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NONAUTONOMOUS NONLINEAR VIBRATIONS OF A CONTINUOUS SELF-EXCITED SYSTEM. A PLATE IN SUPERSONIC FLOW

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1. Introduction

Continuous self-excited systems, which occur in aircraft structures, as well as in other fields of technology, have properties important for structural engineering. Elastic and damping coefficients of such systems depend on the parameter of self-excitation and in the case of external loading or parametrically exciting forces acting on the system, its resonance characteristics also vary with the degree of self-excitation.

Nonautonomous linearized vibration problems of continuous self-excited systems have been considered in a number of papers (cf. for instance, [1], [2], [3]).

Nonlinear analysis of forced vibrations of a continuous self-excited system has been presented in [4] and nonlinear parametric self-excited vibrations in [5]. In the present paper vibrations of a continuous self-excited system will be analysed by way of example of a plate of finite length in plane supersonic flow as shown in Fig. 1. Nonlinear membrane forces induced by the plate motion will be included into analysis.

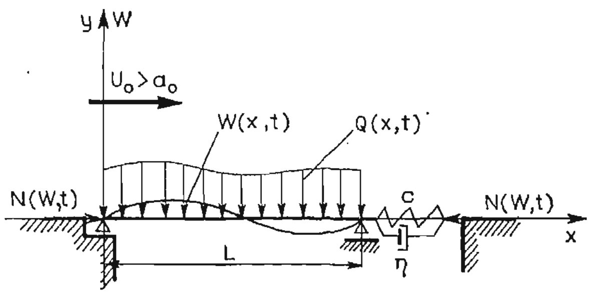

Fig. 1.

The plate is forced by a harmonically varying pressure and periodic in-plane loads parametrically exciting the system.

The phenomena to be investigated include regular and chaotic motions which may occur in such a nonlinear self-excited system (cf. [6]).

2. Equations of the problem

Let us consider vibrations of a plate of length L and infinite width assuming that the plate is exposed to a one-side supersonic flow, the unperturbed velocity of which U_0 is parallel to the x-axis (Fig. 1).

The equation of motion of the plate will be written in the form:

$$D\left(1+\Theta_0\frac{\partial}{\partial t}\right)\frac{\partial^4 W}{\partial x^4}+N(W,t)\frac{\partial^2 W}{\partial x^2}+\varrho_p h\frac{\partial^2 W}{\partial t^2}=\Delta p(W)-Q(x,t), \tag{1}$$

where the pressure difference $\Delta p(W)$ is produced by the normal displacement $W = W(x,t)$ of the plate in supersonic gas stream [1].

$$\Delta p(W) = -\varrho_0 U_0^2\frac{1}{\mu}\left[\frac{\partial W}{\partial x}+\frac{1}{U_0}\left(1-\frac{1}{\mu^2}\right)\frac{\partial W}{\partial t}\right], \tag{2}$$

and

$$\mu=\sqrt{M^2-1},$$

$M = U_0/a_0 > 1$ is Mach number of unperturbed flow.

The variable nonlinear membrane force $N(W,t)$ is assumed as:

$$N(W,t)=N(t)-\left(1+\beta_0\frac{\partial}{\partial t}\right)\frac{c_0}{2}\int_0^L\left(\frac{\partial W}{\partial x}\right)^2 dx. \tag{3}$$

In Eqs. (1) and (3) the viscoelastic Voigt model of the plate material and nonlinear membrane forces are taken into account.

Parametrically exciting part of the membrane forces $N(t)$ is taken in the form:

$$N(t)=N_0+\varepsilon_0\cos p_p t, \tag{4}$$

and external pressure $Q(x,t)$, we assume as:

$$Q(x,t)=Q_1(x)\cos p_e t+Q_2(x)\sin p_e t, \tag{5}$$

where the frequencies p_p and p_e may be different in the general case.

In Eqs. (1), (2), (3) the following symbols have been used:

$D=\dfrac{Eh^3}{12(1-\nu^2)}$ — plate stiffness, c_0 — spring-stiffness coefficient of the plate elastic support, θ_0, β_0 — coefficients of material damping of the plate and its deformable support, ϱ_p, h — density and thickness of the plate, a_0, ϱ_0 — sound velocity and gas density of the unperturbed flow.

3. Transformation of the equations

The problem will be studied in the dimensionless form, assuming that the coordinate x is referred to the length of the plate L, the normal displacement W — to its thickness h, and the time t — to the ratio L/U_0. The equation of motion is then obtained in the form:

$$\left(1+\Theta\frac{\partial}{\partial t}\right)\frac{\partial^4 W}{\partial x^4}+S(W,t)\frac{\partial^2 W}{\partial x^2}+\lambda_1 M^2\frac{\partial^2 W}{\partial t^2}=\lambda_0\Delta p(W)+\lambda_2 P(x,t), \tag{6}$$

where we have:

$$\lambda_0 = 12(1-\nu^2)\frac{L^4}{Eh^4}, \quad \lambda_1 = 12(1-\nu^2)\frac{L^2}{h^2}\frac{\varrho_p a_0^2}{E},$$
$$\lambda_2 = 12(1-\nu^2)\frac{L^4}{h^4}\frac{\varrho_0 a_0^2}{E}, \quad P(x,t) = -\frac{Q(x,t)}{\varrho_0 a_0^2}. \tag{7}$$

The dimensionless in-plane force:

$$S(W,t) = 12(1-\nu^2)\frac{L^2}{Eh^3}N(W,t) = S(t) - \left(1+\beta\frac{\partial}{\partial t}\right)c\int_0^1\left(\frac{\partial W}{\partial x}\right)^2 dx, \tag{8}$$

and the pressure difference:

$$\Delta p(W) = -\varrho_0 U_0^2 \frac{h}{L\mu}\left[\frac{\partial W}{\partial x} + \left(1-\frac{1}{\mu^2}\right)\frac{\partial W}{\partial t}\right]. \tag{9}$$

The solution of Eq. (6) is sought in the form of a series of normalized eigenfunctions $X_j(x)$ of the self-adjoined boundary-value vibration problem of the plate considered in the vacuum with:

$$\Theta = S(W,t) = 0, \tag{10}$$

and it is assumed that the dimensionless load $P(x,t)$ can be expanded in a series of functions $X_j(x)$. Confining ourselves to n terms of these series, we have:

$$W(x,t) = \sum_{j=1}^{n} w_j(t)X_j(x),$$
$$P(x,t) = \sum_{j=1}^{n} h_j(t)X_j(x). \tag{11}$$

For a plate simply supported at both edges — that is, for $x = 0$ and $x = 1$ — the eigenfunctions are:

$$X_j(x) = \sqrt{2}\sin j\pi x, \quad j = 1, 2, 3, \ldots. \tag{12}$$

On substituting (11) and (12) into Eq. (6), taking into account (8), (9) and making use of the Galerkin method, we obtain a set of equations of the following form:

$$\frac{1}{\omega_1^2}\ddot{w}_j + \frac{\eta_j}{\omega_1}\dot{w}_j + u_j^2 w_j + \gamma_1\sum_{k=1}^{n} b_{jk}w_k = -\alpha j^2 w_j \sum_{k=1}^{n} k^2\left(w_k^2 + \frac{\beta_1}{\omega_1}\dot{w}_k w_k\right) + \tag{13}$$
$$+\varepsilon_1 j^2 w_j \cos p_p t + \gamma_0 h_j(t),$$

for $j = 1, 2, 3, \ldots, n$,
where:

$$\omega_1 = \pi^2/M\sqrt{\lambda_1}, \quad \eta_j = \omega_1\left[\gamma_1\left(1-\frac{1}{\mu^2}\right)+\Theta j^4\right],$$
$$u_j^2 = j^2(j^2-\sigma), \quad \gamma_0 = \frac{L^2\varrho_0\lambda_1}{h^2\varrho_p\pi^4}, \tag{14}$$

$$\alpha = 6(1-\nu^2)\frac{Lc_0}{hE}, \quad \sigma = 12(1-\nu^2)\frac{L^2N_0}{\pi^2h^3E},$$

$$\gamma_1 = \frac{L\varrho_0}{h\varrho_p\mu\omega_1^2}, \quad \beta_1 = 2\omega_1\beta, \quad \varepsilon_1 = \frac{\sigma}{N_0}\varepsilon_0.$$

The skew-symmetric coefficients b_{jk}, for $j \neq k$ assume the form:

$$b_{jk} = -b_{kj} = \frac{2jk}{j^2-k^2}[1-(-1)^{j+k}] \tag{15}$$

while for $j = k$, $b_{jk} = 0$.

The set of Eqs. (13) is taken as a starting point for the vibration analysis of the system considered under the assumption that, according to (5), the external pressure is a harmonically varying function:

$$h_j(t) = h_{1j}\cos p_e t + h_{2j}\sin p_e t. \tag{16}$$

4. Analysis of vibrations in the vicinity of critical parameters of the system

Let us assume that the vibrations occur in the vicinity of the critical state of the system under study, and the critical state is considered here as the boundary between damped and self-excited vibrations.

In this connection we shall examine the simplified set of equations which can be obtained by setting the right-hand sides of Eqs. (13) equal to zero.

The critical Mach number is denoted by M_{cr} and ω_{cr} is the real frequency corresponding to M_{cr}.

The reduced time τ is introduced:

$$\tau = \omega_1 t, \tag{17}$$

and Eqs. (13) will be transformed to give:

$$\ddot{v}_j + \eta_{j*}\dot{v}_j + u_j^2 v_j + \gamma_{1*}\sum_{k=1}^{n} b_{jk}v_k = \varepsilon F_j(v_1, \ldots, v_n, \dot{v}_1, \ldots, \dot{v}_n, \tau), \tag{18}$$

for $j = 1, 2, \ldots, n$,
where:

$$F_j(v_1, \ldots, v_n, \dot{v}_1, \ldots, \dot{v}_n, \tau) = (\eta_{j*}-\eta_j)\dot{v}_j + (\gamma_{1*}-\gamma_1)\sum_{k=1}^{n} b_{jk}v_k +$$
$$-\alpha j^2 v_j \sum_{k=1}^{n} k^2(v_k^2 + \beta_1\dot{v}_k v_k) + \varepsilon_1 j^2 v_j \cos q_p\tau + \gamma_0 h_j(\tau), \tag{19}$$

and we have:

$$v_j(\tau) = w_j\left(\frac{\tau}{\omega_1}\right), \quad q_p = p_p/\omega_1, \tag{20}$$

η_{j*} and γ_{1*} denote the values of the coefficients η_j and γ_1 for $M = M_{cr}$, ε is a small parameter.

The solution of the simplified set of Eqs. (18), for $\varepsilon = 0$, can be found as:

$$v_j(\tau) = \xi_j e^{iq\tau}. \tag{21}$$

The coefficients ξ_j satisfy the set of equations:

$$\sum_{k=1}^{n} \xi_k[(u_j^2 - q^2 + iq\eta_{j*})\delta_{jk} + \gamma_{1*} b_{jk}] = 0, \tag{22}$$
$$\text{for } j, k = 1, 2, \ldots, n,$$

where δ_{jk} is the Kronecker delta.

The frequency equation of the simplified Eqs. (18) has the form:

$$D_n = \det\|(u_j^2 - q^2 + iq\eta_{j*})\delta_{jk} + \gamma_{1*} b_{jk}\| = 0, \tag{23}$$
$$j, k = 1, 2, 3, \ldots, n.$$

From Eq. (23) we obtain the critical parameters:

$$M = M_{cr} \quad \text{and} \quad q = q_{1,2} = \pm q_{cr} \tag{24}$$

and the imaginary parts of the remainder roots of Eq. (23) are positive–that is, these roots correspond to damped vibrations.

It results from the above statement that Eqs. (18) satisfy the validity conditions of the asymptotic method of single-frequency analysis [7].

For this reason, we seek the solution of Eqs. (18) in the first-order approximation as:

$$v_j(\tau) = a(\xi_j e^{i\psi} + \bar{\xi}_j e^{-i\psi}), \tag{25}$$
$$\text{for } j = 1, 2, \ldots, n,$$

where:

$$a = a(\tau), \qquad \psi = q\tau + \vartheta(\tau). \tag{26}$$

The first derivatives of these functions can be found from the equations:

$$\dot{a} = \varepsilon A_1(a, \vartheta), \tag{27}$$
$$\dot{\vartheta} = q_{cr} - q + \varepsilon B_1(a, \vartheta),$$

and we also assume that:

$$q_{cr} - q = 0(\varepsilon). \tag{28}$$

On substituting the solution (25) into Eqs. (18), taking into account (26), (27) and employing the method of harmonic balance, we obtain the set of equations:

$$\sum_{k=1}^{n} \xi_k[(u_j^2 - q_{cr}^2 + iq_{cr}\eta_{j*})\delta_{jk} + \gamma_{1*} b_{jk}] = \varepsilon[2q_{cr}(aB_1 - iA_1)\xi_j - \eta_{j*}(A_1 + iaB_1)\xi_j + \Phi_j], \tag{29}$$

for $j = 1, 2, \ldots, n$,
where:

$$\Phi_j = \frac{1}{2\pi} \int_0^{2\pi} F_j(v_1, \ldots, v_n, \dot{v}_1, \ldots, \dot{v}_n, \psi - \vartheta) e^{-i\psi} d\psi. \tag{30}$$

From the set of Eqs. (29) we can determine the functions:

$$A_1 = A_1(a, \vartheta), \quad B_1 = B_1(a, \vartheta)$$

and then we obtain equations of resonance characteristics in the vicinity of the critical state and examine the question of stability of the steady-state vibrations under investigation.

5. Numerical analysis of vibrations

Numerical integration of the set of Eqs. (13) has been also performed in order to study the typical courses of vibrations, phase plane portraits of the motion and resonance characteristics of vibrations.

In this connection Eqs. (13) have been written in the form:

$$\begin{aligned}\ddot{v}_j+\eta_j\dot{v}_j+u_j^2v_j+\gamma_1\sum_{k=1}^{n}b_{jk}v_k = -\alpha j^2v_j\sum_{k=1}^{n}k^2(v_k^2+\beta_1\dot{v}_kv_k)+\\ +\varepsilon_1 j^2v_j\cos q_p\tau+\gamma_0(h_{1j}\cos q_e\tau+h_{2j}\sin q_e\tau),\end{aligned} \tag{31}$$

for $j = 1, 2, 3, \ldots, n$.

Vibrations were calculated for:

$$q_e = q_p/2, \quad q_e = q_p, \quad q_e = 0 \tag{32}$$

and for other relationships between the frequencies q_e and q_p.

The region of frequencies, taken into consideration in the numerical calculations, involved the first two natural frequencies of the system under investigation. In the vicinity of these two frequencies elastic and damping properties of the system depend on the parameter of self-excitation in a significant manner.

6. Examples of numerical analysis

Let us begin with the short analysis of the simplified, linearized nonautonomous vibration problem. The frequency equation of that problem is Eq. (23), which can be written as:

$$D_n = D_n(M, q) = \mathrm{Re}D_n+\mathrm{i}\,\mathrm{Im}D_n = 0. \tag{33}$$

By determining the real roots q in the function of number M from the first and the second of Eq. (33) separately, we can plot diagrams in the plane M, q, which are shown by way of example in Fig. 2 for $n = 4$ and plate of $L/h = 180$. The intersection point of the ines $\mathrm{Re}D_n = 0$ and $\mathrm{Im}D_n = 0$ determines the critical parameters (24). It follows from the computations performed that the location of the line $\mathrm{Re}D_n = 0$ depends in an insignificant manner on the value of the material damping Θ. The line $\mathrm{Im}D_n = 0$ is shifted to the left with increasing values of Θ. This fact results in a decrease in the critical parameters (24).

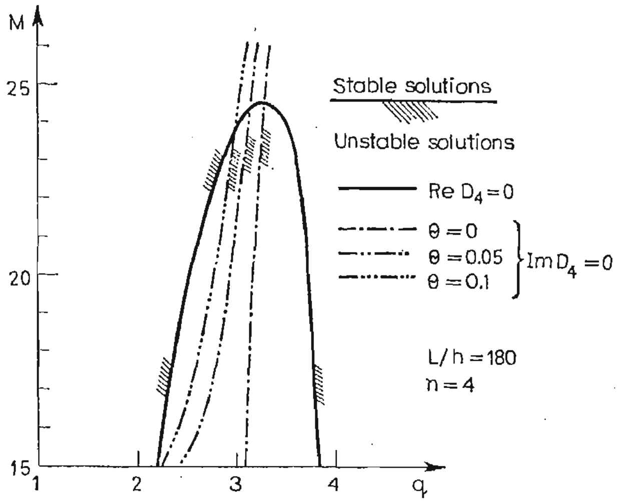

Fig. 2.

The form of the lines (33) for real values of q is also a cause of deformation of the resonance diagrams in the function of Mach number. This is shown by way of example in Fig. 3 for the forced vibration of a plate in supersonic flow (cf. [1]). In this figure $a(0.5)$ denotes the coefficient of dynamic amplification of the amplitude of vibration at the middle of the plate.

It follows from the computation that for $M < M_{cr}$ the maximum amplitudes of forced vibrations approach each other with increasing M and a sharp resonance occurs in the

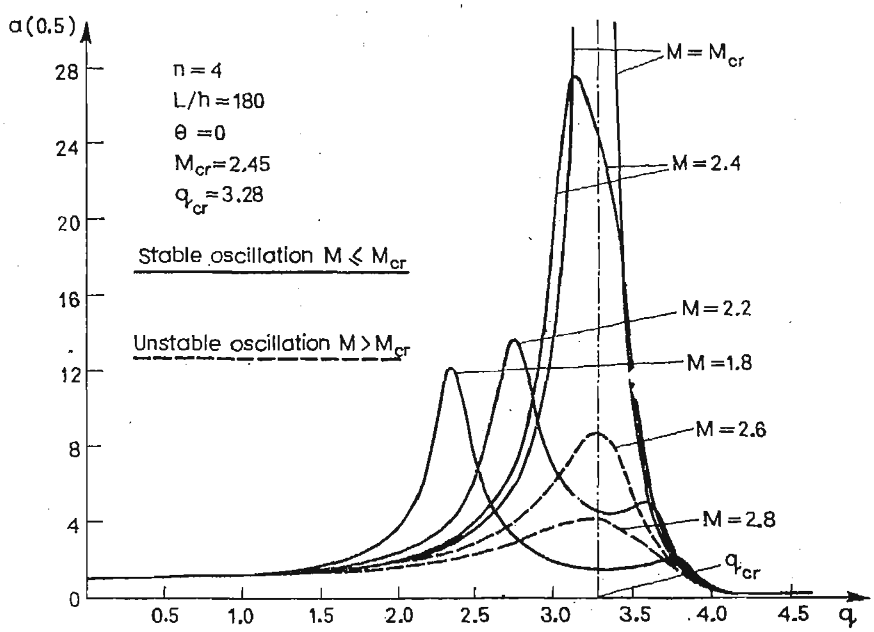

Fig. 3.

critical state (24). In the case of parametrically excited vibrations of a plate in supersonic flow, deformations of regions of parametric resonance have been studied (cf. [1]).

The vibrations have been investigated in the vicinity of the regions of subharmonic resonance corresponding to the first two natural frequencies of the structure.

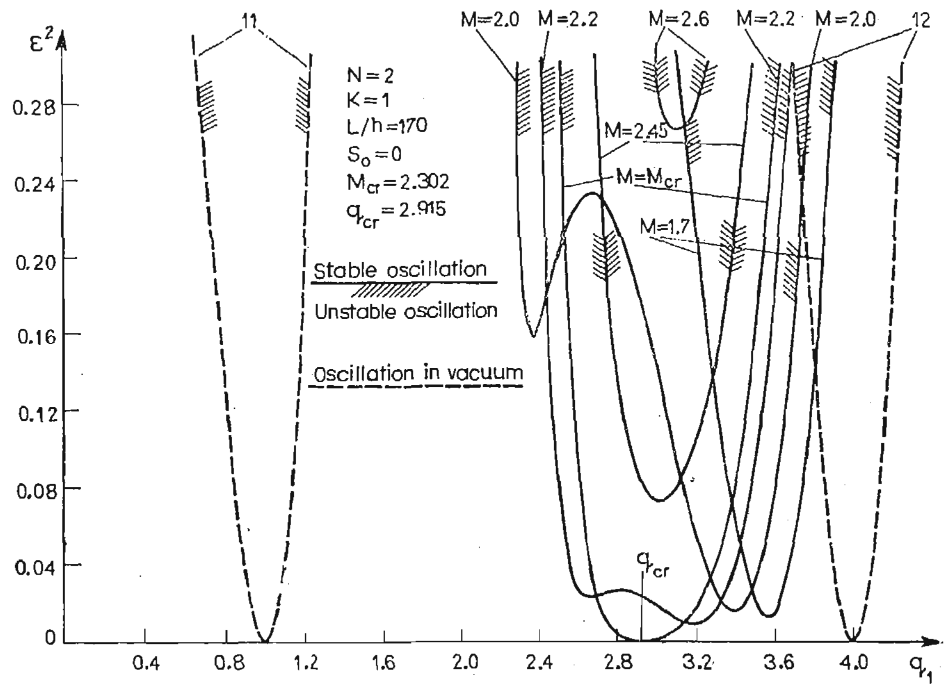

Fig. 4.

Some results of computation of parametrically excited vibrations are shown in Fig. 4, which presents deformations of the regions of unstable vibration on the plane ε^2, q_1, where ε is a reduced amplitude of parametric excitation and $q_1 = p_p/2\omega_1$.

It follows from the computation that in supersonic flow the largest region of unstable vibrations is the first region of subharmonic resonance which corresponds to the second natural frequency. The region corresponding to the first natural frequency undergoes strong degeneration resulting from the action of the flow.

For Mach numbers M approaching M_{cr}, for which the two natural frequencies approach each other, the resonance regions corresponding to these two natural frequencies unite and for $M = M_{cr}$ there occurs a single fundamental region of subharmonic resonance which touches the frequency axis at the point $q_1 = q_{cr}$. For $M > M_{cr}$ the image of regions of unstable vibrations becomes more complicated, this is described in [1].

In the case of the nonlinear system let us consider nonautonomous vibrations in the vicinity of critical parameters.

Equations (18), (19) will be used under the assumption that:

$$\varepsilon_1 = 0, \quad h_j(\tau) = h_{j0}\cos q\tau. \tag{34}$$

This enables us to study the forced vibrations and to determine the resonance characteristics of the system for subcritical and supercritical Mach numbers ($M < M_{cr}$ and $M > M_{cr}$) (cf. [4]).

Some results of computations are shown in Figs. 5 to 10. Diagrams present the ratio:

$$C_w(x) = C(x)/C(x)_{q=2.5}, \tag{35}$$

versus the dimensionless frequency of loading q in the interval $2.5 \leqslant q \leqslant 3.5$ for fixed values of M and x, where $C(x)$ is the amplitude of forced vibrations of the plate under study. The diagrams are plotted for a point with the coordinate $x = 0.75$, since from the computations performed it results that the maximum of the plate displacement occurs near this point (cf. also [1]).

The coefficient α, determining nonlinear properties of the system, plays the role of a parameter of those diagrams.

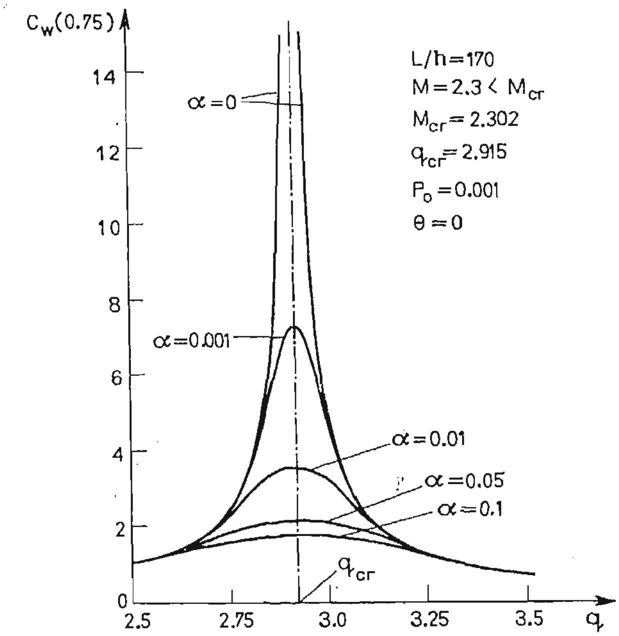

Fig. 5.

Figure 5 shows the results of computation for the Mach number $M = 2.3 < M_{cr}$ and $\Theta = 0$, while in the following three Figs. 6, 7, 8, the diagrams are presented for $M > M_{cr}$ and $\Theta = 0$.

It is seen in Fig. 5 that for a number $M < M_{cr}$ the nonlinear properties of the system considerably decrease the amplitude of forced vibration in the vicinity of the resonance frequency $q = q_{cr}$ even for small values of α.

For $M > M_{cr}$ the system investigated becomes a self-excited one and in the case for $\alpha > 0$ with $P(x, t) \equiv 0$, a stable limit cycle of self-excited vibration occurs with a frequency near to q_{cr}.

For a harmonically varying load $P(x, t)$, $M > M_{cr}$ and $\alpha > 0$, stable periodic forced vibrations appear in the vicinity of q_{cr}. They are shown by bold lines in Figs. 6 to 8.

For a given $\alpha > 0$, the maximum amplitude of stable forced vibrations increases with increasing M and the range of q, in which these vibrations occur, becomes narrower.

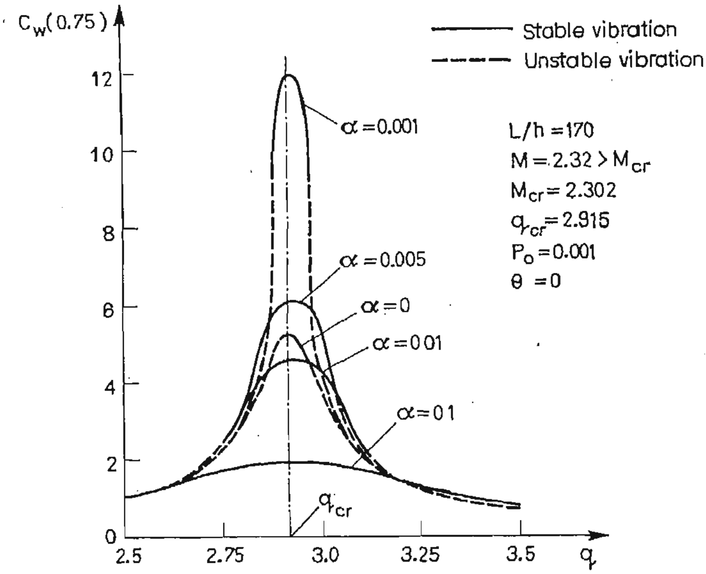

Fig. 6.

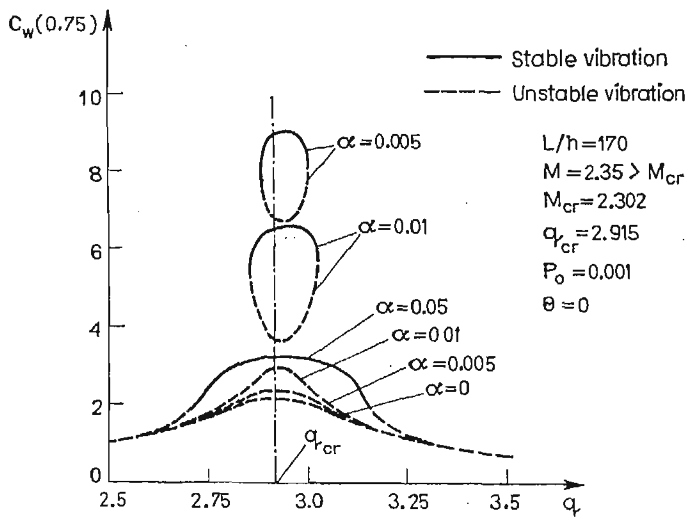

Fig. 7.

By contrast, for a given number $M > M_{cr}$, the maximum amplitude decreases and the range of stable periodic vibrations widens with increasing α.

Figures 9 and 10 show some results of computation performed for the case in which a small material damping $\Theta = 0.05$ is taken into account and $M > M_{cr}$.

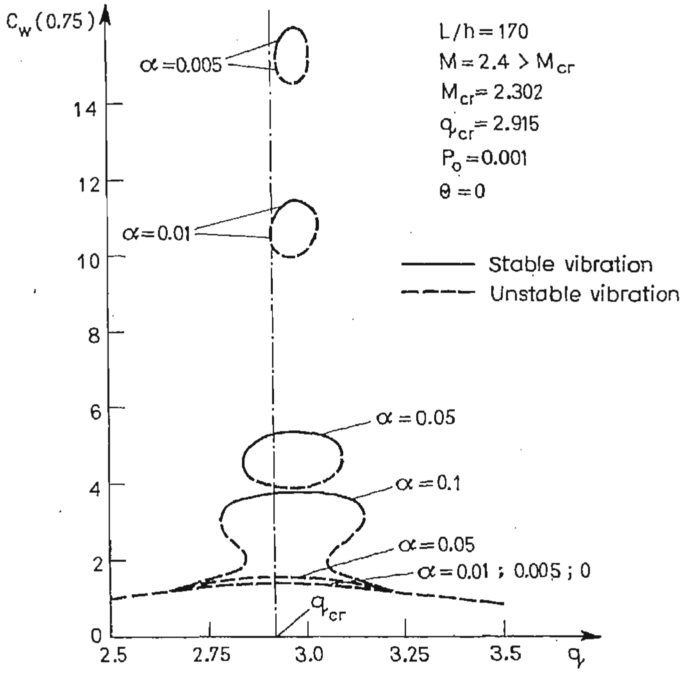
$C_w(0.75)$
L/h = 170
M = 2.4 > M_{cr}
M_{cr} = 2.302
q_{cr} = 2.915
P_o = 0.001
θ = 0
α = 0.005
α = 0.01
Stable vibration
Unstable vibration
α = 0.05
α = 0.1
α = 0.05
α = 0.01 ; 0.005 ; 0
q_{cr}
q

Fig. 8.

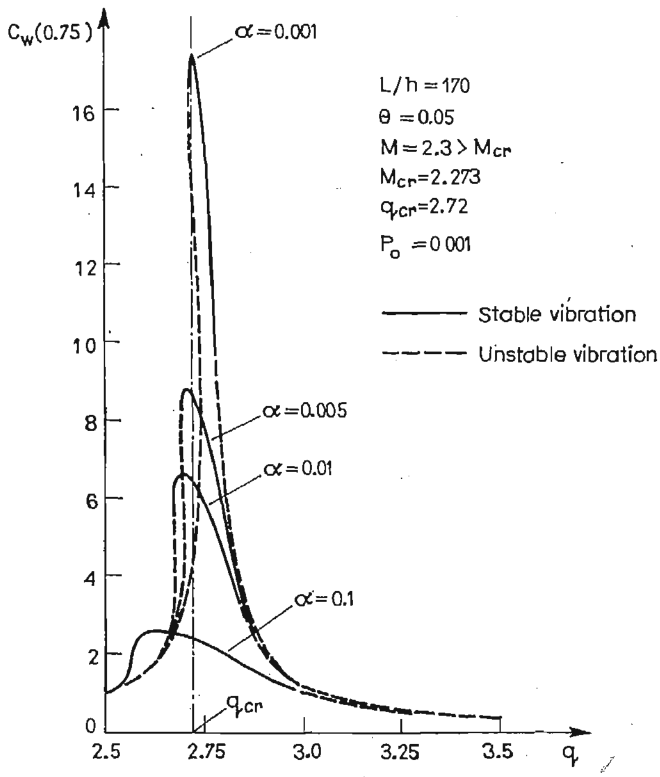
$C_w(0.75)$
α = 0.001
L/h = 170
θ = 0.05
M = 2.3 > M_{cr}
M_{cr} = 2.273
q_{cr} = 2.72
P_o = 0 001
Stable vibration
Unstable vibration
α = 0.005
α = 0.01
α = 0.1
q_{cr}
q

Fig. 9.

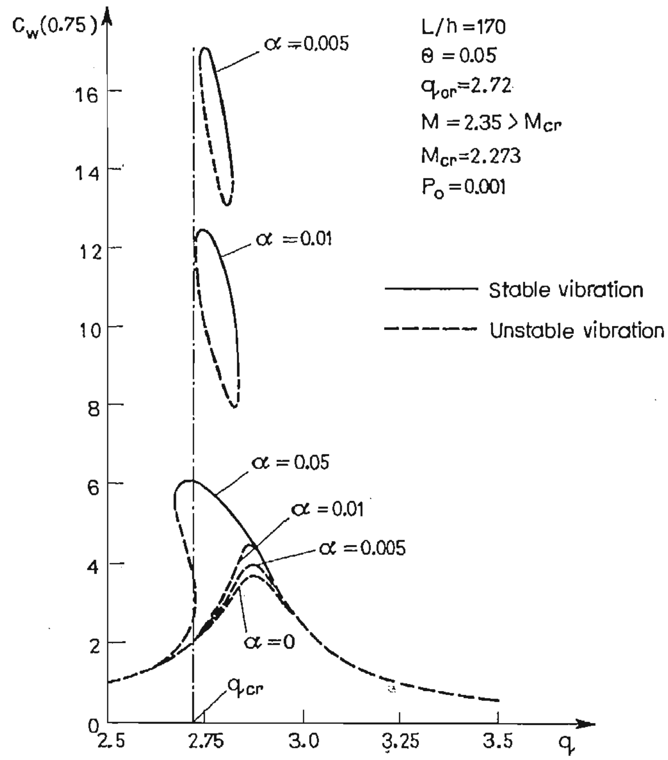

Fig. 10.

Comparing the diagrams in Figs. 9 and 10 with those in Figs. 6 to 8, we find that the introduction of material damping by means of the Voigt model generates much sharper resonance maxima than in the system without material damping. The same phenomenon has been found in the case of forced vibrations of linear aeroelastic self-excited systems (cf. [1]).

7. Concluding remarks

Nonautonomous nonlinear vibrations of continuous self-excited systems have interesting properties which are important for aircraft engineering. Some results of numerical analysis of resonance characteristics of such a system have been presented in this paper. The equations derived in the paper enable us also to study other kinds of nonautonomous vibrations including regular and chaotic motions of the system. This will be presented in further publications.

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Р е з ю м е

НЕАВТОНОМНЫЕ НЕЛИНЕЙНЫЕ КОЛЕБАНИЯ АВТОКОЛЕБАТЕЛЬНОЙ СИСТЕМЫ С РАСПРЕДЕЛЕННЫМИ ПАРАМЕТРАМИ. ПЛАСТИНКА В СВЕРХЗВУКОВОМ ПОТОКЕ

Рассмотрены неавтономные нелинейные колебания автоколебательной системы с распределенными параметрами на примере пластинки конечной длины в плоском сверхзвуковом потоке нагруженной одновременно гармонически меняющимся давлением и параметрически возбуждающими силами в её плоскости. В уравнении пластинки учтено нелинейность упругих и неупругих сил вызванных нагрузкой в плоскости пластинки.

Решение предлагается в виде ряда по собственным функциям колебаний пластинки в вакууме. Применен асимптотический метод одночастного анализа колебаний для исследования резонансных характеристик в окрестности критических параметров автоколебательной системы и метод численного интегрирования уравнений движения.

Представлены результаты примерных численных расчётов резонансных характеристик исследуемой системы.

S t r e s z c z e n i e

NIEAUTONOMICZNE NIELINIOWE DRGANIA CIĄGŁEGO UKŁADU SAMOWZBUDNEGO. PŁYTA W OPŁYWIE NADDŹWIĘKOWYM

Rozpatrzono nieautonomiczne nieliniowe drgania ciągłego układu samowzbudnego na przykładzie płyty o skończonej długości w płaskim opływie naddźwiękowym obciążonej parametrycznie pobudzającymi siłami w płaszczyźnie płyty i harmonicznie zmiennym ciśnieniem. W równaniu płyty uwzględniono nieliniowość sił sprężystych i tłumiących spowodowanych siłami napięcia w płaszczyźnie płyty.

Rozwiązania poszukiwano w postaci rozwinięcia względem funkcji własnych zlinearyzowanego problemu drgań płyty w próżni. Omówiono asymptotyczną metodę jednoczęstościowej analizy drgań w celu wyznaczenia rezonansowych charakterystyk w otoczeniu krytycznych parametrów samowzbudnych drgań układu oraz metodę numerycznego całkowania równań ruchu.

Przedstawiono wyniki przykładowych obliczeń numerycznych charakterystyk rezonansowych badanego układu.

Praca wpłynęła do Redakcji dnia 8 grudnia 1987 roku.