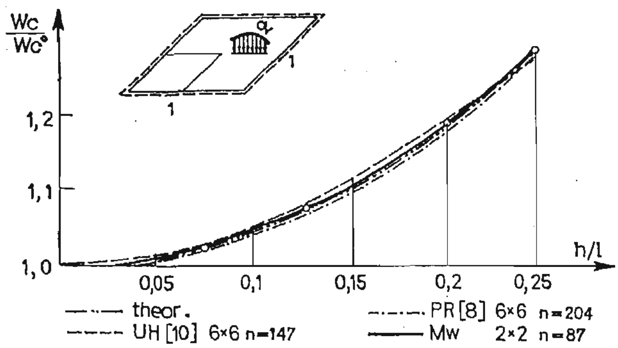

Fig. 6. Central displacement of simply supported square thick plate subjected to constant distributed load

With the increase of the thickness of the plate the relative deflections of the plate increase (Figs. 6 and 7). Besides the results obtained by our element Mw, we present in the figures the results obtained by Prior et al. [8] (Pr) with stiffness element, Chang-Chun Wu [10] (UH) — with a hybrid element, and Rao et al. [9] (Ra) — with a triangular stiffness element. The number n besides the particular results denotes the number of equations by means of which the results were obtained.

From Fig. 6 one can see that our mixed element (Mw) gives excellent results, with values somewhat higher than the theoretical ones, while the stiffnes element (Pr) gives similar accuracy, with values below the exact. The results in the case of concentrated force (Fig. 7) take the values between those obtained by the other. In region of thin plates the results are not so good, since the rough mesh was used (2×2). If the refined mesh (4×4) is used then the results fall down close to the exact ones (doted line in the figure).

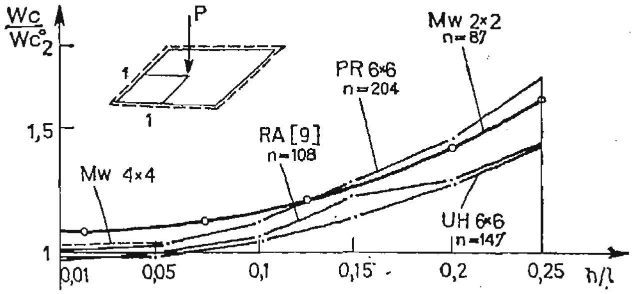

Fig. 7. Central displacement of simply supported square thick plate subjected to concentrated force

A disadvantage of the stiffness elements for analysis of thick plates is that they give bad results for thin plates. In the case of thin plates there is the so called "locking" phenomenon. Contrary to that, our mixed element gives good results for thick and thin plates, regardless of their thickness.

4. Further development and conclusions

The three-dimensional element presented here represents our first step in the development of mixed three-dimensional elements. Due to its shape, the practical application of the element is limited. The purpose of the development of the element was to show the way of development and expected accuracy of the mixed elements.

The accuracy of the element is very good. In the case of plane stress problem the element gives the same results as the corresponding plane stress element [2]. The plane stress element is one of the best elements available at present. In the analysis of plate bending the three-dimensional element gives the same results as the plate bending element with assumption that the moments and displacements are independent [7]. The plate bending element gives also very good results.

Such results obtained by means of the presented element are encouraging for further research of the three-dimensional mixed elements. As the next step in the development of mixed elements should be the development of a general hexahedron element (Fig. 8a) with 36 d.o.f. It would be convenient if such an element to be taken with displacements along the edges as the degrees of freedom. The same element could be developed with curved boundaries. The development of the element could be based on the assumption that the stresses and displacements are independent. It was shown that such an assumption is correct and has advantage in the exclusion of the parasitic stresses. The final aim would be to develope such elements explicitly.

The best mixed element that can be developed is the in Fig. 8b, with curved boundaries and 60 d.o.f. Such an element corresponds to our plane stress isoparametric element

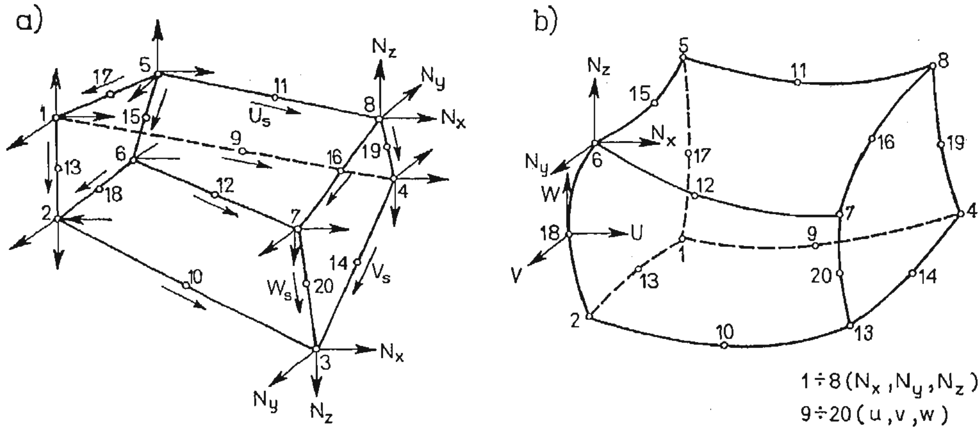

Fig. 8. Possible three-dimensional elements: (a) with 36 d.o.f. (b) with 60 d.o.f.

with 16 d.o.f. [2]. This plane stress element seems to be the best plane stress element available at present. Thus, the corresponding three-dimensional element should be expected to give excellent results also. The development of such an element is in progress.

The reduced three-dimensional element with 16 d.o.f. succesfully can be applied for analysis of plate bending. The element gives results of very good accuracy, unconditionally stable, regardless the stiffness of the plate. Now we will try to eliminate the rotations as d.o.f., although they are internal d.o.f., and in such a way develope a 12 d.o.f. plate bending element for analysis of thick plates.

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Р е з ю м е

ТРЁХМЕРНЫЕ СМЕШАННЫЕ КОНЕЧНЫЕ ЭЛЕМЕНТЫ

Следующие параметры являются неизвестными в элементе: компоненты напряжений в угловых узлах (8×3) и перемещения граний (12×1). Элемент развитый при помощи непосредственного метода, без применения вариационного принципа. Известные „паразитные напряжения" исключаются автоматически. Сведенный трёхмерный элемент дает очень хорошие результаты в анализе толстых и тонких плит. В работе показан путь развития смешанных элементов и ожидаемая точность. Далее рассмотрены изопараметрические элементы любых форм (рис. 8, а - б). Найлучшим ожидаемым элементом является показаный на рис. 8б элемент с 60 степенями свободы, который теперь разработывается.

S t r e s z c z e n i e

TRÓJWYMIAROWE MIESZANE ELEMENTY SKOŃCZONE

Nieznanymi parametrami elementu są składowe naprężenia w węzłach narożnych (8×3) i przemieszczenia krawędzi (12×1). Elementy zostały rozwinięte z pomocą metody bezpośredniej bez zastosowania zasady wariacyjnej. „Naprężenia parametryczne" zostały wyeliminowane automatycznie. Zredukowane trójwymiarowe elementy dają znakomite wyniki w analizie płyt grubych i cienkich.

Celem pracy jest pokazanie sposobu rozwoju mieszanych elementów i spodziewanej dokładności ich stosowania. Oprócz tego rozpatrujemy izoparametryczne elementy o dowolnych kształtach (rys. 8a - b). Najlepszym jest element pokazany na rys. 8b z 60 stopniami swobody. Jest on obecnie rozwijany.

Praca wpłynęła do Redakcji dnia 29 czerwca 1987 roku.