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ON PROPERTIES OF THERMO-DIFFUSIVE STRESSES IN SOLIDS

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1. Introduction

Considering the problems of thermo-diffusion in solid bodies we are interested, as a rule, in finding the distribution of stresses. The effect of uneven heating and that of mass diffusion may result in stress concentration. Likewise there are the cases for which the stresses generated by thermal and (or) diffusive effects can be singular.

In this paper we shall not dwell on the dynamical cases. We shall point out the two-dimensional distributions of thermo-diffusive effects resulting in solid body deformation only. It is shown that in the case of simply connected bodies there are no stresses while for multiconnected bodies the problem can be reduced to that of Volterra's dislocations, determining the character of the stresses. Next we show the features of stresses for the three dimensional layered bodies. Finally we discuss the character of stresses in solids with cracks, taking as an example a disc shaped crack opened by a flux of heat and that of mass diffusion. The stress intensity factor depends on the distribution of known temperature and the distribution of diffusion concentration on the crack surfaces.

2. Basic equations

As our point of departure we take the equations of thermo-diffusion, i.e. the generalized Navier equations, the equation of heat conduction (Fourier's law) and Fick's equation. We have the following system of partial differential equations:

$$(1-2\nu)\nabla^2 \boldsymbol{u} + \operatorname{grad}\operatorname{div}\boldsymbol{u} = 2(1+\nu)(\alpha_\Theta \operatorname{grad}\Theta + \alpha_c \operatorname{grad} c), \tag{2.1}$$

$$\nabla^2\Theta = 0, \quad \nabla^2 c = 0, \tag{2.2}$$

where $\boldsymbol{u} = (u, v, w)$ is the displacement vector, ν — Poisson's ratio, μ, λ Lamé's constants, $\gamma_\Theta = (3\lambda+2\mu)\alpha_\Theta$, $\gamma_c = (3\lambda+2\mu)\alpha_c$, $\Theta(x, y, z)$ change of temperature with respect to the natural state, $c(x, y, z)$ — concentration of diffusing mass, α_Θ, α_c coefficients of the linear thermal and diffusive expansion, respectively. In the considered case the constitutive equations (generalized Duhamel-Neumann relations), in absolute notation, take the following form:

$$\boldsymbol{\sigma} = 2\boldsymbol{\varepsilon} + (\lambda \operatorname{div}\boldsymbol{u} - \gamma_\Theta\Theta - \gamma_c c)\mathbf{1}, \tag{2.3}$$

where $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$, $\mathbf{1}$ — denote the stress, strain, and unit tensors, respectively.

3. Two-dimensional state of strain

For two-dimensional state of strain Eqs. (2.3) reduce in a cartesian coordinate system to the form:

$$\sigma_{\alpha\beta} = 2\mu\varepsilon_{\alpha\beta}+(\lambda u_{\gamma,\gamma}-\gamma_\Theta\Theta-\gamma_c c)\delta_{\alpha\beta},\ \alpha,\beta,\gamma = 1,2, \tag{3.1}$$

and:

$$\sigma_{33} = \lambda(u_{1,1}+u_{2,2})-\gamma_\Theta\Theta-\gamma_c c, \tag{3.2}$$

since $\varepsilon_{33} = 0$. Here $u_1 \equiv u$, $u_2 \equiv v$, $u_3 \equiv w$.
Let us assume the representation:

$$u_\alpha = u'_\alpha+(1+\nu)(\alpha_\Theta u^*_\alpha+\alpha_c\tilde{u}_\alpha). \tag{3.3}$$

We can impose on u^*_α and $\tilde{u}_\alpha$ ($\alpha = 1,2$) the additional conditions, namely:

$$u^*_{1,1} = u^*_{2,2} = \Theta(x,y), \quad u^*_{1,2} = -u^*_{2,1},$$

and (3.4)

$$\tilde{u}_{1,1} = \tilde{u}_{2,2} = c(x,y), \quad \tilde{u}_{1,2} = -\tilde{u}_{2,1}.$$

Then substituting (3.3) into Eqs. (3.1) we obtain:

$$\sigma_{\alpha\beta} = \mu(u'_{\alpha,\beta}+u'_{\beta,\alpha})+\lambda\delta_{\alpha\beta}u'_{\gamma,\gamma}. \tag{3.5}$$

In a similar way, substituting into Navier's equations (3.1), we obtain the system of homogeneous equations:

$$\mu u'_{\alpha,\beta\beta}+(\lambda+\mu)u'_{\beta,\beta\alpha} = 0. \tag{3.6}$$

It is evident from (3.5) that for vanishing tractions $u'_\alpha \equiv 0$. Thus we obtain for simply connected bodies:

$$u_\alpha = (1+\nu)(\alpha_\Theta u^*_\alpha+\alpha_c\tilde{u}_\alpha),$$
$$\sigma_{\alpha\beta} = 0,\ \alpha,\beta = 1,2, \text{ and} \tag{3.7}$$

$$\sigma_{33} = -2\mu(1+\nu)(\alpha_\Theta\Theta+\alpha_c c). \tag{3.8}$$

Conclusions:

1. For two-dimensional state of strain and uneven heating and (or) diffusing mass concentration penetrating through the boundary there are no stresses except σ_{zz} in a simply connected body bounded by any (non-intersecting) contour. The displacements can be found from Eqs. (3.7) while u^*_α and $\tilde{u}_\alpha$ from conditions (3.4),
2. This is a generalization of the result given by Muskhelishvili [1] in the case of heat conduction.
3. The same is true for an infinite body with a flux of heat and (or) diffusing mass penetrating through the boundary of a single hole of any shape.
4. The result holds for simply connected two-dimensional solids and plane strain only. In the case of multiconnected regions the problem can be reduced to Volterra's distorsions. Then in the expressions for $u^*_1+iu^*_2$ and $\tilde{u}_1+i\tilde{u}_2$ logarithmic terms appear (compare [1], §46).
5. In the classical theory of elasticity it is shown that the two-dimensional stress cases differ by magnitude of constants occurring in the equations. In the case of thermo-diffusive

effects it is difficult to expect that for two-dimensional stress the heat conduction and diffusion of mass could be in plane only. Thus it does not make much sense to consider the two-dimensional stress to be analogous to the two-dimensional strain.

4. Stresses in solids bounded by a plane

We assume that the bounding plane is free from tractions and that over certain domains Ω_Θ and Ω_c there act the fluxes of heat and of mass diffusion, respectively. The system of partial differential equations of thermodiffusion in elastic solids can be reduced by applying the exponential Fourier transform:

$$\begin{aligned} \bar{f}(\xi, \eta) &= \mathcal{F}[f(x, y); \; x \to \xi, y \to \eta], \\ f(x, y) &= \mathcal{F}^{-1}[\bar{f}(\xi, \eta); \; \xi \to x, \eta \to y], \end{aligned} \tag{4.1}$$

to the following system of the linear ordinary differential equations in the transformed space:

$$\begin{aligned} &[(1-2\nu)(D^2-\eta^2)-2(1-\nu)\xi^2]\bar{u}-\xi\eta\bar{v}-\mathrm{i}\xi D\bar{w} = -2(1+\nu)\mathrm{i}\xi(\alpha_\Theta\bar{\Theta}+\alpha_c\bar{c}), \\ &-\xi\eta\bar{u}+[(1-2\nu)(D^2-\xi^2)-2(1-\nu)\eta^2]\bar{v}-\mathrm{i}\eta D\bar{w} = -2(1+\nu)\mathrm{i}\eta(\alpha_\Theta\bar{\Theta}+\alpha_c\bar{c}), \\ &-\mathrm{i}\xi D\bar{u}-\mathrm{i}\eta D\bar{v}+[2(1-\nu)D^2-(1-2\nu)(\xi^2+\eta^2)]\bar{w} = 2(1+\nu)(\alpha_\Theta D\bar{\Theta}+\alpha_c D\bar{c}), \\ &(D^2-\xi^2-\eta^2)\bar{\Theta}=0, \qquad (D^2-\xi^2-\eta^2)\bar{c}=0, \end{aligned} \tag{4.2}$$

where $D \equiv \dfrac{d}{dz}$.

The solution to this system of differential equations, with the regularity conditions at infinity taken into account, takes the following form:

$$\begin{aligned} \bar{u} &= (A_1+z\sqrt{\xi^2+\eta^2}B_1)\exp(-z\sqrt{\xi^2+\eta^2}), \\ \bar{v} &= (A_2+z\sqrt{\xi^2+\eta^2}B_2)\exp(-z\sqrt{\xi^2+\eta^2}), \\ \bar{w} &= (A_3+z\sqrt{\xi^2+\eta^2}B_3)\exp(-z\sqrt{\xi^2+\eta^2}), \\ \bar{\Theta} &= A\exp(-z\sqrt{\xi^2+\eta^2}), \\ \bar{c} &= A_c\exp(-z\sqrt{\xi^2+\eta^2}) \end{aligned} \tag{4.3}$$

with the relationships:

$$\begin{gathered} \xi B_1+\eta B_2 = \mathrm{i}\sqrt{\xi^2+\eta^2}B_3, \qquad \xi B_1 = \eta B_2, \\ \sqrt{\xi^2+\eta^2}[2(1-\nu)B_3-A_3] = (1+\nu)(\alpha_\Theta A_\Theta+\alpha_c A_c), \\ \xi A_1+\eta A_2+\mathrm{i}\sqrt{\xi^2+\eta^2}[(3-4\nu)B_3-A_3] = 2\mathrm{i}(1+\nu)(\alpha_\Theta A_\Theta+\alpha_c A_c). \end{gathered}$$

In the case when the shear stress components disappear on the plane $z = 0$ we obtain:

$$\eta A_1 = \xi A_2, \quad \mathrm{i}\sqrt{\xi^2+\eta^2}(B_3-A_3) = \eta A_2+\xi A_1. \tag{4.4}$$

If we also assume that the normal component of the stress tensor vanishes on $z = 0$, we obtain the condition:

$$\sqrt{\xi^2+\eta^2}A_3+(1+\nu)(\alpha_\Theta A_\Theta+\alpha_c A_c) = 0. \tag{4.5}$$

This condition results from the formula for the transform of the normal stress tensor component:

$$\bar{\sigma}_{zz} = \frac{2\mu}{1-2\nu}\,[(1-\nu)D\bar{w} - i\nu(\xi\bar{u}+\eta\bar{v}) - (1+\nu)(\alpha_\Theta\bar{\Theta}+\alpha_c\bar{c})] =$$

$$= -\frac{\mu}{1-\nu}\left[\sqrt{\xi^2+\eta^2}A_3+(1+\nu)(\alpha_\Theta A_\Theta+\alpha_c A_c)\right]\left(1+\sqrt{\xi^2+\eta^2}\,z\right)\cdot \tag{4.6}$$

$$\cdot\exp\left(-z\sqrt{\xi^2+\eta^2}\right).$$

It is evident from Eq. (4.6) that normal stresses are identically zero in the whole space Though σ_{zz} stress tensor component vanishes in the entire solid, the stress components σ_{xx} and σ_{yy} exist. The corresponding results for the thermal stresses were obtained by Sternberg and McDowell [3] and W. Nowacki [4].

5. Stresses generated by thermodiffusion in solid with a crack

In the case of axial symmetry the system of partial differential equations (3.1) can be reduced by means of the Hankel transforms of the zero and the first order to a system of ordinary differential equations [7]. The solution can be written down in the form of the following Hankel's integrals:

$$\begin{aligned}
u &= \int_0^\infty \left\{\frac{1+\eta\zeta}{2(1-\nu)}[\psi(\eta)+\varphi(\eta)]-\psi(\eta)\right\}\exp(-\zeta\eta)J_1(\eta\varrho)\,d\eta,\\
w &= \int_0^\infty \left\{\psi(\eta)+\frac{\eta\zeta}{2(1-\nu)}[\psi(\eta)+\varphi(\eta)]\right\}\exp(-\zeta\eta)J_0(\eta\varrho)\,d\eta,\\
c &= \frac{1}{\alpha_c(1+\nu)a}\int_0^\infty \eta\varphi_1(\eta)\exp(-\zeta\eta)J_0(\eta\varrho)\,d\eta,\\
\Theta &= \frac{1}{\alpha_\Theta(1+\nu)a}\int_0^\infty \eta\varphi_2(\eta)\exp(-\zeta\eta)J_0(\eta\varrho)\,d\eta,\\
& r=\varrho a, \quad z=\zeta a, \quad \varphi=\varphi_1+\varphi_2,
\end{aligned} \tag{5.1}$$

and the z component of the stress tensor:

$$\sigma_{zz} = -\frac{\mu}{(1-\nu)a}\int_0^\infty [\psi(\eta)+\varphi(\eta)](1+\eta\zeta)\exp(-\zeta\eta)J_0(\eta\varrho)\,d\eta. \tag{5.2}$$

The above solution is valid for the boundary conditions $\sigma_{rz}(r, 0) = 0$, $r\in[0,\infty)$, $z = 0$. Function $\psi(\eta)$ can be determined from the remaining mechanical boundary condition on $z = 0$ while $\varphi_1(\eta)$ and $\varphi_2(\eta)$ from the thermal and diffusion boundary conditions, respectively. The solution to the problem is obtained from the corresponding dual integral equations when on the crack surface temperature and diffusion of mass are prescribed.

In the case when the crack surface is traction free the stresses around the crack are generated by the distribution of uneven heating and (or) mass diffusion through the crack surfaces. Here an important remark should be made. For the traction free surfaces the crack is openend only provided the sum $\alpha_c c_0 + \alpha_\Theta \vartheta_0$ is negative. If it is positive we deal with a source of heat and that of mass diffusion in an infinite solid and there is neither crack opening nor non zero stress intensity factor.

Let us take an example. Over the crack surface $\Omega = \{z = 0, r \in [0, a)\}$ there act a flux of heat $Q = -Q_c$ and a flux of mass diffusion $\mathfrak{M} = -M_\Theta$. Then we obtain the solution:

$$u = \frac{1+\nu}{2(1-\nu)} a^2(\alpha_c c_0 + \alpha_\Theta \vartheta_0) \int_0^\infty \left\{ [\eta\zeta - (1-2\nu)] \left[\eta^{-2} J_1(\eta) - \frac{1}{2}\eta^{-1}\cos\eta \right] - \right.$$
$$\left. -(1+\eta\zeta)\eta^{-2} J_1(\eta) \right\} J_1(\varrho\eta) \exp(-\eta\zeta)\, d\eta,$$

$$w = \frac{1}{2}(1+\nu) a^2(\alpha_c c_0 + \alpha_\Theta \vartheta_0) \int_0^\infty [2\eta^{-2} J_1(\eta) - \eta^{-1}\cos\eta -$$
$$- \frac{\zeta}{2(1-\nu)} \cos\eta] J_0(\varrho\eta) \exp(-\zeta\eta)\, d\eta, \tag{5.3}$$

$$\sigma_{zz} = -\frac{\mu(1+\nu)}{2(1-\nu)} a(\alpha_c c_0 + \alpha_\Theta \vartheta_0) f(\varrho, \zeta),$$

$$f(\varrho, \zeta) = \int_0^\infty (1+\eta\zeta) \cos\eta J_0 \varphi\eta) \exp(-\eta\zeta)\, d\eta =$$
$$= R^{-1} \cos\frac{1}{2}\Phi + R^{-3}\zeta \left[\zeta\cos\frac{3}{2}\Phi + \sin\frac{3}{2}\Phi \right],$$

where:

$$R^4 = (\varrho^2 + \zeta^2 - 1)^2 + 4\zeta^2, \quad \tan\Phi = \frac{2\zeta}{\varrho^2 + \zeta^2 - 1}.$$

We have the special cases, namely:

$$f(0, \zeta) = \frac{2\zeta^3}{(1+\zeta^2)^2}, \quad f(\varrho, 0) = H(\varrho - 1)(\varrho^2 - 1)^{-1/2}. \tag{5.4}$$

The stress intensity factor assumes the value:

$$K_I = \frac{\mu}{2} \frac{1+\nu}{1-\nu} [\alpha_c Q_c + \alpha_\Theta M_\Theta] a^{3/2}. \tag{5.5}$$

In a similar way we can find the stress intensity factors in all the cases for which the classical "mechanical" solution is known.

References

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Резюме

О СВОЙСТВАХ ТЕРМО-ДИФФУЗИОННЫХ НАПРЯЖЕНИЙ

В работе рассмотрены некоторые задачи теории напряжений возникающие как результат действия потока тепла и диффузии массы. Обобщена известна задача Н. И. Мусхелишвили, найдены распределения напряжений от потоков тепла и диффузии на некоторой части плоскости ограничивающей тело, а также найден коэффициент интенсивности напряжений в случае дискообразной трещины.

Streszczenie

O WŁASNOŚCIACH NAPRĘŻEŃ OD TERMODYFUZJI

W pracy przedstawiono kilka zadań teorii naprężeń wywołanych strumieniem ciepła i dyfuzji masy. Uogólniono znane zagadnienie N. I. Muscheliszwilego, znaleziono naprężenia, gdy strumień ciepła i dyfuzji masy działa na części płaszczyzny ograniczającej ciepło. Wyprowadzono również wzór na współczynnik intensywności naprężeń w przypadku szczeliny osiowo symetrycznej.

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