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PHYSICAL CORRECTNESS OF COSSERAT-TYPE MODELS OF HONEYCOMB GRID PLATES

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1. Introduction

Formulated via phenomenological considerations micropolar theory of elasticity (cf. [1]) can be applied for continuum description of dense, regular grids. This has been noted and applied by Woźniak et al. in numerous papers pertaining to lattice-type shells and plates, cf. [2]. Woźniak's approach is based on variational methods; an adequacy ofthe proposed differential model of a body with additional degrees of freedom is „a priori" assumed. In the first-order approximation (see Sec. III in [2]) an in-plane plate motion is described by means of three independent functions approximating displacements and rotations of nodes. The governing equations of the theory have a similar form to those of the plane-stress theory of micropolar media. Therefore the Woźniak's approach is a heuristic one thus the recalled above procedure does not allow us to perform a physical correctness analysis of the model provided appropriate numerical tests are not carried out.

In the case of simple layout grid plates (in which neighbourhoods of all nodes are congruent) Woźniak's algorithm leads to one set of effective constants describing elastic properties of the structure. However in the case of complex layout grids, one can derive at least two sets of **B** and **C** tensors (cf. [3]). In the present paper an attempt is made to elucidate questions concerning the mentioned difficulties in formulation of Cosserat-type models of complex geometry lattice plates. An attention will be focused on hexagonal grids belonging to the class of complex layout structures. In order to have a new look at Woźniak's continuum models results of the work [4] (pertaining to differential models due to Rogula-Kunin's approach) are applied.

It is easy to note that Cosserat-type equilibrium equations expressed in terms of displacements cannot be obtained by asymptotic method, e.g. by formal simplifications (neglect of terms of higher order) of equilibrium equations found in [4] by Rogula-Kunin's procedure; thus a simple correspondence between the latter and Woźniak's-type equations is not valid. This fact is obvious since differential models derived in [4] (just contrary to theories outlined in [3]) do not satisfy stability conditions.

In Sec. 5 a simple modification of the second-order differential approximation (obtained in [4]) will be proposed. The aim of the procedures is to formulate a well-established

Cosserat-type equations, so called $\varkappa$- versions. In Sec. 6 an attempt is undertaken to examine a range of applicability of the latter versions as well as of two variants resulting from Woźniak's concept, see [3].

2. Basic assumptions

A subject of our considerations is a plane-stress statical problem of a honeycomb grid composed of bars whose axes constitute hexagons of sides being equal to l, cf. Fig. 1. In order to make final results as clear as possible the bars are assumed to be prismatic (thus their heights h are constant while their depth is of unit dimension) and made of an isotropic, elastic material whose properties are characterised by Young modulus E and

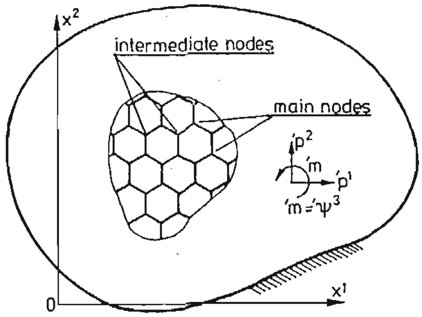

Fig. 1

Poisson's ratio ν. External loads (subjected to lattice joints only) are assumed to yield plane-stress plate response hence the external forces are supposed to be subjected in-plane while moments should be normal to the mid-surface of the grid. A slenderness ratio of lattice rods is defined by, cf. [3]

$$\eta = l^2/h^2.$$

A parameter ϱ defined as a quotient of opening's diameter to a spacing between centres of neighbouring openings, see [3], reads

$$\varrho = \left(\sqrt{3\eta}-1\right)/\sqrt{3\eta} \tag{2.1}$$

As it has been pointed out in [3] $EJ/l^3 = E/12\eta^{3/2}$ where J denotes a moment of inertia of a constituent bar's cross section. Lattice rods are assumed to be sufficiently slender so as to known methods of the theory of structures could be applied. Analogously to [3, 4] lattice nodes are divided into two families of main and intermediate ones, see Fig. 1. Displacements and rotations of main nodes are approximated by continuous functions $u^\alpha(x^\beta)$ and φ (x^β). External loads subjected to main and intermediate nodes are characterised by functions F^α, M, $\overset{*}{F}{}^\alpha$, $\overset{*}{M}$, respectively.

3. Two versions of Woźniak-type continuum descriptions

3.1. Equilibrium equations in terms of displacements. A set of the title equations has a form similar to that known from plane-stress problem of micropolar media, see [3]

$$\begin{aligned}
&[(2\mu+\lambda)\partial_1^2+(\mu+\alpha)\partial_2^2]u^1+(\lambda+\mu-\alpha)\partial_1\partial_2u^2+[B(\partial_1^2-\partial_2^2)+2\alpha\partial_2]\varphi+{}'p^1 = 0,\\
&(\lambda+\mu-\alpha)\partial_1\partial_2u^1+[(2\mu+\lambda)\partial_2^2+(\mu+\alpha)\partial_1^2]u^2+[-2B\partial_1\partial_2-2\alpha\partial_1]\varphi+{}'p^2 = 0,\\
&[B(\partial_1^2-\partial_2^2)-2\alpha\partial_2]u^1+[-2B\partial_1\partial_2+2\alpha\partial_1]u^2+[C(\partial_1^2+\partial_2^2)-4\alpha]\varphi+{}'Y^3 = 0,
\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
&{}'p^\alpha = p^\alpha+\partial_\beta \overset{*}{p}{}^{\beta\alpha}, \qquad {}'Y^3 = Y^3+\partial_\alpha \overset{*}{m}{}^\alpha+e_{\alpha\beta}\overset{*}{p}{}^{\alpha\beta},\\
&(p^\alpha, \overset{*}{p}{}^\alpha, Y^3, \overset{*}{Y}{}^3) = (F^\alpha, \overset{*}{F}{}^\alpha, M, \overset{*}{M})/P, \qquad p = 1{,}5\sqrt{3}\,l^2.
\end{aligned} \tag{3.2}$$

Ricci tensor is denoted by $e_{\alpha\beta}$, an area of a recurrent hexagon amounts to P. The effective moduli λ, μ and α are uniquely defined in both Woźniak-type versions examined in [3] whereas B and C moduli can be defined twofold

i) Klemm-Woźniak's version (Sec. 3, [3])

$$\check{B} = \frac{2\sqrt{3}\cdot\eta}{\bar{\eta}+1}\cdot\frac{EJ}{l^2}, \qquad \check{C} = \frac{\sqrt{3}(3\eta+\bar{\eta}+1)}{3(\bar{\eta}+1)}\cdot\frac{EJ}{l}. \tag{3.3}$$

ii) the second, author's variant (Sec. 4, [3])

$$\hat{B} = \frac{2\sqrt{3}\cdot\eta\cdot(3\eta-\bar{\eta})}{(\bar{\eta}+1)\cdot(3\eta+\bar{\eta})}\cdot\frac{EJ}{l^2}, \qquad \hat{C}' = \frac{\sqrt{3}\cdot[(3\eta-\bar{\eta})^2+(3\eta+\bar{\eta})]}{3(\bar{\eta}+1)\cdot(\bar{\eta}+3\eta)}\cdot\frac{EJ}{l} \tag{3.4}$$

where $\bar{\eta} = \eta+2.4(1+\nu)$.

The tensors $\overset{*}{p}{}^{\alpha\beta}$ and $\overset{*}{m}{}^\alpha$ are dependent upon the loads subjected to intermediate nodes; their definitions are given in [3], [6].

3.2. Strain energy as a positive definite function. Strain energy of the structure is postive definite provided, [3]

$$\mu > 0, \qquad \alpha > 0, \qquad \mu+\lambda > 0, \qquad C > 0, \qquad B^2 < C\cdot\mu. \tag{3.5}$$

4. Unstable quasicontinuum micropolar-type equilibrium equations

4.1. Derivation of governing equations. Focus attention on Cosserat-type equations (3.1). On noting that $B \sim l$, $C \sim l^2$ one can make following remarks:

a) two first equilibrium equations involve zero-order (with respect to powers of l) terms of displacement-type and a first-order term relevant to nodal rotations;

b) the last equation involves first-order terms of displacement-type and a second-order term depending upon the rotations.

The procedures put forward in [3] did not explain why:

a) two first equations do not involve first-order terms being dependent on displacements. Does it yield from approximations only or result from specific properties of the hexagonal grid?

b) the last equation does not involve second-order terms of displacement-type; also herein the same question arises.

Answers to the above questions are supplied by quasicontinuum considerations, [4]. Compare consequent second-order equations ((6.1) in [4]) with Eqs. (3.1) of Cosserat-type theory. First of all it can be stated that moduli λ, μ and α (being involved in both compared sets of equations) have been identically defined (cf. $(3.8)_{1-3}$ in [3] and $(6.2)_{1-3}$ in[4]). Thus both approaches (based on Woźniak's [3] and Rogula-Kunin's, [4], concepts) result in the same definitions of elastic moduli λ, μ and α being dependent upon slenderness ratio of bars only and thus being independent of the internode spacing l.

Note that equations similar to the Cosserat-type (3.1) can be derived from the second-order Eqs. ((6.1) in [4]) provided in the latters all the terms involving derivatives up to the second order are retained:

$$\begin{aligned}
&[(2\mu+\lambda)\partial_1^2+(\mu+\alpha)\partial_2^2]u^1+(\lambda+\mu-\alpha)\partial_1\partial_2u^2+[B^0(\partial_1^2-\partial_2^2)+2\alpha\partial_2]\varphi+\mathring{p}^1 = 0,\\
&(\lambda+\mu-\alpha)\partial_1\partial_2u^1+[(2\mu+\lambda)\partial_2^2+(\mu+\alpha)\partial_1^2]u^2+[-2B^0\partial_1\partial_2-2\alpha\partial_1]\varphi+\mathring{p}^2 = 0,\\
&[B^0(\partial_1^2-\partial_2^2)-2\alpha\partial_2]u^1+[-2B^0\partial_1\partial_2+2\alpha\partial_1]u^2+[C^0(\partial_1^2+\partial_2^2)-4\alpha]\varphi+\mathring{Y}^3 = 0,
\end{aligned} \tag{4.1}$$

where

$$B^0 = \beta l, \qquad C^0 = \gamma l^2. \tag{4.2}$$

Moduli β and γ have been defined by Eqs. $(6.2)_{4,6}$ in [4]. Hence we obtain

$$\begin{aligned}
B^0 &= \frac{\sqrt{3}}{2}\cdot\frac{\eta}{\bar\eta}\left(\frac{3\eta-\bar\eta}{3\eta+\bar\eta}+\frac{3\bar\eta-1}{\bar\eta+1}\right)\cdot\frac{EJ}{l^2},\\
C^0 &= \frac{\sqrt{3}}{\bar\eta}\cdot\left[\frac{(3\eta-\bar\eta)^2}{3(3\eta+\bar\eta)}-\frac{\eta}{1+\bar\eta}\right]\cdot\frac{EJ}{l},
\end{aligned} \tag{4.3}$$

Quantities $\mathring{p}^\alpha$ and $\mathring{Y}^3$ are equal to functions $'p^\alpha$ and $'Y^3$ employed in [4]. In a zero-order approximation we have

$$\mathring{p}^\alpha = p^\alpha+\overset{*}{p}{}^\alpha+\frac{3\eta}{3\eta+\bar\eta}\mathrm{e}^{\alpha\beta}\partial_\beta\overset{*}{Y}{}^3, \qquad \mathring{Y}^3 = Y^3+\frac{\bar\eta-3\eta}{\bar\eta+3\eta}\cdot\overset{*}{Y}{}^3. \tag{4.4}$$

The derivation of Eqs. (4.1) (which will be called further quasicontinuum micropolar-type equilibrium equations) violates accuracy principles formulated in [4] where the approximation procedure has been called consequent provided all the terms proportional to $l^p, p \leqslant s$, (s is fixed) are being retained. In the next section an improved accuracy analysis will be presented. The approximations of governing equations correspond to a certain form of density of strain energy of the structure. Thus various approximations of three equilibrium equations are reflected in the form of the one scalar function which stands for the energy of the grid. This method of error analysis is not new, the idea was originated by Koiter in the paper [5] pertaining to the Kirchoff-Love shell theory and up till now it is often applied to the accuracy analysis of so called improved theories describing plate and shell behaviour, see [6].

4.2. The micropolar-type approximation as a model of „moderate" rotations. The energy criterion. Strain energy of the infinite hexagonal grid amounts to ,

$$E_c = \frac{1}{2} P \sum_{\mathbf{m},\mathbf{n}}{}' w_{\mathbf{n}}^{\alpha} \Phi_{\alpha\beta}^{(\mathbf{n}-\mathbf{m})} w_{\mathbf{m}}^{\beta}, \qquad \alpha, \beta = 1, 2, 3$$

where the hexagon's area of a side l has been denoted by P, $P = 1.5\sqrt{3} \cdot l^2$; $w^{\alpha}, = u^{\alpha}$, $w^3 = \varphi$; the summation has been implied over all main nodes which constitute a plain (Bravais-type) lattice of the grid. The values of $\Phi_{\alpha\beta}^{(\mathbf{n})}$ functions were given in [4]. On passing to k-representation we arrive at (the proof is omitted here)

$$E_c = \int_{\hat{P}} \hat{e}(\mathbf{k}) d^2\mathbf{k}, \qquad \hat{e}(\mathbf{k}) = \frac{1}{8\pi^2 \cdot P} \bar{\hat{w}}^{\alpha}(\mathbf{k}) \hat{\Phi}_{\alpha\beta}(\mathbf{k}) \cdot \hat{w}^{\beta}(\mathbf{k}),$$

$$d^2\mathbf{k} = 4 \cdot \pi^2 \cdot dk_1 dk_2 / P$$

where the domain of the unit cell of the reciprocal lattice has been denoted by $\hat{P}$; a discrete Fourier transform of a discrete-argument function $f^{\mathbf{m}}$ has been denoted by $\hat{f}(\mathbf{k})$. Certain approximations of of $\hat{\Phi}_{\alpha\beta}(\mathbf{k})$, cf. [4], result in differential models, particularly (as it will be shown further) — in Cosserat-type models.

Introduce dimensionless variables $\Psi^{\alpha} = u^{\alpha} / L$, where $[L] = m$. Let $L = l/\varepsilon$. The density of strain energy of the grid

$$\hat{e}(\mathbf{k}) = \frac{1}{8\pi^2 P} \left\{ \frac{l^2}{\varepsilon^2} (\bar{\hat{\Psi}}^{\alpha} \hat{\Phi}_{\alpha\beta} \hat{\Psi}^{\beta}) + \frac{l}{\varepsilon} [\bar{\hat{\Psi}}^{\alpha} \hat{\Phi}_{\alpha 3} \hat{\varphi} + \bar{\hat{\varphi}} \hat{\Phi}_{3\alpha} \hat{\Psi}^{\alpha}] + (\bar{\hat{\varphi}} \hat{\Phi}_{33} \hat{\varphi}) \right\}, \qquad \alpha, \beta = 1, 2$$

can be rearranged to the form

$$\hat{e}(\mathbf{k}) = \frac{1}{4\pi^2} \left\{ \frac{1}{2} a_{11} \cdot |\hat{\Psi}^1|^2 + \frac{1}{2} a_{22} \cdot |\hat{\Psi}^2|^2 + \frac{1}{2} a_{33} \cdot |\hat{\varphi}|^2 \right\} + \\ + \frac{1}{4\pi^2} \operatorname{Re} \{ a_{12} \bar{\hat{\Psi}}^1 \hat{\Psi}^2 + a_{13} \bar{\hat{\Psi}}^1 \hat{\varphi} + a_{23} \cdot \bar{\hat{\Psi}}^2 \hat{\varphi} \}. \tag{4.5}$$

Let us define dimensionless quantities τ and θ by means of the formulae

$$\tau = l|\mathbf{k}|, \qquad \cos\Theta = k_1/|\mathbf{k}|, \qquad \sin\Theta = k_2/|\mathbf{k}|.$$

Within the fourth-order approximation (with respect to the powers of k_{α}) the coefficients $a_{\alpha\beta}$ can be expressed by means of the following equations, cf. Eqs. (6.1) in [4]

$$a_{11} = \frac{\tau^2}{\varepsilon^2} \left[(\mu+\alpha) + (\lambda+\mu-\alpha)\cos^2\Theta - \frac{3\tau^2}{16}(\mu+\alpha) - \frac{\tau^2}{4}(\lambda+\mu-\alpha)\cos^4\Theta \right],$$

$$a_{22} = \frac{\tau^2}{\varepsilon^2} \left[(\lambda+\mu-\alpha)\sin^2\Theta + (\mu+\alpha) - \frac{3\tau^2}{16}(\mu+\alpha) - \frac{3\tau^2}{16}(\lambda+\mu-\alpha) \cdot \right. \\ \left. \cdot \left(-\frac{1}{3}\cos^4\Theta + \sin^4\Theta + 2\sin^2\Theta\cos^2\Theta \right) \right],$$

$$a_{33} = 4\alpha + \gamma\tau^2 - \frac{3}{16}\tau^4 \cdot \gamma,$$

$$a_{12} = \frac{\tau^2}{\varepsilon^2} \left[(\lambda+\mu-\alpha)\cos\Theta\sin\Theta - \frac{\tau^2}{8}(\lambda+\mu-\alpha)\cos\Theta\sin\Theta(\cos^2\Theta + 3\sin^2\Theta) + \right. \\ \left. + i\tau\delta \cdot \cos\Theta(\cos^2\Theta - 3\sin^2\Theta) \right],$$

$$a_{13} = \frac{\tau}{\varepsilon}\left[\tau\beta\cos 2\Theta - \frac{\tau^3\beta}{16}(5\cos^4\Theta - 3\sin^4\Theta - 6\cos^2\Theta\sin^2\Theta) + \right.$$

$$\left. + i\left(-2\alpha\sin\Theta + \frac{3}{4}\tau^2\sin\Theta\right)\right],$$

$$a_{23} = \frac{\tau}{\varepsilon}\left[-\tau\beta\sin 2\Theta + \frac{\tau^3}{8}\beta\sin 2\Theta(\cos^2\Theta + 3\sin^2\Theta) + i\left(2\alpha\cos\Theta - \frac{3}{4}\tau^2\alpha\cos\Theta\right)\right].$$

The δ modulus has been defined by $(6.2)_5$ in [4]. Let λ_d means a wavelength of the deformation pattern in $\mathbf{k}$ direction; $\lambda_d = 2\pi/|\mathbf{k}|$.
Thus the magnitude τ is:

$$\tau = 2\pi l/\lambda_\alpha = \frac{2\pi}{\sqrt{3}}\frac{b}{\lambda_d} \approx 3.6275\frac{b}{\lambda_d}$$

where $b = l\sqrt{3}$ stands for the spacing of main nodes. From now on the τ quantity will be supposed to be less than one, $\tau < 1$; this yields $\lambda_d > 3.6275\ b$. The parameters τ and ε are interrelated by means of the formula $\tau/\varepsilon = 2\pi L/\lambda_d$.

Let $\hat{\varphi}_0$ and $\hat{u}_0$ stand for the absolute values of the transforms $\hat{\varphi}$ and max $\hat{u}^\alpha$ measured at the fixed node O of the grid. The parameter $\hat{\varphi}_0 \cdot l/\hat{u}_0 = (\hat{\varphi}_0/\hat{\Psi}_0)\ \varepsilon$ determines the relation between rotation and displacements of the node O. Depending upon the assumed estimates of value of this parameter three types of equations describing hexagonal grid behaviour can be distinguished:

a) $\hat{\varphi}_0\ \varepsilon/\hat{\Psi}_0 \sim \tau^2$ — the state (model) of infinitesimal rotations,
b) $\hat{\varphi}_0\ \varepsilon/\hat{\Psi}_0 \sim \tau$ — the state (model) of small rotations,
c) $\hat{\varphi}_0\ \varepsilon \sim \hat{\Psi}_0$ — the state (model) of moderate rotations.

The relation $a \sim b$ (which reads: a is of the same order as b) should be understood in the sense similar to that used in the literature devoted to the thin shell theories, cf. [5, 6].

Within the frames of the mentioned deformation classes differential models of an arbitrary accuracy order can be formulated. A brief analysis of approximation of strain energy density $\hat{e}$ in two cases b) and c) will be carried out below. The case a) will not be dealt with here.

Ad b) On inserting $\hat{\varphi}_0 \sim \frac{\tau}{\varepsilon}\hat{\psi}_0$ into (4.5) we have

$$\hat{e}(\mathbf{k}) \sim \left[\left(\frac{\tau}{\varepsilon}\right)^2 \sum_{m=0}^{\infty} e^{(b)}_{(m)}(\Theta)\tau^m\right]|\hat{\Psi}_0|^2.$$

Three first terms of this expansion correspond to the approximation of $\hat{e}$ which yield a second-order model derived in [4], Sec. 6, Eqs. (6.1). The presented derivation provides a deeper insight into the assumptions (implicitly and tacitly assumed in [4]) which are a basis of this model.

On neglecting all the terms except for the two first ones the first-order model, cf. [4], Sec. 7, occurs.

The first term of the expansion is related to the zero-order, asymptotic or Horvay's theory, see [4], Sec. 8.

Ad c) On substituting $\hat{\varphi}_0 \sim \frac{1}{\varepsilon}\hat{\Psi}_0$ into (4.5) we obtain

$$\hat{e}(\mathbf{k}) \sim \left[\frac{1}{\varepsilon^2}\sum_{m=0}^{\infty} e^{(c)}_{(m)}(\Theta)\cdot\tau^m\right]|\hat{\Psi}_0|^2.$$

By neglecting the terms of higher order than second (i.e. proportional to τ^p, $p > 2$) we arrive at the expansion which corresponds to the micropolar-type approximation. Also herein it can be pointed out that the approach presented has revealed and elucidated assumptions which constitute a basis of the Cosserat-type models of fine hexagonal networks.

4.3. Stability. Necessary and sufficient stability conditions of Eqs. (4.1) (in the spirit of Kunin, [7]) will be arrived at. According to this definition stability of equilibrium is satisfied provided an energy expressed in terms of the wave vector components k_α is positive definite. Stability implies both existence and uniqueness of solutions.

The Eqs. (4.1) are stable in the considered meaning when and only when the matrix

$$\mathbf{\Lambda}(x, y) = \begin{bmatrix} (2\mu+\lambda)x^2+(\mu+\alpha)y^2 & (\lambda+\mu-\alpha)xy & B^0(x^2-y^2)+2\alpha y\cdot i \\ (\lambda+\mu-\alpha)xy & (2\mu+\lambda)y^2+(\mu+\alpha)x^2 & -(2B^0xy+2\alpha xi) \\ B^0(x^2-y^2)-2\alpha yi & -(2B^0x\cdot y-2\alpha\cdot x\cdot i) & C^0(x^2+y^2)+4\alpha \end{bmatrix}$$

is positive definite for arbitrary $x, y \in R$. It can be shown (cf. [8]) that the above condition can be reduced to the system of inequalities involving effective moduli λ, μ, α, B^0 and C^0

$$\mu > 0 \wedge \alpha > 0 \wedge 2\mu+\lambda > 0 \wedge$$
$$\left[\left(\alpha < \mu+\lambda \wedge C^0 > \frac{(B^0)^2}{\mu+\alpha}\right) \vee \left(\alpha > \mu+\lambda \wedge C^0 > \frac{(B^0)^2}{2\mu+\lambda}\right)\right]. \tag{4.6}$$

By virtue of the definitions $(6.2)_{1-3}$, [4], of λ, μ and α moduli it can be stated that $\alpha < \mu+\lambda$. The last condition $(4.6)_3$ reduces to the form

$$C^0 > C^0_s = (B^0)^2/(\mu+\alpha). \tag{4.7}$$

The moduli λ, μ and α satisfy the conditions $(4.6)_{1-3}$ whereas the inequality (4.7) is not fulfilled for real grids. Therefore Eqs. (4.1) obtained by formal (allthough justified in the previous section) simplifications of the second-order Eqs. (6.1), [4], are unstable in the meaning of Kunin.

4.4. Strong ellipticity. Strong ellipticity of a partial-differential equation system implies (see [9]) the solutions featured by the properties similar to those known from a classical theory of well-established boundary value elliptic problems involving a one function to be sought. If boundary conditions are admissible the strong ellipticity suffices for existence, uniqueness and continuous dependence the solution upon the boundary conditions.

Consider a correctly supported hexagonal grid plate. Solutions are unique and always exist as it clearly follows from the theory of structures. This continuum theories ought to ensure (apart from specific cases which are not dealt with here) the solutions to be unique that holds good provided the moduli λ, μ, α, B^0 and C^0 satosfy the strong ellipticity condition.

The set of Eqs. (4.1) is strongly elliptic when and only when the matrix

$$\Xi(x, y) = \begin{bmatrix} (2\mu+\lambda)x^2+(\mu+\alpha)\cdot y^2 & (\lambda+\mu-\alpha)xy & B^0(x^2-y^2) \\ (\lambda+\mu-\alpha)\cdot x\cdot y & (2\mu+\lambda)y^2+(\mu+\alpha)x^2 & -2B^0xy \\ B^0(x^2-y^2) & -2B^0xy & C^0(x^2+y^2) \end{bmatrix}$$

is positive definite for arbitrary $x, y \in R$. On using Sylvester theorem necessary and sufficient conditions of strong ellipticity, [8]

$$2\mu+\lambda > 0 \wedge \mu+\alpha > 0 \wedge$$
$$\left[\left(\alpha < \mu+\lambda \wedge C^0 > \frac{(B^0)^2}{\mu+\alpha}\right) \vee \left(\alpha > \mu+\lambda \wedge C^0 > \frac{(B^0)^2}{2\mu+\lambda}\right)\right] \tag{4.8}$$

are arrived at. Therefore stability implies strong ellipticity condition so that ellipticity analysis does not yield additional restrictions imposed on moduli B^0 and C^0. Thus Eqs. (4.1) are not strongly elliptic.

5. Formulation of stable quasicontinuum Cosserat-type $\varkappa$-models

5.1. Modification of the modulus C^0. In the preceding sections instability and non-ellipticity of Eqs. (4.1), which approximate difference equilibrium equations (3.4), [4], on their solutions, have been shown. In order to construct a stable system of equations (which will be called $\varkappa$ — equations) a modification of the last equation $(3.4)_3$, [4], expressing a balance of moments of the main node **i**, will be carried out. The modified equation reads

$$\mathscr{L}_{31}(\mathbf{u}^1)+\mathscr{L}_{32}(\mathbf{u}^2)+\mathscr{L}_{33}(\boldsymbol{\varphi})+\mathscr{L}_3(\mathbf{F}_1, \mathbf{F}_2, \mathbf{M}) = 0, \tag{5.1}$$

where

$$\begin{gathered} \mathbf{u}^\alpha = (u^\alpha_K, u^\alpha), \quad \boldsymbol{\varphi} = (\varphi_K, \varphi), \quad \alpha = 1, 2, \quad K = I, \ldots, VI, \\ \mathbf{F}_\alpha = (\overset{*}{F}{}^\alpha_{\mathrm{a}}, \overset{*}{F}{}^\alpha_{\mathrm{b}}, \overset{*}{F}{}^\alpha_{\mathrm{c}}, F^\alpha), \qquad \mathbf{M} = (\overset{*}{M}_{\mathrm{a}}, \overset{*}{M}_{\mathrm{b}}, \overset{*}{M}_{\mathrm{c}}, M) \end{gathered} \tag{5.2}$$

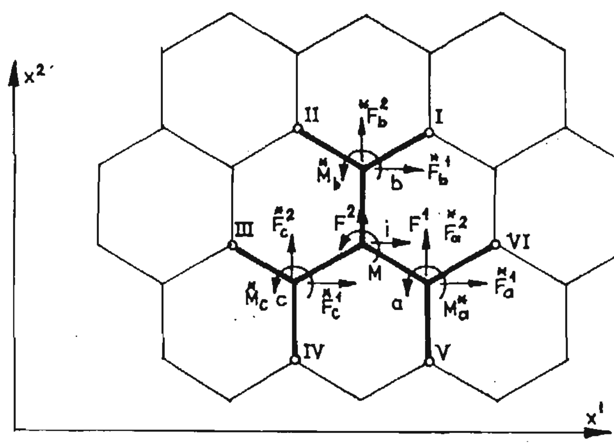

Fig. 2

Quantities (u_K^α, φ_K), $K = \mathrm{I}, \ldots, \mathrm{VI}$, denote displacements of main nodes surrounding the main node **i** to which Eq. (5.1) is referred, cf. Fig. 2, (u^α, φ) mean displacements in the **i** node; $\overset{*}{F}_J$, $\overset{*}{M}_J$, $J = \mathbf{a}, \mathbf{b}, \mathbf{c}$, denote forces and moments subjected to intermediate modes surrounding **i**; F^α, M stand for similar quantities referred to the latter node .By means of $\mathscr{L}_{3J}$, $\mathscr{L}_3$ difference operators determined by the coefficients $\Phi_{3\beta}^{(m)}$ and $S_{3\beta}^{(m)\varepsilon}$ (see Eqs. (3.5) in [4]) are denoted.

The object of the modification is an operator $\mathscr{L}_{33}$

$$\mathscr{L}_{33}(\boldsymbol{\varphi}) = \left\{\left[\frac{4}{\sqrt{3}}\cdot\frac{\eta}{\bar{\eta}(1+\bar{\eta})} - \frac{2}{\sqrt{3}}\cdot\frac{3\eta+\bar{\eta}}{\bar{\eta}} + \frac{2}{3\sqrt{3}}\frac{(3\eta-\bar{\eta})^2}{\bar{\eta}(3\eta+\bar{\eta})}\right]\varphi + \right.$$

$$\left. + \left[\frac{-2\eta}{3\sqrt{3}\,\bar{\eta}(1+\bar{\eta})} + \frac{2}{9\sqrt{3}}\frac{(3\eta-\bar{\eta})^2}{\bar{\eta}(\bar{\eta}+3\eta)}\right]\cdot\sum_{J=I}^{VI}\varphi_J\right\}\frac{EJ}{l^3} \tag{5.3}$$

Let the differential expression

$$\overset{\circ}{C}_{(\varkappa)}\nabla^2\varphi(x^\sigma) - 4\alpha\varphi(x^\sigma), \qquad \nabla^2 = \partial_1^2+\partial_2^2, \qquad \sigma = 1,2 \tag{5.4}$$

be approximated by a weighted difference expression $\mathscr{L}_{33}(\varphi)$

$$\mathscr{L}_{33}(\boldsymbol{\varphi}) = \overset{\circ}{C}_{(\varkappa)}\nabla_R^2\boldsymbol{\varphi} - 4\alpha\tilde{\boldsymbol{\varphi}} \tag{5.5}$$

where

$$\nabla_R^2\boldsymbol{\varphi} = \frac{1}{3l^2}\left[-4\varphi + \frac{2}{3}\sum_{J=I}^{VI}\varphi_J\right], \qquad \tilde{\boldsymbol{\varphi}} = \varkappa\varphi + \frac{1-\varkappa}{6}\sum_{J=I}^{VI}\varphi_J. \tag{5.6}$$

The parameter $\varkappa$ is taken from the interval [0,1] hence the weighted coefficients are assumed to be positive. The expression $\tilde{\boldsymbol{\varphi}}$ approximates the value $\varphi(\mathbf{i})$ with an error of order l^2 The expression $\nabla_R^2\boldsymbol{\varphi}$ approximates the laplacian $\nabla^2\varphi$ with an error of fourth order. Thus the RHS of Eq. (5.5) approximates (5.4) with an error of second order. By equating (5.3) with the RHS of (5.5) two relations involving $C^0_{(\varkappa)}$ and α are obtained. The first one yield the known definition $(6.2)_3$, [4], of the modulus α. The second one results in ,

$$C^\circ_{(\varkappa)} = C^\circ + 3(1-\varkappa)\alpha l^2$$

$$C^0_{(\varkappa)} = \left\{\frac{\sqrt{3}}{3\bar{\eta}(\bar{\eta}+3\eta)}[9\cdot\eta^2 + 6\cdot(2-3\varkappa)\eta\cdot\bar{\eta} + \bar{\eta}^2] - \frac{\sqrt{3}\,\eta}{\bar{\eta}(1+\bar{\eta})}\right\}\frac{EJ}{l}. \tag{5.7}$$

If $\varkappa = 1$ we have $C^0_{(1)} = C^0$. Thus a simple generalisation of the definition $(4.3)_2$ is found. The modulus $C^0_{(\varkappa)}$ (being dependent upon $\varkappa$ parameter) varies considerably when $\varkappa$ changes its value from zero to one. If $\varkappa = 1/3$

$$C^0_{(1/3)} = \frac{\sqrt{3}\,(3\eta+\bar{\eta}+1)}{3(\bar{\eta}+1)}\cdot\frac{EJ}{l} = C^\wedge \tag{5.8}$$

and thus the Woźniak-Klemm's modulus $C^\wedge$, see $(3.3)_2$, occurs. It is worth mentioning that the modulus $C^\wedge$, $(3.4)_2$, resulting from the second version of constitutive equations, differs inconsiderably from $C^0_{(1)} = C^\circ$ provided the bars are sufficiently slender $(\bar{\eta} \approx \eta)$

$$C^0_{(1)} = \left(\frac{\sqrt{3}}{3} - \frac{\sqrt{3}}{1+\eta}\right)\cdot\frac{EJ}{l} \approx \frac{\sqrt{3}}{3}\cdot\frac{EJ}{l} = C^{\wedge} \tag{5.9}$$

5.2. Stability condition. Lower and upper bounds of the modulus $C^0_{\varkappa()}$. Examine for what values of $\varkappa$ the modified system of Eqs. (4.1) (viz. in which a quantity C^0 is substituted by $C^0_{(\varkappa)}$) satisfies stability as well as ellipticity conditions. Stability (or strong ellipticity) condition (4.7):

$$C^0_{(\varkappa)} > C^0_s = \frac{(B^0)^2}{\mu+\alpha}$$

implies the domain of variation of $\varkappa$ parameter to be decreased to the interval $[0, \varkappa_s)$ being dependent upon the slenderness ratio η.
If $\bar{\eta} \approx \eta$ we can evaluate

$$\varkappa_s = \frac{249\eta^2+266\eta+17}{324\eta^2+360\eta+36} \approx 0.7 \quad \text{for} \quad \eta \gg 25.$$

One can require the $C^0_{(\varkappa)}$ modulus to satisfy inequality $(3.5)_3$ resulting from positive determination of strain energy expressed in terms of strain components $\gamma_{\alpha\beta}$ and $\varkappa_\alpha$.

$$C^0_{(\varkappa)} > C^0_{p.d.} = (B^0)^2/\mu \tag{5.10}$$

that decreases the upper bound of $\varkappa$: $\varkappa \in [0, \varkappa_d)$ where $\varkappa_d < \varkappa_s$. The definition of $\varkappa_d$ expressed in terms of η will not be reported here.

The upper bound of $C^0_{(\varkappa)}$: $C^0_{(\varkappa)} < C^0_{(0)}$ does not follow from physical considerations but from the condition of positivity of weighted coefficients in $(5.6)_2$. Thus the modulus $C^0_{(\varkappa)}$ can vary in the limits ,

$$C^0_{p.d.} < C^0_{(\varkappa)} < C^0_{(0)} \quad \text{or} \quad C^0_s < C^0_{(\varkappa)} > C^0_{(0)}. \tag{5.11}$$

5.3. K-representation interpretation of the proposed „stabilisation procedure". Remarks on the range of applicability of the micropolar-type $\varkappa$-models. The proposed modification of $\mathscr{L}_{33}$ operator can be interpreted as an approximation of the function $P^{-1}\hat{\Phi}_{33}(\mathbf{k})$ defined by Eq. (3.5) in [4]. Accuracy analysis of this approximation is outlined below.

The function $P^{-1}\hat{\Phi}_{33}(\mathbf{k})$ can be expressed by the formula

$$P^{-1}\hat{\Phi}_{33}(\mathbf{k}) = \Phi^{(0)}_{33} + 2\Phi^{(I)}_{33}\cdot\sum_{J=I}^{III}\cos\left(l\sqrt{3}\mathbf{k}\mathbf{t}_J\right) = 4\alpha - \frac{4}{9}\gamma\sum_{J=I}^{III}\left[\cos\left(l\sqrt{3}\mathbf{k}\mathbf{t}_J\right)-1\right] \tag{5.12}$$

where the relations

$$6\Phi^{(I)}_{33} + \Phi^{(0)}_{33} = 4\alpha, \qquad -4.5\Phi^{(I)}_{33} = \gamma$$

have been applied. The constants α and γ have been defined by Eqs. $(6.2)_{3,6}$ in [4]. Components of $\mathbf{t}_J$ vectors are, see Fig. 1 in [4]

$$\mathbf{t}_I = \left(-0.5, -\sqrt{3}/2\right), \quad \mathbf{t}_{II} = \left(0.5, -\sqrt{3}/2\right), \quad \mathbf{t}_{III} = (1, 0).$$

It can be shown that the function $\hat{\Phi}_{33}(\mathbf{k})$ varies almost independently of the wave vector direction provided $\tau = l\cdot|\mathbf{k}| < 3$, thus,

$$\sum_{J=I}^{III} \cos(l\sqrt{3}\,\mathbf{k}\mathbf{t}_J) \approx \sum_{J=I}^{III} \cos(l\sqrt{3}\,(0, |\mathbf{k}|)\mathbf{t}_J) = 2\cdot\cos(1.5\tau)+1$$

hence

$$P^{-1}\hat{\Phi}_{33}(\mathbf{k}) = 4\alpha - \frac{8}{9}\gamma[\cos(1.5\tau)-1], \tag{5.13}$$

The modification proposed in Sec. 5.1 corresponds to the approximation

$$P^{-1}\hat{\Phi}_{33}(\mathbf{k}) = 4\alpha + \gamma_{(\varkappa)} l^2|\mathbf{k}|^2 = 4\alpha + \gamma_{(\varkappa)}\cdot\tau^2 \tag{5.14}$$

where $\gamma_{(\varkappa)} = C^0_{(\varkappa)}/l^2, \quad \gamma_{(1)} = \gamma, \quad \gamma_{(\varkappa)} > \gamma, \quad \text{if} \quad \varkappa \in (0, 1)$.
Fig. 3 displays variations of the function f

$$f(\tau) = \frac{1}{\alpha\cdot P}\hat{\Phi}_{33}(\mathbf{k}) \approx 4 + \frac{8}{9}\frac{\gamma}{\alpha}\cdot(1-\cos(1.5\tau)) \tag{5.15}$$

and its approximation (Eq. 5.14)

$$g_{(\varkappa)}(\tau) \approx \frac{1}{\alpha P}\hat{\Phi}_{33}(\mathbf{k}) \approx 4 + \frac{\gamma}{\alpha}\frac{\gamma_{(\varkappa)}}{\gamma}\tau^2 \tag{5.16}$$

The diagrams in Fig. 3, are found for $\eta \approx \bar{\eta} = 50$ ($l/h \approx 7$) thus $\gamma/\alpha = 0.6275$.

As it was possible to suspect it can be stated that $g_{(1)}$ curve yields the best approximation of f. For wave vectors satisfying the condition $\tau < 1.5$ a relative error of approximation of f by $g_{(1)}$ is less than 10% and less than 2.5% provided $\tau < 1$. If an accuracy analysis is confined to the behaviour of the function $\hat{\Phi}_{33}(\mathbf{k})$ one can conclude that unstable Cosserat-type model which employs the set of moduli λ, μ, α, B^0, $C^0_{(1)}$ approximates „rotational waves" (i.e. two-variable continuous functions interpolating rotations of main nodes) of lengths $\lambda_d = \frac{2\pi}{|\mathbf{k}|} > 2\pi l$ with some per cent errors only. Considerably worse result is yielded from the set of moduli (λ, μ, α, B^0, $C^0_{(0)}$). If $\tau < 0.25$ a relative error approximation of f by $g_{(0)}$ is less than 4.7%. The latter condition means that „rotational waves" of lengths $\lambda_d > 8\pi l \approx 23.14\, l$ are admitted with 5% error whereas the wave patterns of $\lambda_d > 4\cdot\pi\cdot l$ are related to 18% of error. Therefore according to the choice of the parameter $\varkappa \in (0, 1)$ the „rotational wave patterns" of lengths $\lambda_d > \lambda_{(\varkappa)}$ where $\lambda_{(\varkappa)} \in (6.28\, l, 25.14\, l)$ are admissible.

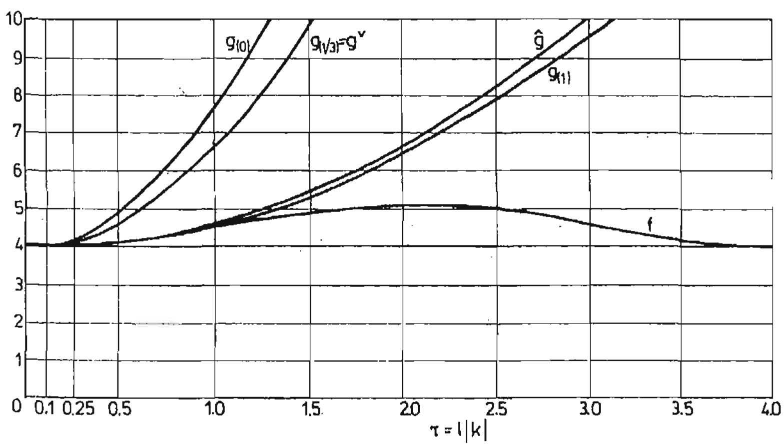

Fig. 3

The above analysis is somewhat incomplete since our attention has been focused on the function $\hat{\Phi}_{33}(\mathbf{k})$ only. To make-up the considerations a complete analysis of approximations of all functions $\hat{\Phi}_{\alpha\beta}$ should be given.

Mention yet that the constant function $h(\tau) = 4$ supplies a better approximation of the function f that $g_{(\varkappa)}$ functions provided ,

$$\gamma_{(\varkappa)}/\gamma > 2. \tag{5.17}$$

Therefore $\varkappa$ parameter cannot be to small; if not the carried out modification of $\hat{\Phi}_{33}$ induces the greater error than a simple neglect of the term $C^0\nabla^2\varphi$ in $(4.1)_3$. It is easy to show that lower bounds of C^0: $C^0_{p.d.}$ and C^0_s do not satisfy (5.17), i.e. $C^0_{p.d.} > 2C^0_{(1)} = 2\cdot\gamma\cdot l^2$. Nevertheless it is more purposeful to retain the term $C^0\nabla^2\varphi$ in $(4.1)_3$ than to neglect it (and further to eliminate rotations φ from Eqs. $(4.1)_{1,2}$, see [4], Sec. 6) in order to formulate well-established stable theory. However the stability is achieved at the sacrifice of the approximation condition.

6. Comparison of Woźniak's and $\varkappa$-models. Further remarks on accuracy analysis

6.1. Governing equations. Governing equations (3.1) and (4.1) have similar forms. As it has been mentioned is Sec. 4 both systems of partial differential equations involve the same moduli λ, μ and α. Qualitative differences distinguish between two sets of moduli B and C; to this topic Sec. 6.4 will be devoted. Essential quantitative differences occur in functions approximating effects of external loads. Definitions (4.4) of $\overset{\circ}{p}{}^\alpha$ and $\overset{\circ}{Y}{}^3$ cannot be rearranged to the form (3.2), viz. Eqs. (3.2) do not involve $\overset{*}{p}{}^\alpha$ but their derivatives only. The latter fact can be treated as a shortcoming of the theory. The definitions of $\overset{\circ}{Y}{}^3$ and $'Y^3$ are also different. In the case of slender rods ($l/h > 7$, $\eta > 49$, say) $'Y^3$ depends inconsiderably on $\overset{*}{Y}{}^3$ and, if $\eta \to \infty$ we have $'Y^3 \to Y^3$; on the other hand in $\varkappa$-models there is: $Y^3 \to Y^3 - 0.5\overset{*}{Y}{}^3$ provided $\eta \to \infty$.

6.2. Existence and uniqueness of solutions. The existence and uniqueness of solutions are ensured by:

a) stability (4.6) or strong ellipticity (4.8) conditions — in the case of $\varkappa$-models;

b) positive definiteness of strain energy expressed in terms of γ and $\varkappa$ tensors (3.5) — in both approaches due to Woźniak's concept.

It is worth stressing that the latter condition is stronger than the former.

6.3. Boundary conditions. In Sec. 4 boundary conditions of $\varkappa$-versions have not been formulated. However, since the governing equations of these models have similar form to Woźniak's equations it seems reasonable to subject the solutions to similar boundary conditions, cf. Eqs. (7.3) in [3].

6.4. Analysis of moduli B and C, In the paper [3] (cf. Sec. 3 of the present work) two versions of constitutive equations of Woźniak-type models resulting in different moduli B ($B^{\vee}$, $B^{\wedge}$) and $C(C^{\vee}$, $C^{\wedge})$ have been proposed. Considerations based on the quasicontinuum approach (Sec. 4) yield the one definition (4.3) of B^0 and C^0. In Sec. 5 the one-parameter definition (5.7) of the modulus $C^0_{(\varkappa)}$ has been derived.

Variations of moduli $B(\varrho)$, $\varrho \in [0.8, 1]$ (ratio ϱ being defined by (2.1)) are plotted in Fig. 5 in the case of $\nu = 0{,}3$ (so that $\bar{\eta} = \eta + 3.12$). The following inequalities hold true

$$B^\wedge(\varrho) < B^0(\varrho) < B^\vee(\varrho), \qquad \varrho \in [0.8, 1].$$

Differences between $B^\wedge$ and B^0 are considerable. If one assumes B^0 as exact difinition of B deviations of $B^\wedge$ from B^0 attain to 100% relative error.

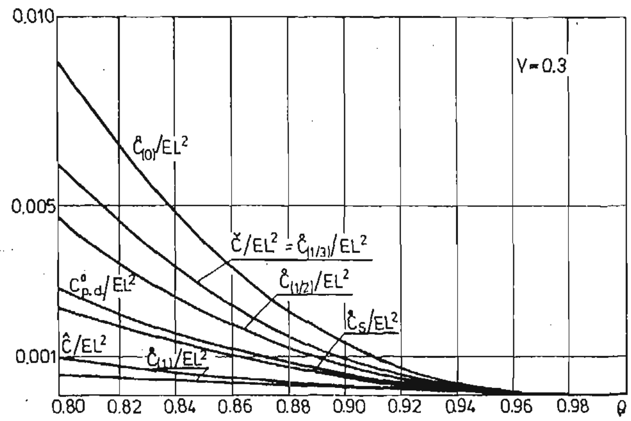

Fig. 4

Consider variations of moduli $C^0_{(\varkappa)}(\varrho)$, $C^\wedge(\varrho)$ and $C^\vee(\varrho)$. The family of curves $C^0_{(\varkappa)}(\varrho)$ covers the area between $C^0_{(0)}(\varrho)$ and $C^0_s(\varrho)$ or $C^0_{p.d.}(\varrho)$ curves. In Fig. 4 two curves $C^\vee(\varrho) = C^0_{(1/3)}$ and $C^0_{(1/2)}$ lieing within the mentioned „admissible" area and a curve $C^\wedge(\varrho)$ outside this region are plotted.

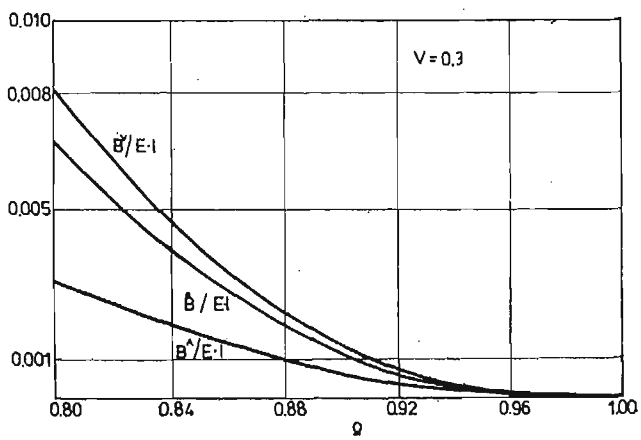

Fig. 5

6.5. A range of applicability of two models due to Woźniak's concept. Analogously to the method applied to $\varkappa$-versions, cf. Sec. 5.3, an accuracy analysis of the two models due to Woźniak's concept can be carried out:

i) Woźniak-Klemm's variant.

Examine approximations of functions $\hat{\Phi}_{\alpha\beta}(\mathbf{k})$ (see [4]) corresponding to Eqs. (3.1) of I version in which moduli λ, μ, α, $B^{\vee}$ and $C^{\vee}$ are employed. An analysis will be restricted to $\hat{\Phi}_{3k}$ and $\hat{\Phi}_{k3}$ ($k = 1, 2, 3$) functions approximations of which are non-trivial i.e. do not result from neglect of higher order terms in the expansions (5.2), [4]. By virtue of an equality $\overline{\hat{\Phi}}_{3k} = \hat{\Phi}_{k3}$ it is sufficient to analyse three functions $\hat{\Phi}_{k3}$, $k = 1, 2, 3$, only.

Approximation analysis of $\hat{\Phi}_{33}(\mathbf{k})$ outlined in Sec. 5.3 makes it possible to evaluate the errors of approximation of this function induced by Woźniak's models. Namely, bearing in mind that Eq. (5.8) holds good, an analysis of approximation by the I model reduces to the analysis of the case of $\varkappa = 1/3$. A curve $g_{(1/3)}$ is displayed in Fig. 3. On assuming the f function to be approximated by $g_{(1/3)}$ with a 12% error a quantity $\lambda_{(\varkappa)}$ (cf. Sec. 5.3) amounts to $\lambda_{(1/3)} = 4 \cdot \pi \cdot l \approx 7.26b$. Thus only smooth regular long „rotational waves" φ of $\lambda_d > \lambda_{(1/3)}$ may be admitted.

Consider a function $\hat{\Phi}_{13}(\mathbf{k})$ and its approximation related to the I version. First few terms of $\hat{\Phi}_{13}$ written out explicitly read,

$$P^{-1}\frac{l}{\alpha}\hat{\Phi}_{13}(\mathbf{k}) \approxeq \frac{\beta}{\alpha}\left[\tau^2\cos 2\Theta - \frac{\tau^4}{16}(\cos^4\Theta - 3\sin^4\Theta - 6\cos^2\Theta\sin^2\Theta)\right] + \tag{6.1}$$

$$-i\frac{\tau}{4}\sin\Theta(8-3\tau^2)$$

where (acc. to notations introduced in Sec. 4.2) $l\cdot\mathbf{k} = \tau(\cos\Theta, \sin\Theta)$ whereas the following expression

$$P^{-1}\frac{l}{\alpha}\hat{\Phi}_{13}(\mathbf{k}) \approx \frac{\beta^{\vee}}{\alpha}\tau^2\cos 2\Theta - 2i\tau\sin\Theta, \qquad \beta^{\vee} = B^{\vee}/l \tag{6.2}$$

corresponds to the first (I) version. The approximation errors being induced by (6.2) depend considerably upon the wave vector direction and attain the greatest values when $k_2 = 0$. One can show that if lattice bars are slender ($\eta > 50$, say) a relative error of evaluation of absolute values of complex function $\hat{\Phi}_{13}$ is not less than 15% for arbitrary $k_1 > 0$ and amounts to 17% if $\cos\Theta = 0.25$. Thus no matter how the functions u^{α} and φ are regular approximation errors of $|\hat{\Phi}_{13}|$ are at least about 15%. In the case of $k_1 = 0$ the errors induced by (6.2) are less than in the previous case and tend to zero provided $k_2 \to 0$. Deformations of $\mathbf{k} = (0, k_2)$ direction and of wavelengths $\lambda_d > 8\pi l$ are associated with 4.6% error of evaluating of $|\hat{\Phi}_{13}|$, and if $\lambda_d > 4\pi l$ the errors increase to 15%.

Consider the errors of evaluating of $|\hat{\Phi}_{23}(\mathbf{k})|$. The first few terms of $\hat{\Phi}_{23}$ are

$$P^{-1}\frac{l}{\alpha}\hat{\Phi}_{23}(\mathbf{k}) \approx \frac{\beta}{\alpha}\tau^2\cos\Theta\sin\Theta\left[-2+\frac{\tau^2}{4}(\cos^2\Theta+3\sin^2\Theta)\right]+$$

$$+i\left[2\tau\cos\Theta - \frac{3}{4}\tau^3\cos\Theta\right]. \tag{6.3}$$

Equations of the (I) version result from the approximation

$$P^{-1}\frac{l}{\alpha}\hat{\Phi}_{23}(\mathbf{k}) \approx \frac{-2\beta^{\vee}}{\alpha}\tau^2\cos\Theta\sin\Theta+2i\tau\cos\Theta \tag{6.4}$$

If $k_1 = 0$ the approximation errors vanish. For $k_2 = 0$ a wavelength $8\cdot\pi l$ corresponds to 2.2% error of evaluating of $|\hat{\Phi}_{23}|$; if $\lambda_d = 4\pi l$ an error amounts to 8.5%. In the case of $k_1 = k_2$ ($\Theta = \pi/4$) wavelengths $\lambda_d > 8\pi l$ correspond to 5% error, and if $\lambda_d > 4\pi l$ — to 15% error.

The above analysis proves that approximation errors of $|\hat{\Phi}_{k3}|$, $k = 1, 2, 3$, are of the same order (12%—15%) in the case of wavelengths $\lambda_d > 4\pi l$.

ii) Variant II.

In this version approximation of $|\hat{\Phi}_{33}|$ is considerably better than in the first (I) version. A curve $\hat{g}$

$$\hat{g}(\tau) = \frac{1}{\alpha P}\hat{\Phi}_{33}(\mathbf{k}) = 4+\frac{\gamma}{\alpha}\frac{\hat{\gamma}}{\gamma}\tau^2, \qquad \hat{\gamma} = \hat{C}/l^2$$

approximates the function f (cf. Fig. 5.2) with an error of 12% provided $\lambda_d > \frac{4}{3}\pi l \approx 2.42$b, and with 3.2% error for $\lambda_d > 2\pi l$.

Consider $\hat{\Phi}_{13}$ function. According to the wave vector direction a relative error of evaluating of $|\hat{\Phi}_{13}|$ is less or great than that induced by Klemm-Woźniak's equations. In the case of $k_1 = 0$, $k_2 > 0$ an error of 9% is associated with deformations of wavelengths $\lambda_d > 4\cdot\pi l$ (in I version — 15%). The error decreases to 0% provided $k_2 \to 0$. Assume that $k_1 > 0$, $k_2 = 0$. If $k_1 \to 0$ the error tends to 43% and decreases (what is a paradox) to 17% for $\lambda_d = 2\pi l$. In the case f of $k_1 = k_2$ the errors induced by both (I) and (II) versions are identical.

Consider approximations of $|\hat{\Phi}_{23}(\mathbf{k})|$. In both cases: $k_1 = 0$, $k_2 > 0$ and $k_1 > 0$, $k_2 = 0$ the errors are equal to those induced by the I version: 2.2% for $\lambda_d > 8\pi l$ and 8.5% if $\lambda_d > 4\pi l$. In the case of $k_1 = k_2$ an approximation error of $|\hat{\Phi}_{23}|$ is equal to 16.5% provided $\lambda_d > 8\pi l$ thus it is greater than in the I version where this error is about 5%.

Therefore approximation errors of $|\hat{\Phi}_{3k}|$ hesitate: from inconsiderable errors induced by the approximation of $|\hat{\Phi}_{33}|$ to essential errors related to $\hat{\Phi}_{13}$ in the particular case of $k_2 = 0$. Note that in the case of deformations relevant to the vector $\mathbf{k} = (0, k_2)$, displacement waves of $\lambda_d > 4\pi l$ correspond to 10% errors what is rather a small value if one takes into account that a wavelength $\lambda_d d$ min $= 4\pi l$ is relatively short with respect to the internode distance. Displacement waves of the direction $\mathbf{k} = (k_1, 0)$ are related to the errors of about 40%. Thus the II variant is characterized by the lack of symmetry of approximations; directions k_1 and k_2 are not equivalent here.

7. Concluding remarks

Three Cosserat-type models for fine hexagonal grids have been analysed:

i) wariant I (due to Woźniak and Klemm) involving moduli λ, μ, α, $\check{B}$ and $\check{C}$

ii) wariant II in which moduli λ, μ, α, $\hat{B}$ and $\hat{C}$ are employed

iii) $\varkappa$-models with moduli λ, μ, α, B^0 and $C^0_{(\varkappa)}$, $\varkappa \in [0, \varkappa_s)$.

In order to estimate ranges of applicability of the mentioned models approximation errors of $\hat{\Phi}_{3\beta}$ and $\hat{\Phi}_{\beta 3}$, $\beta = 1, 2, 3$, induced by each approach, have been examined.

Ad i) It has been shown that approximation errors of $|\hat{\Phi}_{k3}|$ are equal to $\sim 15\%$ for deformation patterns of wavelengths $\lambda_d > 4\pi l$. This value of errors results from the, approximation of all of the functions $\hat{\Phi}_{k3}$, $k = 1, 2, 3$

Ad ii) The considerable approximation errors pertain to the functions $\hat{\Phi}_{\alpha 3}$, $\alpha = 1, 2$ These errors depend upon the wave vector direction. To wave deformation patterns of $\mathbf{k} = (k_1, 0)$ a 40% error of evaluation of $|\hat{\Phi}_{13}|$ is related

Ad iii) The essential errors occur in approximation of the function $\hat{\Phi}_{33}$ and strongly depend on a choice of a parameter $\varkappa$. In the limiti ng case of $\varkappa = 0$ an 18% error is related *to rotational wave patterns of* $\lambda_d > 4\pi l$ *whereas if* $\varkappa = 1/3$ *an analogous error is equal to* 12%.

The presented error analysis is obviously simplified since no relations between resulting errors of the sought functions u^α and φ and errors of approximations imposed on functions $\hat{\Phi}_{ij}(\mathbf{k})$ have been given. Nevertheless the proposed accuracy analysis allows us to formulate the following remarks.

1. It is impossible to suspect which of the versions considered (i ÷ iii) induces the greatest errors of evaluating displacements u^1 and u^2 whereas it is almost apparent that the (ii) version should yield the best evaluation of nodal rotations φ.

2. The unstable variant $\varkappa = 1$ (iii) can be employed for examining local effects resulting for instance from an influence of concentrated nodal loads.

3. The version (iii) allows us to consider an effect of changes of the parameter $\varkappa$ on solutions of boundary value problems. Computations performed for several values of $\varkappa \in (0, \varkappa_s)$ yield results which are divisible into two groups (a) and (b). Results of (a) type are stable with respect to variations of $\varkappa$ whereas the (b)-results do not satisfy the latter condition. Apart from this, a zero-order Horvay's asymptotic model (see [4]) involving displacements u^1, u^2 only, can be employed. Results of (a) type vary inconsiderably thus only these results are reliable. Nevertheless results of (b) type are of great interest, because despite the fact that their values are evaluated incorrectly, valuable qualitative information is obtained. In cases of simple states of stress, to (a) group displacements belong while rotations belong to (b) group.

The remarks formulated above are confirmed in [10] in the specific case of infinitely long grid strip of hexagonal structure subjected to longitudinal forces. Specifically, the remark 1 occured to be accurate; allthough approximations of nodal rotations are charged with errors, the II version seems to induce the smallest ones. Nonetheless it should be pointed out here that the Cosserat-type models provide the description qualitatively correct, particularly the alternate vanishing variation of rotations of main and intermediate nodes lieing along the lines perpendicular to the strip's horizontal axis being successfully predicted.

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Р е з ю м е

АНАЛИЗ ФИЗИЧЕСКОЙ ПРАВИЛЬНОСТИ МОДЕЛЕЙ ТИПА КОССЕРАТОВ ДЛЯ СЕТЧАТЫХ ГЕКСАГОНАЛЬНЫХ ПЛАСТИНОК

В работе рассматривается проблема физической корректности микрополярной модели стержневых гексагональных плястинок. Проанализированы три варианта таких теорий: две версии модели Возняка и псевдоконтинуальные $\varkappa$-модели получены путём соответствующих модификаций уравнений Рогули — Кунина.

Упругие свойства решётки определяются при помощи эффективных модулей λ, μ, α, B и C. Первые три константы определены однозначно, в то время как „микрополярные" модули B и C забисят от выборя варианта метода континуальной описи решётки. Кроме того, анализирòванные теории разным образом учитывают влияние нагрузок в „посредственных" узлах конструкции.

Каждой версии описи типа Коссератов отвечает некоторая, аипроксимация функции $\Phi_{\alpha\beta}(\mathbf{k})$ в окрестности $\mathbf{k} = 0$. Представлен анализ приближений тех функции с помощю которого формулируются гипотезы касающиеся границ применения дифференцияльных моделей.

S t r e s z c z e n i e

ANALIZA FIZYCZNEJ POPRAWNOŚCI MODELI TYPU COSSERATÓW PRĘTOWYCH TARCZ HEKSAGONALNYCH

W pracy podjęto problem fizycznej poprawności modelu mikropolarnego prętowej tarczy heksagonalnej. Dokonano analizy trzech wariantów teorii wykorzystującej ten model: dwie wersje teorii Woźniaka oraz pseudokontynualne $\varkappa$ — modele otrzymane drógą modyfikacji równań Roguli i Kunina.

Właściwości siatki są określone za pomocą modułów zastępczych λ, μ, α, B i C. Pierwsze trzy stałe są jednoznacznie określone, podczas gdy moduły „mikropolarne" B i C zależą od wyboru wariantu metody kontynualnego opisu tarczy. Ponadto analizowane wersje w różny sposób uwzględniają obciążenia przyłożone do węzłów pośrednich.

Każdej wersji opisu typu Cosseratów odpowiada pewna aproksymacja funkcji $\hat{\Phi}_{\alpha\beta}(\mathbf{k})$ w otoczeniu $\mathbf{k} = 0$. Przeprowadzono analizę aproksymacji tych funkcji i na jej podstawie sformułowano hipotezy dotyczące obszarów stosowalności omawianych modeli różniczkowych.

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