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ON INVARIANCE OF FINITE ELEMENT APPROXIMATIONS

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1. Introduction

In order that some physical law is a law of the nature, it can not depend on the choice of the coordinate system where it is applied. In view of the fact that these laws are represented by mathematical equations, this means that the form of natural laws (i.e. their equations) do not depend on the system in which they are formulated — they are invariant with respect to the operation of the change of the coordinate system. If one understands these laws as relations between mathematical objects, invariant in the sense of tensor calculus, the invariant mathematical objects will be the tensor fields, while the natural laws will be described by the tensor equations.

On the other hand, in the applications of the theory we are most frequently forced to use the approximations of natural laws; however, this is not the reason to desist from the request that these approximative laws would be "natural" too. After all, what we call "the natural laws" are only the approximative forms of the true laws of the nature, and nevertheless we request theirs invariance! This request, if we stay on the natural laws described by the tensor equations, would mean that the approximations of tensor fields which take part in these equations, must be invariant under coordinate transformations.

We shall see in the next section what are the repercussions of the request for invariance of finite element approximations in Euclidean space.

2. Invariant finite element approximations in Euclidean space

Let us start from the following interpolation formula for one vector function:

$$\boldsymbol{v}(x^a) = P^K(x^a)\boldsymbol{v}(x^a_K) = P^K(x^a)\boldsymbol{v}_K. \tag{1}$$

where P^K are interpolation functions, and x^a are arbitrary curvilinear coordinates in (three-dimensional) Euclidean space; Einstein's summation convention for diagonally repeated indices will be used; index K relates to the points in the space where the values of the vector function were done. There is nothing new in the vector representation (1) and it is quoted in this form for example by Oden ((7.48) in [2]), but immediately rejected as "less accurate" than "the usual approximation" ((7.51) in [2]). However, let us look for the coordinate form of the representation (1); after the multiplication with base vectors, we shall have:

$$\{v(x^a)\cdot g^b(x^a)=\}v^b(x^a)=P^K(x^a)v(x^a_K)\cdot g^b(x^a)$$
$$=P^K(x^a)v^c(x^a_K)g_c(x^a_{(K)})\cdot g^b(x^a)$$
$$=P^K(x^a)g^b_c(x^a, x^a_{(K)})v^c_K. \quad (2)$$

since the scalar product of the base vectors at different points of the space is equal to the Euclidean shifters ([1], p. 806); placement of index K in the parentheses in (2) means that the summation convention is not applied to the corresponding member — in summation over K this member is simply associated to the other members with this index.

There is another way to obtain representation (2). Let us start from the usual expression in the rectangular Cartesian system for the approximation of coordinates of the vector function under consideration:

$$v^j(z^i)=Q^K(z^i)v^j(z^i_K)=Q^K(z^i)v^j_K. \quad (3)$$

where now Q^K are some interpolation functions. In order to give to this expression an invariant form, we introduce arbitrary generalized coordinates:

$$x^a=x^a(z^i). \quad (4)$$

Under this coordinate transformation we shall have:

$$v^c(x^a)\left.\frac{\partial z^j}{\partial x^c}\right|_{x^a}=Q^K[z^i(x^a)]v^c(x^a_K)\left.\frac{\partial z_j}{\partial x^c}\right|_{x^a_{(K)}}=R^K(x^a)v^c_K\left.\frac{\partial z^j}{\partial x^c}\right|_{x^a_{(K)}} \quad (5)$$

or, after multiplication by $\partial x^b/\partial z^j|_{x^a}$:

$$v^b(x^a)=R^K(x^a)\left.\frac{\partial x^b}{\partial z^j}\right|_{x^a}\left.\frac{\partial z^j}{\partial x^c}\right|_{x^a_{(K)}}v^c(x^a_k)=R^K(x^a)g^b_c(x^a, x^a_{(K)})v^c_K \quad (6)$$

and that is the same formula as (2); (we used the fact that the shifters are given by:

$$g^b_c(x^a, x^a_K)=\left.\frac{\partial x^b}{\partial z^j}\right|_{x^a}\left.\frac{\partial z^j}{\partial x^c}\right|_{x^a_K} \quad (7)$$

s. [1], p. 807). In this way, the equivalence between vectorial and coordinate approach in obtaining the invariant approximation is proved. Anyhow, one can say that the interpolation (6) reduces to:

$$v^b(x^a)=R^K(x^a)v^b_K(x^a). \quad (8)$$

i.e. to the summation at the point x^a shifted nodal values of a vector function. In any case, the shifters (which has not appeared in (13.94) at Oden, when the vector-valued representation has been used) are introduced in a natural way in the approximations of vector function — by passing on the curvilinear coordinates; this is just the consequence of the request that the interpolation procedure must be invariant. It is clear that this invariant process can be also extended on the tensor fields.

Only in rectangular Cartesian coordinates, when the shifters are the Kronecker delta, the expression (2), i.e. (6) reduces to the usual finite element approximation for the coordinates of a vector field:

$$v^b(x^a)=P^K(x^a)v^b(x^a_K)=P^K(x^a)v^b_K. \quad (9)$$

However, the approximation (9) (which is simpler than (2) or (6), since it does not include the shifters) has not the above-mentioned property of invariance. Let us perform in (9) the transvection with base vectors, and we shall have the following representation of a vector field:

$$\boldsymbol{v}(x^a) = v^b(x^a)g_b(x^a) = P^K(x^a)v^b(x^a_K)g_b(x^a) \tag{10}$$

(we emphasize that geometrically is incorrect to indicate the expression:

$$v^b(x^a_K)g_b(x^a) \tag{11}$$

as the value of a vector field at the point x^a_K; cf. (7.45) in [2]). If, by the transformation:

$$y^p = y^p(x^a) \tag{12}$$

we introduce another curvilinear coordinates y^p, we can also write (10) in the form:

$$\begin{aligned} v^q(y^p)\boldsymbol{h}_q(y^p) &= \boldsymbol{v}(y^p) = \boldsymbol{v}[x^a(y^p)] \\ &= P^K[x^a(y^p)] \left.\frac{\partial x^b}{\partial y^q}\right|_{y^p_{(K)}} v^q(y^p_K \left.\frac{\partial y^r}{\partial x^b}\right|_{y^p} \boldsymbol{h}_r(y^p) \\ &= Q^K(y^p)v^q(y^p_K)\left.\frac{\partial x^b}{\partial y^q}\right|_{y_{(K)}} \left.\frac{\partial y^r}{\partial x^b}\right|_{y^p} \boldsymbol{h}_r(y^p); \end{aligned} \tag{13}$$

in general case, this is different from the representation obtained by starting from the approximation for coordinates of vector field analogous to (9), but in the system of curvilinear coordinates (12). Consequently, the approximation in the form of (9) is not really invariant under the transformations of the coordinate system. This means that the form (9) would not be used (except in Cartesian orthogonal coordinates) in approximations of one natural law, if we request its invariance.

3. Example: comparison of two methods of interpolation

For the sake of comparison of two approaches (the usual and the invariant one), we shall consider a vector field defined on a cylindrical surface. Let us prescribe the values of the field at the points A, B, C and D (s. Fig. 1), so that in the cylindrical polar system:

$$v^2(x^a_A) = v^2(x^a_B) = v^2(x^a_C) = v^2(x^a_D) = 0 \tag{14}$$

and:

$$v^3(x^a_A) = v^3(x^a_B) = v^3(x^a_C) = v^3(x^a_D) = 0. \tag{15}$$

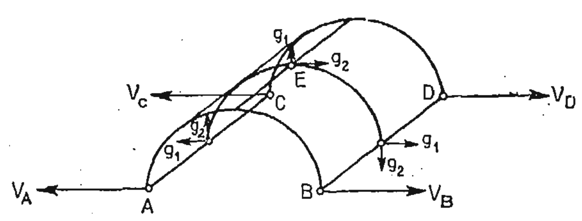

Fig. 1.

Regardless of the interpolation functions assumed in these approximation procedures, these two approaches will be essentially different. To be assured in that, we will first determine the value of the vector field at the point E by the usual approximation (10); it will be:

$$v^1(x_E^a) = v(x_E^a)g^1(x_E^a) = P^K(x_E^a)v^b(x_K^a)g_b(x_E^a)g^1(x_E^a) = P^K(x_E^a)v^1(x_K^a) \neq 0 \tag{16}$$

and:

$$v^2(x_E^a) = v(x_E^a)g^2(x_E^a) = P^K(x_B^a)v^b(x_K^a)g_b(x_E^a)g^2(x_E^a) = P^K(x_E^a)v^2(x_K^a) = 0; \tag{17}$$

here we use the orthogonality of cylindrical coordinates, and in (17) we use the assumption (14) too. However, if we use the invariant approximation in the form of (2) for the vector field in question, we shall have:

$$\begin{aligned} v^1(x_E^a) = v(x_E^a)g^1(x_E^a) &= P^K(x_E^a)v^b(x_K^a)g_b(x_{(K)}^a)g^1(x_E^a) \\ &= P^K(x_E^a)v^2(x_K^a)g_2(x_{(K)}^a)g^1(x_E^a) = 0 \end{aligned} \tag{18}$$

and:

$$\begin{aligned} v^2(x_E^a) = v(x_E^a)g^2(x_E^a) &= P^K(x_E^a)v^b(x_K^a)g_b(x_{(K)}^a)g^2(x_E^a) \\ &= P^K(x_E^a)v^1(x_K^a)g_1(x_{(K)}^a)g^2(x_E^a) \neq 0; \end{aligned} \tag{19}$$

here we have used the fact that:

$$g^1(x_E^a) \perp g_1(x_A^a), g_1(x_B^a), g_1(x_C^a), g_1(x_D^a) \tag{20}$$

and:

$$g^2(x_E^a) \perp g_2(x_A^a), g_2(x_B^a), g_2(x_C^a), g_2(x_D^a), \tag{21}$$

as well as the assumption (14).

Generally, one can say that the first approximation procedure gives the field of radially distributed vectors, while the second one gives the field of vectors parallel to the prescribed vectors at the points A, B, C and D.

4. Concluding remarks and future work

The basic conclusion is the following: the usage of the shifting operators in a coordinate form of approximations of vector and tensor fields in an arbitrary curvilinear coordinate system in (three-dimensional) Euclidean space is necessary if we want to realize the invariance of the approximative form of a natural law in which these fields take part. Only in Cartesian orthogonal coordinates these approximations coincide with usual expressions for the approximation of coordinates of vector and tensor fields.

The dwelling upon Euclidean space has, on the one hand, its reasons in the fact that we have been primarely interested in (finite element) approximations in such physical theory like mechanics of continua. More definitely, the necessity of the consistent introduction of shifters into interpolation formulae was appeared in the three-field theory [3] (in the case of the use of these formulae in curvilinear coordinates). On the other hand, it is difficult to speak about operators of parallel displacement (in the sense of the abovementioned shifters) in non-euclidean spaces.

The acceptance of the presented procedure of the invariant interpolation will request, for example, to carry out the finite element equations of motion in arbitrary curvilinear coordinates. However, the naturalness of this interpolation is not the guarantee of its simplicity — the shifters, in which variables are not separate, will arise explicitly in it. In any case, the presented approach would be justified in numerical examples, in the sense that we will essay to explain some effects by the consistent introducing of shifting operators.

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Р е з ю м е

ИНВАРИАНТНЫЕ АПРОКСИМАЦИИ КОНЕЧНЫХ ЭЛЕМЕНТОВ

Фазические законы будут „натуральными" тогда когда они инвариантны (ковариантны) при трансформации системы координат. Мы можем обсуждать такую инвариантность также в случае приближений натуральных законов если приймём, что эти законы описаны тензорными уравнениями (в 3-мерном) пространстве Евклида. В результате мы получаем явную форму евклидовых шифтеров (в общей криволинейной системе координат) для приближений векторного, или в общем случае, тензорного поля.

S t r e s z c z e n i e

O NIEZMIENNICZOŚCI PRZYBLIŻEŃ ELEMENTÓW SKOŃCZONYCH

Prawa fizyczne są „naturalne" jeżeli są niezmiennicze (kowariantne) względem transformacji układu współrzędnych. Możemy badać taką niezmienniczość również w przypadku przybliżeń naturalnych praw jeżeli założymy, że prawa te są opisane równaniami tensorowymi (w trójwymiarowej) przestrzeni Euklidesa. W wyniku otrzymujemy jawną postać (w dowolnym krzywoliniowym układzie współrzędnych) euklidesowych przybliżeń wektorowego, lub w ogólnym przypadku, tensorowego pola.

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