MECHANIKA
TEORETYCZNA
I STOSOWANA
3-4, 22 (1984)

CONSTRAINTS IN CONSTITUTIVE RELATIONS OF MECHANICS

CZESŁAW WOŹNIAK (Warszawa)

University of Warsaw
Faculty of Mathematics, Computer Sciences and Mechanics
Institute of Mechanics

Introduction

Constraints in mechanics are usually understood as the known restrictions imposed on the class of motions of a certain material system; they are due either to internal properties of a system (internal constrains) or to the influence of certain exterior objects or external fields (external constraints). Moreover, constraint imposed on motions is maintained by what are called reaction forces which can be internal (for internal constraints) or external (in the case of external constraints). As a rule, kinematic constraints together with the suitable reaction forces are analysed within the theory of constitutive relations of mechanics, i.e., within relations which characterize either internal (material) properties of the body under consideration or interactions between the body and its exterior.

So far, different special cases of constraints have been analysed independently in different problems of mechanics; the complete list of pertinent references is rather extensive and will not be given here. For the discussion of constraints in Hamiltonian and Lagrangian mechanics the reader is referred to [1] where the further references can be found. Internal constraints have been studied within the theory of constitutive relations of continuum mechanics; for the basic assumptions of the theory cf. [2]. The concept of constraints has been also applied in order to simplify the analytical form of problems in the elasticity theory, [3], and to obtain relations of structural mechanics (cf. [4], where the list of suitable references is given).

The main aim of the paper is to develop a general approach to the concept of constrains in discrete and continuum mechanics and to obtain and analyse the general form of constitutive relations in which the constraints are involved. It must be stressed that constitutive relations we are to deal with, describe not only material properties of bodies but also interactions between a body and external fields. The main attention in the paper will be given to these aspects of constitutive relations which are due to the constraints.

The concepts of constraint and that of the constitutive relations subject to constraints will be introduced and analysed in their abstract form, i.e., independently of any special class of problems in mechanics. Such approach, after suitable interpretations of the obtained relations, enables to formulate problems in which constraints are imposed not only on the kinematical fields but also on the internal and external forces as well as on any other field encountered in mechanics. Moreover, putting aside certain non physical situations, no regularity of any kind will be imposed on the sets of fields which are admissible by constraints. Hence different problems with involved form of constraints can be formulated on the basis of the general results obtained in the paper. The method of constraints which is developed here, constitutes the useful tool for the formation of new constitutive relations by imposing constraints on the known constitutive relation. The proposed approach is applied in order to obtain certain new classes of ideal materials within continuum mechanics; for further applications the reader is referred to [5] where some examples of unilateral internal constraints for strains and stress are discussed. Applications to structural mechanics will be given in papers [6, 7].

1. Constraints and reactions

We start from a class of mappings which are assumed to describe within mechanics all time evolutions of a quantity related to a certain material system. Throughout the paper we shall confine ourselves to mappings which can be represented by finite systems of real-valued functions defined on the time axis R and which are continuous and have continuous time derivatives for a.e. $t \in R$[1]. To introduce the class of mappings under consideration we shall assume that there is known the n-th dimensional manifold M of the class C^1. The tagent bundie to M will be denoted by TM, the cotangent bundle by T^*M; for every $m \in M$ the suitable tangent and cotangent spaces will be denoted by $T_m M$, $T_m^* M$, respectively. Moreover, τ_M and τ_M^* will stand for the natural projections of TM, T^*M, respectively, onto M. The dual pairing between $T_m M$ and $T_m^* M$ (for an arbitrary $m \in M$) will be denoted by $(T_m M, \langle \cdot, \cdot \rangle, T_m^* M)$. Let all mappings describing a time evolution of a certain quantity related to the material system under consideration be represented by elements of the known (topological) space $\Phi(R, M)$ of functions defined a.e. on R and with values in M. Hence the mappings we are to deal with are

$$\varphi : R \ni t \to \varphi(t) \in M \tag{1.1}$$

for some $\varphi \in \Phi(R, M)$. We shall also assume that the R.H.S. derivatives $\varphi'(t)$ exist for every $t \in R$.

The intuitive concept of constraints is closely related to the fact that in many problems under consideration not every $\varphi \in \Phi(R, M)$ describes certain physical situation and that in different situations we have to deal with different subsets of $\Phi(R, M)$. Thus, from a formal point of view, we are tempted to define constraints as certain proper subsets of $\Phi(R, M)$. However, such treatment of constraints is not based on the physical meaning of this concept. Firstly, not every restriction of $\Phi(R, M)$ has the physical sense of constraints[2]. Secondly, the choice of the space $\Phi(R, M)$ itself can be interpreted as introducing constraints in their intuitive meaning. To avoid any ambiguity, we shall introduce the following definition of contraints.

---

[1] Thus within continuum mechanics we confine ourselves to the situations in which there is involved only sufficiently small neighborhood of an arbitrary but fixed material particle.

[2] Such restriction can be introduced, for example, by imposing extra smoothness conditions on the space $\Phi(R, M)$ of mappings (1.1).

Definition 1. By *TM*-constraint we shall mean the multifunction

$$\mathscr{C}: R \ni t \to \mathscr{C}(t) \subset TM, \tag{1.2}$$

such that, under the denotations

$$\begin{gathered} A(m, t) \equiv \mathscr{C}(t) \cap T_m M, \quad (m, t) \in M \times R, \\ DA(t) \equiv \{m \in M | A(m, t) \neq \phi\} = \tau_M \mathscr{C}(t), \quad t \in R, \end{gathered} \tag{1.3}$$

the following conditions hold

$$\begin{gathered} (\forall t \in R)(\forall m \in DA(t))(\forall v \in A(m, t))(\exists f \in C^1(-\varepsilon, \varepsilon), \varepsilon > 0)[m = \\ = f(O), v = f'(0), f'(\lambda) \in A(f(\lambda), t+\lambda) \quad \text{for every} \quad \lambda \in [0, \varepsilon)], \\ (\forall t \in R)[DA(t) \neq \phi]. \end{gathered} \tag{1.4}$$

The multifunction

$$A: M \times R \ni (m, t) \to A(m, t) \subset TM, \tag{1.5}$$

such that

$$(\forall (m, t) \in M \times R)[A(m, t) \subset T_m M] \tag{1.6}$$

and satisfying Eqs. (1.4), will be called *TM*-constraint multifunction.

Corollary 1. Every *TM*-constraint multifunction defines *TM*-contraint and conversely. *TM*-constraint determined by *TM*-constraint multifunction A will be denoted by $\mathscr{C}_A$.

Corollary 2. Every *TM*-constraint multifunction (1.5) determines the subset $\Phi_A(R, M)$ of $\Phi(R, M)$ defined by

$$\Phi_A(R, M) := \{\varphi \in \Phi(R, M) | \varphi(t) \in DA(t), \quad \varphi'(t) \in A(\varphi(t), t), \quad t \in R\}, \tag{1.7}$$

where, by virtue of Eq. (1.4), the subset $\Phi_A(R, M)$ is not empty.

Remark 1. If M is a differentiable submanifold of a certain C^1-manifold M_1, then *TM*-constraint can also be interpreted as TM_1-constraint. Analogously, if M_0 is a differentiable submanifold of M, then *TM*-constraint can be interpreted as TM_0-constraint provided that $\mathscr{C}(t) \subset TM_0$ for every $t \in R$. Thus the concept of constraint is strictly related to the choice of the differentiable manifold M. This manifold in problems of mechanics, as a rule, is introduced by the known class $\Phi(R, M)$ of mappings[3]. The *TM*-constraint $\mathscr{C}$ we deal with will be called generalized since no regularity of any kind (apart from conditions given by Eq. (1.4)) is imposed on the non-empty subsets $\mathscr{C}(t)$ of TM.

Remark 2. In many problems of mechanics we deal with situations in which *TM*-constraint $\mathscr{C}$ (or *TM*-constraint multifunction A) is not known a priori but depends on certain element of a non empty set Ξ, i.e., $\mathscr{C} = \mathscr{C}_\xi$, $\xi \in \Xi$. If $\overline{\overline{\Xi}} > 1$ then $\mathscr{C}_\xi$, $\xi \in \Xi$, will be referred to as the implicit *TM*-constraints and if Ξ is one element set then we return to Definition 1 of (explicit) *TM*-constraint. For implicit constraints instead of *TM*-constraint multifunction (1.5) we shall introduce implicit *TM*-constraint multifunction

$$A: M \times R \times \Xi \ni (m, t, \xi) \to A(m, t, \xi) \subset TM, \tag{1.8}$$

such that $A(\cdot, \cdot, \xi)$ is, for every $\xi \in \Xi$, the known constraint multifunction.

The concept of constraints in mechanics is related not only to the restrictions imposed on the class of mappings (leading from $\Phi(R, M)$ to $\Phi_A(R, M)$), but also to the existence

---

[3] That is why *TM*-constraints $\mathscr{C}$ will be identified neither with TM_1-constraint nor with TM_0-constraint.

of certain fields which are treated as „maintaining" the constraint and are said to be „reactions" to constraint. In order to introduce such „reactions" we shall firstly define the sets

$$T_{A(m,t)}(v) := \{w \in T_v(T_m M) | w = g'(0), \quad v = g(0), \quad g(\lambda) \in A(m,t) \text{ for every } \lambda \in [0, \varepsilon), \varepsilon > 0, \text{ and some } g \in C^1((-\varepsilon, \varepsilon), T_m M)\},$$

and then, taking into account the canonical isomorphisms $\lambda_v: T_v(T_m M) \to T_m M$, we obtain the cones

$$K_{A(m,t)}(v) \equiv \lambda_v T_{A(m,t)}(v) \tag{1.9}$$

of directions tangent to $A(m, t)$ in $T_m M$ at the points $v \in A(m, t)$. Mind, that cones $K_{A(m,t)}(v)$ are empty if $v \in T_m M \setminus A(m, t)$ or $m \in M \setminus DA(t)$. Now we shall formulate the following

Definition 2. By a reaction cone of TM-constraints $\mathscr{C}: R \ni t \to \mathscr{C}(t) \subset TM$ at a time instant $t \in R$, at a point $m \in DA(t)$ and for an element $v \in A(m, t)$, we shall mean a cone in $T_m^* M$ given by

$$K^*_{A(m,t)}(v) := \{v^* \in T_m^* M | \langle w, v^* \rangle \geqslant 0 \quad \text{for every} \quad w \in K_{A(m,t)}(v)\}, \tag{1.10}$$

where, as usual, $A(m, t) = \mathscr{C}(t) \cap T_m M$, $m \in M$, $t \in R$.

Remark 3. Elements of every non-empty reaction cone for TM-constraint will be called reactions to constraint. It can be seen that every reaction cone $K^*_{A(m,t)}(v)$ is closed in T^*M and conjugate to the cone $K_{A(m,t)}(v)$ of directions tangent to $A(m, t)$ at $v \in A(m, t)$. For an arbitrary time instant $t \in R$ these cones are not empty if and only if $m \in DA(t)$, $v \in A(m, t)$.

Example of interpretation. Let M be a configuration space and $\varphi: R \ni t \to \varphi(t) \in M$ stands for a motion of a certain material system. For an arbitrary TM-constraint $\mathscr{C}_A$ we interpret $DA(t)$ as a set of all configurations which are „admissible" by constraint $\mathscr{C}_A$ at a time instant $t \in R$. At the same time $A(m, t)$ is a set of all velocities which are „admissible" by constraint $\mathscr{C}_A$ at a configuration $m \in DA(t)$. Every motion φ is „admissible" by constrains if and only if $\varphi \in \Phi_A(R, M)$; we can here assume that $\Phi(R, M) = D^1(R, M)$. Elements of $K_{A(m,t)}(v)$ now play the role of what can be called „virtual displacements". The cones of „virtual displacements" have been introduced only in order to define conjugate cones $K^*_{A(m,t)}(v)$ in $T_m^* M$, $m \in M$, which are called the reaction cones. Elements of $K^*_{A(m,t)}(v)$, for a certain motion $\varphi \in \Phi_A(R, M)$ and for $m = \varphi(t)$, $v = \dot{\varphi}(t)$, can be interpreted as reactions due to the constraints, which can act on the moving material system under consideration at the time instant t (at the configuration $\varphi(t)$ and the generalized velocity $v = \dot{\varphi}(t)$). Hence we see that now elements of $T_m^* M$ can be interpreted as certain generalized forces which can act on the moving system in its configuration $m = \varphi(t)$.

To complete the Section we discuss different cases of TM-constraints from the point of view of reactions. TM-constraint $\mathscr{C} = \mathscr{C}_A$ will be called taut or stretched at $t \in R$, $m \in DA(t)$ and for $v \in A(m, t)$, if and only if $\{0\}$ is a proper subset of the reaction cone $K^*_{A(m,t)}(v)$; otherwise the relation $K^*_{A(m,t)}(v) = \{0\}$ holds and $\mathscr{C}_A$ will be called untaut or unstretched at $t \in R$, $m \in DA(t)$ and for $v \in A(m, t)$. Define

$$K^*(m, t) \equiv \bigcup_{v \in A(m,t)} K^*_{A(m,t)}(v).$$

TM-constraint $\mathscr{C}_A$ will be termed reactive at $t \in R, m \in DA(t)$, if $K^*(m, t) \supset\supset \{0\}$, and will be termed unreactive if $K^*(m, t) = \{0\}$. TM-constraint $\mathscr{C}_A$ will be called nonreactive if it is nonreactive for every $t \in R$ and every $m \in DA(t)$; otherwise it will be called reactive.

Corollary. TM-constraints are nonreactive if $\tau_M^{-1}(\tau_M \mathscr{C}(t)) = \mathscr{C}(t)$ for every $t \in R$ or if $A(m, t) = T_m M$ for every $m \in DA(t)$ and every $t \in R$.

It must be emphasized that the term „constraint" in the known terminology of mechanics is reserved, as a rule, for situations in which TM-constraint is reactive.

Let TM-constraint $\mathscr{C}_A$ at $t \in R, m \in DA(t)$ and for $v \in A(m, t)$ be taut. Then the constraint will be called:

1° bilateral if $K^*_{A(m,t)}(v)$ is a linear subspace of $T^*_m M$ and $K^*_{A(m,t)}(v) \neq \{0\}$,

2° unilateral if $K^*_{A(m,t)}(v)$ does not contain any linear subspace of $T^*_m M$ different from $\{0\}$ and $K^*_{A(m,t)}(v) \neq \{0\}$,

3° combined if $K^*_{A(m,t)}(v)$ is not a linear subspace of $T^*_m M$ but contains such a subspace different from $\{0\}$.

This terminology is based on the terminology used in analytical mechanics. In mechanics we also deal with what are called „ideal" constraints, in which the total work of the reaction of these constraints on any virtual displacement is equal to zero. This case in a general approach given in the paper is represented by the condition: $\langle w, v^* \rangle = 0$ for every $w \in K_{A(m,t)}(v)$, where $v^* \in K^*_{A(m,t)}(v)$ and $v \in A(m, t)$. Hence it follows that the ideal constraint coincides with the bilateral constraint in the sense which was introduced above. However, from the point of view of applications of the theory developed in the paper, it is better to introduce the term „ideal constraints" as describing all situations in which „constraints" of the form: $v \in A(m, t), m \in DA(t)$, are „maintained" by the reactions $v^* \in K^*_{A(m,t)}(v)$. Thus the theory we are to develop can be treated as the theory of „ideal" constraints.

2. Constitutive relations

Time evolutions of material systems in mechanics are described by what are called dynamical processes; throughout the paper we shall confine ourselves to the processes represented by pairs of functions $(\varphi, \psi) \in \Phi(R, M) \times \Psi(R, T^*M)$, such that $\psi(t) \in T^*_{\varphi(t)} M$ for every $t \in R$, where $\Psi(R, T^*M)$ is the known functional (topological) space of functions defined a.e. on R and with values in T^*M. Thus every dynamical process under consideration, will be represented by

$$R \ni t \to (\varphi(t), \psi(t)) \in M \times T^*M, \tag{2.1}$$

where $\varphi \in \Phi(R, M)$, $\psi \in \Psi(R, T^*M)$ and $\psi(t) \in T^*_{\varphi(t)} M$ for a.e. $t \in R$. Moreover, in problems under consideration, every process (2.1) has to satisfy certain relation $\varrho \subset \Phi(R, M) \times \Psi(R, T^*M)$ which, roughly speaking, characterizes properties of the physical object or phenomenon which is analysed in this problem. Thus we shall assume that

$$(\varphi, \psi) \in \varrho \subset \Phi(R, M) \times \Psi(R, T^*M), \tag{2.2}$$

and refer ϱ to as the constitutive relation. To be more exact, to every material system we shall assign certain set $\mathscr{R}$ of constitutive relations of the form (2.2)[4] such that:

1° Every $\varrho \in \mathscr{R}$ is either the internal constitutive relation, i.e., it describes the „material" properties of the system (i.e., all these properties which are independent of any external field), or the external constitutive relation, describing the interaction between the system and its exterior.

2° Every $\varrho \in \mathscr{R}$ satisfies the principle of determinism, i.e., for every $\varrho \in \mathscr{R}$ there exist relations

$$\eta_t \subset \Phi(\bar{R}_+, M) \times \Psi(\bar{R}_+, T^*M), \qquad t \in R,$$

such that $(\varphi, \psi) \in \varrho$ if and only if $(\varphi^{(t)}, \psi^{(t)}) \in \eta_t$ for almost every $t \in R$, where $\varphi^{(t)}(s) \equiv \equiv \varphi(t-s)$, $\psi^{(t)}(s) \equiv \psi(t-s)$, $s \geqslant 0$. If ϱ is an internal relation then, as a rule, η_t is assumed to be constant for every $t \in R$.

Remark 1. Constitutive relation, apart from $\varphi \in \Phi(R, M)$ and $\psi \in \Psi(R, T^*M)$, can also involve elements δ of a certain set Δ which is not specified here. To take this fact into account we shall tacitly assume that $\varrho = \varrho_\delta$ for some $\delta \in \Delta$. Thus the constitutive relation involving δ will be represented not by a single relation ϱ but rather by a family ϱ_δ, $\delta \in \Delta$, of such relations.

Remark 2. The term „constitutive relation" is usually restricted to the description of material properties only of the system under consideration. Throughout the paper the constitutive relations are not restricted to relations describing internal properties of bodies (as internal constitutive relations) but also describe interactions between the body and external fields or objects (external constitutive relations).

In the sequel we shall deal only with what will be called TM-constrained constitutive relations.

Definition 3. Constitutive relation $\varrho \subset \Phi(R, M) \times \Psi(R, T^*M)$ will be termed TM-constrained if and only if there exists TM-constraint $\mathscr{C} = \mathscr{C}_A$ (here A is a constraint multifunction), such that dom $\varrho = \Phi_A(R, M)$, where $\Phi_A(R, M)$ is a nonempty subset of $\Phi(R, M)$ given by Eq. (1.7).

The foregoing definition yields an interrelation between the concept of a constraint and that of a constitutive relation (internal or external). From now on by a constitutive relation we shall mean TM-constrained constitutive relation, including also the trivial case in which $\Phi_A(R, M) = \Phi(R, M)$, i.e., in which $A(m, t) = T_m M$ for every $m \in M$, $t \in R$.

Now the question arises what restrictions have to be imposed on the form of constitutive relations due to the existence of constrains. To answer this question we shall formulate the following:

Principle of Constraints. Every TM-constrained constitutive relation $\varrho \subset \Phi(R, M) \times \times \Psi(R, T^*M)$, dom$\varrho = \Phi_A(R, M)$, has to satisfy the condition

$$(\forall \varphi \in \Phi_A(R, M))(\forall r \in R_A(\varphi))[[(\varphi, \psi) \in \varrho] \Rightarrow [(\varphi, \psi \pm r) \in \varrho]], \tag{2.3}$$

where we have denoted

$$R_A(\varphi) := \{r \in \Psi(R, T^*M) | r(t) \in K^*_{A(\varphi(t), t)}(\varphi'(t)) \quad \text{for a.e.} \quad t \in R\}, \tag{2.4}$$

---

[4] For different constitutive relations sets M, $\Phi(R, M)$, $\Psi(R, T^*M)$ can be different.

and where the sign „+" (the sign „−") has to be used if ϱ is an external (an internal) constitutive relation.

The principle of constraints emphasizes the formal difference between external and internal constitutive relations; rougly speaking, the external constitutive relation is „unsusceptible" on the reaction $r \in R_A(\varphi)$ to constraints $\mathscr{C}_A$, while the internal constitutive relation is „unsusceptible" on any „action" on constraints $-r(t)$, $t \in R$, where $r \in R_A(\varphi)$. Hence, from a purely formal point of view, to every external constitutive relation $(\varphi, \psi) \in \varrho$ we can uniquely assign the internal constitutive relation $\tilde{\varrho}$, putting $(\varphi, \tilde{\psi}) \in \tilde{\varrho}$ iff $(\varphi, -\tilde{\psi}) \in \varrho$, i.e., replacing function ψ by a function $\tilde{\psi}$.

To discuss the consequences of the principle of constraints let us introduce the multioperator

$$Mr\varrho : \Phi(R, M) \to 2^{\Psi(R, T^*M)},$$

putting

$$Mr\varrho(\varphi) := \{\psi \in \Psi(R, T^*M) | (\varphi, \psi) \in \varrho\}.$$

It follows that for every $r \in R_A(\varphi)$ we obtain $\psi \pm r \in Mr\varrho(\varphi)$ provided that $\psi \in Mr\varrho(\varphi)$, where the sign „+" (the sign „−") is related to the external (the internal) constitutive relation. Introducing now an arbitrary multifunction

$$\tilde{E} : \Phi(R, M) \to 2^{\Psi(R, T^*M)}, \tag{2.5}$$

such that

$$\operatorname{dom} \tilde{E} := \{\varphi \in \Phi(R, M) | \tilde{E}(\varphi) \neq \phi\} = \Phi_A(R, M), \tag{2.6}$$

we obtain $Mr\varrho(\varphi) = \tilde{E}(\varphi) \pm R_A(\varphi)$ and arrive at the following form of TM-constrained constitutive relation

$$\psi \in \tilde{E}(\varphi) \pm R_A(\varphi), \quad \varphi \in \Phi_A(R, M), \tag{2.7}$$

where the sign „+" and „−" are related to the case in which we deal with an external or internal constitutive relation, respectively. Mind, that relation (2.7), in which $\tilde{E}(\cdot)$ is an arbitrary multifunction (2.5) satisfying Eq. (2.6), fulfils identically the principle of constraints.

Using the principle of determinism, mentioned above, we assume that there exist the multifunctions

$$(\varphi^{(t)}, \varphi'(t)) \to E_t(\varphi^{(t)}, \varphi'(t)) \subset T^*_m M, \quad m \in M, \quad t \in R, \tag{2.8}$$

such that $\varphi(t) = m$ and

$$\psi(t) \in E_t(\varphi^{(t)}, \varphi'(t)) \pm r(t) \quad \text{for some} \quad r \in R_A(\varphi), \quad t \in R.$$

Taking into account Eqs. (2.4), (1.7) we obtain finally the following general form of TM-constrained external constitutive relation[5]

$$\begin{aligned} &\psi(t) \in E_t(\varphi^{(t)}, \varphi'(t)) + K^*_{A(\varphi(t), t)}(\varphi'(t)), \\ &\varphi(t) \in DA(t), \quad \varphi'(t) \in A(\varphi(t), t); \quad t \in R. \end{aligned} \tag{2.9}$$

Moreover, for internal constitutive relations, the subsets $E_t(\varphi^{(t)}, \varphi'(t))$ of $T^*_{\varphi(t)}M$, for an arbitrary but fixed history $\varphi^{(t)}$, are time independent. For such relations we also assume that

[5] Mind, that $E_t(\varphi^{(t)}, \varphi'(t))$ is a subset of $T^*_{\varphi(t)}M$.

the time does not enter the constraint: $A(m, t) = A(m)$, $t \in R$. Thus the general form of TM-constrained internal constitutive relation is given by

$$\begin{aligned} &\psi(t) \in E(\varphi^{(t)}, \varphi'(t)) - K^*_{A(\varphi(t))}(\varphi'(t)), \\ &\varphi(t) \in DA, \quad \varphi'(t) \in A(\varphi(t)); \quad t \in R, \end{aligned} \tag{2.10}$$

where

$$(\varphi^{(t)}, \varphi'(t)) \to E(\varphi^{(t)}, \varphi'(t)) \subset T^*_m M, \quad m = \varphi(t),$$

constitutes a special case of a multioperator (2.8). Summing up, we formulate the following

Proposition. Every TM-constrained external constitutive relation $(\varphi, \psi) \in \varrho \subset \Phi(R, M) \times \Psi(R, T^*M)$ has a form (2.9) in which $A: M \times R \ni (m, t) \to A(m, t) \subset TM$ is TM-constraint multifunction and $E_t(\cdot)$ are multioperators such that $E_t(\varphi^{(t)}, \varphi'(t)) \neq \phi$ if $\varphi^{(t)} \in \Phi(\bar{R}_+, M)$ and $\varphi(t-s) \in DA(t-s)$, $\varphi'(t-s) \in A(\varphi(t-s), t-s)$ for every $t \in R$ and $s \geqslant 0$. Every TM-constrained internal constitutive relation $(\varphi, \psi) \in \varrho \subset \Phi(R, M) \times \Psi(R, T^*M)$ has a form (2.10), in which $A: M \ni m \to A(m) \subset TM$ is TM-constraint (time independent) multifunction and $E(\cdot)$ is a multioperator such that $E(\varphi^{(t)}, \varphi'(t)) \neq \phi$ if $\varphi^{(t)} \in \Phi(\bar{R}_+, M)$ and $\varphi(t-s) \in DA$, $\varphi'(t-s) \in A(\varphi(t-s))$ for every $t \in R$ and $s \geqslant 0$.

Conclusion 1. If for a certain TM-constrained constitutive relation the suitable TM-constraints are nonreactive, then the principle of constraints is satisfied identically. In this case TM-constrained external constitutive relation is given by

$$\psi(t) \in E_t(\varphi^{(t)}, \varphi'(t)), \quad \varphi(t) \in DA(t), \quad \varphi'(t) \in A(\varphi(t), t),$$

for a.e. $t \in R$, and TM-constrained internal constitutive relation has a form

$$\psi(t) \in E(\varphi^{(t)}, \varphi'(t)), \quad \varphi(t) \in DA, \quad \varphi'(t) \in A(\varphi(t)),$$

for a.e. $t \in R$.

Conclusion 2. If TM-constrained constitutive relation $\varrho \in \Phi(R, M) \times \Psi(R, T^*M)$ is a functional relation (defined on the subset $\Phi_A(R, M)$ of $\Phi(R, M)$, i.e., if $\psi = \varrho\varphi$, $\varphi \in \Phi_A(R, M)$), then TM-constraints $\mathscr{C}_A$ are unreactive.

Example of interpretation. To illustrate the foregoing analysis we can assume that M is a space of all 3×3 symmetric matrices and that DA is a subset of all positive definite matrices representing the values $\varphi(t)$ of the Cauchy-Green deformation tensor at an arbitrary time instant. Moreover, let every $\psi(t)$ be treated as the value of the second Piola-Kirchhoff stress tensor. Then Eqs. (2.10) represent constraints for deformations and Eq. $(2.10)_1$ stands for a suitable stress-strain relations.

Remark 3. Conditions $(2.9)_2$, $(2.10)_2$ are implied by conditions $(2.9)_3$, $(2.10)_3$, respectively, since

$$DA(t) := \{m \in M | A(m, t) \neq \phi\},$$

$$DA := \{m \in M | A(m) \neq \phi\}.$$

Remark 4. The requirements formulated in the foregoing proposition represent only necessary conditions imposed on constrained constitutive relations. The sufficient conditions can be formulated only for some special classes of constitutive relations.

At the end of the Section we shall formulate some alternative forms of TM-constrained constitutive relations for the case in which

$$[E_t(\varphi^{(t)}, \varphi'(t)) \neq \phi] \Rightarrow [E_t(\varphi^{(t)}, \varphi'(t)) = \{F_t(\varphi^{(t)}, \varphi'(t))\}],$$
$$[E(\varphi^{(t)}, \varphi'(t)) \neq \phi] \Rightarrow [E(\varphi^{(t)}, \varphi'(t)) = \{F(\varphi^{(t)}, \varphi'(t))\}],$$

for every $t \in R$ i.e., in which $E_t(\varphi^{(t)}, \varphi'(t))$, $E(\varphi^{(t)}, \varphi'(t))$ are singletons or empty sets only. Combining together external and internal relations, we obtain from Eqs. (2.9), 2.10)

$$\begin{gathered} \psi(t) = F_t(\varphi^{(t)}, \varphi'(t)) \pm r(t), \qquad r(t) \in K^*_{A(\varphi(t), t)}(\varphi'(t)), \\ \varphi(t) \in DA(t), \qquad \varphi'(t) \in A(\varphi(t), t); \qquad t \in R, \end{gathered} \tag{2.11}$$

where for internal relations the sign „−" has to be taken into account and F_t, $DA(t)$, $A(\varphi(t), t)$ have to be replaced by F, DA and $A(\varphi(t))$, respectively. Analogously, we also obtain

$$\begin{gathered} \psi(t) = F_t(\varphi^{(t)}, \varphi'(t)) - r(t), \\ \langle w, r(t) \rangle \geqslant 0, \qquad w \in K_{A(\varphi(t), t)}(\varphi'(t)), \\ \varphi(t) \in DA(t), \qquad \varphi'(t) \in A(\varphi(t), t); \qquad t \in R, \end{gathered} \tag{2.12}$$

for an internal constitutive relation. If TM-constraints in constitutive relations (2.11) - (2.14) are nonreactive then we obtain $\psi(t) = F_t(\varphi^{(t)}, \varphi'(t))$ or $\psi(t) = F(\varphi^{(t)}, \varphi'(t))$ for the external or internal TM-constrained constitutive relation, respectively.

3. The method of constraints

The principle of constraints postulated in Sec. 2 makes it possible to formulate an approach leading from the known constitutive relation $\varrho \subset \Phi(R, M) \times \Psi(R, T^*M)$, to a new relation $\varrho_B \subset \Phi(R, M) \times \Psi(R, T^*M)$, where $B: M \times R \ni (m, t) \to B(m, t) \subset TM$ is a certain TM-constraint multifunction. The general idea of this approach is based, rougly speaking, on the imposing TM-constraint $\mathscr{C}_B$ on the relation ϱ. The approach outlined below will be referred to as the method of constraints and can be treated as a certain generalization of the method of internal constraints, [3].

We start from the known TM-constrained constitutive relation ϱ which will be given by

$$\psi \in Mr\varrho(\varphi), \tag{3.1}$$

with $\operatorname{dom}\varrho = \Phi_A(R, M)$ and where $A: M \times R \ni (m, t) \to A(m, t) \subset TM$ is the known TM-constraint multifunction. Putting

$$\mathscr{C}_A(t) = \bigcup_{m \in M} A(m, t), \qquad t \in R,$$

we obtain TM-constraint $\mathscr{C} = \mathscr{C}_A$.

Now assume that there is known the TM-constraint multifunction $B: M \times R \ni (m, t) \to$ $\to B(m, t) \subset TM$. This multifunction, for every $t \in R$, determines the non-empty subset $\mathscr{C}_B$ of TM:

$$\mathscr{C}_B(t) = \bigcup_{m \in M} B(m, t), \qquad t \in R.$$

Let us also assume that the conditions

$$\mathscr{C}_A(t)\cap\mathscr{C}_B(t) \neq \phi,$$
$$\mathscr{C}_A(t)\cap\mathscr{C}_B(t) \text{ is closed in } \mathscr{C}_A(t), \tag{3.2}$$

hold for every $t \in R$, and $\mathscr{C}_A(t)\cap\mathscr{C}_B(t) \subset\subset\mathscr{C}_A(t)$ for some $t \in R$. Define the relation $\varrho|_B \subset \Phi(R, M)\times\Psi(R, T^*M)$, putting

$$(\varphi, \psi) \in \varrho|_B \Leftrightarrow [(\varphi, \psi) \in \varrho] \wedge [\varphi \in \Phi_B(R, M)]. \tag{3.3}$$

Relation $\varrho/_B$ is not empty and may be not *TM*-constrained constitutive relation since it may not satisfy the principle of constraints.

Taking into account Eq. (3.3) we shall define the new relation $\varrho_B \subset \Phi(R, M)\times \times\Psi(R, T^*M)$ by means of

$$(\varphi, \psi) \in \varrho_B \Leftrightarrow \big(\exists r \in R_B(\varphi)\big)[(\varphi, \psi\pm r) \in \varrho|_B], \tag{3.4}$$

where we use the sign „+" if ϱ is the internal relation and the sign „−" if ϱ is the external relation. Introducing the multioperator

$$Mr\varrho|_B(\varphi) := \{\psi \in \Psi(R, T^*M)|(\varphi, \psi) \in \varrho|_B\},$$

we obtain from Eq. (3.4) that

$$\psi\pm r \in Mr\varrho|_B(\varphi) \quad \text{for some} \quad r \in R_B(\varphi)$$

with the same meaning of sign as in Eq. (3.4).
Thus we conclude that $(\varphi, \psi) \in \varrho_B$ if and only if

$$\psi \in Mr\varrho|_B(\varphi)\pm R_B(\varphi), \tag{3.5}$$

where now the sign „+" (the sign „−") is valid if ϱ is the external (the internal) constitutive relation. By virtue of Eqs. (2.7), (3.2)÷(3.5) we can formulate now the following

Conclusion. Relation ϱ_B, obtained from *TM*-constrained constitutive relation ϱ by means of Eqs. (3.3), (3.4), is *TM*-constrained (constitutive) relation with reacting *TM*-constraint $\mathscr{C} = \mathscr{C}_B$[6].

The procedure leading from *TM*-constrained constitutive relation $\varrho \subset \Phi(R, M)\times \times\Psi(R, T^*M)$ to *TM*-constrainted constitutive relation $\varrho_B \subset \Phi(R, M)\times\Psi(R, T^*M)$ will be called the method of constraints. Roughly speaking, the relation ϱ_B has been obtained by imposing *TM*-constraints $\mathscr{C}_B$ on the relation ϱ.

Now taking into account Eqs. (2.9) and applying to Eq. (3.5) the procedure analogous to that leading from Eq. (2.7) to Eqs. (2.9), we obtain

$$\begin{gathered}\psi(t) \in E_t\big(\varphi^{(t)}, \varphi'(t)\big)|_B+K^*_{A(\varphi(t),t)}\big(\varphi'(t)\big)+K^*_{B(\varphi(t),t)}\big(\varphi'(t)\big),\\ \varphi(t) \in DA(t)\cap DB(t), \quad \varphi'(t) \in A\big(\varphi(t), t\big)\cap B\big(\varphi(t), t\big),\end{gathered} \tag{3.6}$$

for $t \in R$. Eqs. (3.6) represent an external *TM*-constrained constitutive relation ϱ_B; here multioperator $E_t(\cdot)/_B$ is obtained from $E_t(\cdot)$ by restricting its domain only to such

[6] We can only assume that ϱ_B is the constitutive relation if ϱ is such a relation. In fact ϱ_B satisfies only sufficient conditions of being constitutive relation, formulated in Sec. 2.

$\varphi \in \Phi(R, M)$ which satisfy Eqs. $(3.6)_{2,3}$. Analogously, taking into account Eqs. (2.10), (3.5) we arrive at

$$\begin{aligned} &\psi(t) \in E(\varphi^{(t)}, \varphi'(t))|_B - K^*_{A(\varphi(t),t)}(\varphi'(t)) - K^*_{B(\varphi(t),t)}(\varphi'(t)), \\ &\varphi(t) \in DA \cap DB, \quad \varphi'(t) \in A(\varphi(t)) \cap B(\varphi(t)), \end{aligned} \tag{3.7}$$

for $t \in R$. Eqs. (3.7) represent an internal TM-constrained constitutive relation ϱ_B.

Summing up, we conclude that the method of constraints leads from constitutive relations (2.9) and (2.10) to constitutive relations (3.6) and (3.7), respectively, Let us also observe, that to TM-constrained relation ϱ_B are assigned TM-constraints $\mathscr{C}_{A \cap B}$, given by $\mathscr{C}_{A \cap B}(t) \equiv \mathscr{C}_A(t) \cap \mathscr{C}_B(t)$, $t \in R$, where $A \cap B$ stands here for TM-constraint multifunction defined by

$$(A \cap B)(m, t) \equiv A(m, t) \cap B(m, t), \quad (m, t) \in M \times R.$$

The foregoing multifunction also enables to rewrite Eqs. $(3.6)_1$, $(3.7)_1$ to more compact form corresponding to that of Eqs. $(2.9)_1$, $(2.10)_1$, respectively.

4. Special cases of constraints

So far we have analysed TM-constraints $\mathscr{C}$ in which the subsets $\mathscr{C}(t)$ of TM were restricted exclusively by condition (1.4). In this Section we are to define and to discuss more special cases of constraints which are often encountered in different problems of mechanics.

To begin with we shall introduce the important concept of what are called holonomic constraints. Roughly speaking, by holonomic TM-constraint we shall mean the constraint $\mathscr{C}$ in which for every $t \in R$ all subsets $A(m, t) = \mathscr{C}(t) \cap T_m M$ are uniquely defined by means of a certain non-empty subset $H(t)$ of M, $t \in R$. To be more exact, let us assume that there is known the multifunction

$$H: R \ni t \to H(t) \subset M \tag{4.1}$$

and define for every $t \in R$, $m \in M$, the subsets $F_H(m, t)$ of $C^1(R, M)$, given by

$$\begin{aligned} F_H(m, t) := \{f \in C^1(R, M) | f(t) = m, \quad & f(t+\lambda) \in H(t+\lambda) \\ \text{for} \quad \lambda \in [0, \varepsilon) \quad \text{and some} \quad & \varepsilon > 0\}. \end{aligned}$$

For every $m \in M \setminus H$ there is $F_H(m, t) \neq \phi$, $t \in R$.

Definition 4. TM-constraint $\mathscr{C} = \mathscr{C}_A$ will be called holonomic if and only if constraint multifunction $A(\cdot)$ is defined by

$$A(m, t) := \{v \in T_m M | v = f'(t) \quad \text{for some} \quad f \in F_H(m, t)\}, \tag{4.2}$$

where $H: R \ni t \to H(t) \subset M$ is a multifunction satisfying the condition

$$(\forall t \in R)(\forall m \in H(t))[F_H(m, t) \neq \phi]. \tag{4.3}$$

TM-constraint $\mathscr{C}$ will be called scleronomic if and only if $\mathscr{C}(t)$ is constant for every $t \in R$; otherwise they will be called rheonomic. TM-constraint will be called holonomic — scleronomic if it is both holonomic and scleronomic.

Conclusion. Holonomic — scleronomic TM-constraint is uniquely determined by an arbitrary non-empty subset H of M.

Proposition 1. The holonomic-scleronomic TM-constraint multifunction is given by

$$A(m, t) = A(m) = T_H(m), \quad t \in R, \quad m \in M, \tag{4.4}$$

where H is an arbitrary non-empty subset of M and $T_H(m)$ is a cone of all directions tangent to H at m (empty if $m \in M \setminus H$):

$$T_H(m) := \{v \in T_m M | v = g'(0), \quad m = g(0), \quad g(\lambda) \in H$$
$$\text{for every } \lambda \in [0, \varepsilon), \varepsilon > 0, \quad \text{and some } g \in \mathscr{C}^1((-\varepsilon, \varepsilon), M)\}.$$

Eq. (4.4) can be obtained from Eq. (4.2) and from a definition of a set $F_H(m, t)$, taking into account that $H(t) = H$ for every $t \in R$. It must be emphasized that in general no regularity of any kind has to be imposed on the non-empty subset H of M, which uniquely determines holonomic-scleronomic constraint.

Corollary 1. If H is a differentiable submanifold of M determining holonomic-scleronomic constraint $\mathscr{C}$ then $\mathscr{C} = TH$ and $\tau_H^{-1}(\tau_H \mathscr{C}) = \mathscr{C}$.

From now on we are to deal exclusively with holonomic-scleronomic TM-constraints.

Proposition 2. If for some $m \in H$ the cone $A(m) = T_H(m)$ is convex in $T_m M$, then the reaction cones $K^*_{A(m)}(v)$, $v \in A(m) = T_H(m)$, are determined by

$$K^*_{A(m)}(v) := \{v^* \in T^*_m M | \langle u, v^* \rangle \geqslant \langle v, v^* \rangle \quad \text{for every} \quad u \in T_H(m)\}. \tag{4.5}$$

In order to prove the foregoing proposition let us observe that for every $v \in A(m) = T_H(m)$, where $A(m)$ is convex in $T_m M$, we obtain (cf. Eq. (1.9)):

$$K_{A(m)}(v) = \mathrm{con}[T_H(m) - v],$$

where we have used the known denotation

$$\mathrm{con}\Omega := \{x \in V | x = \lambda \bar{x}, \bar{x} \in \Omega, \lambda \geqslant 0\},$$

for an arbitrary subset Ω in a vector space V. Now taking into account Eq. (1.10) we also conclude that $v^* \in K^*_{A(m)}(v)$ if and only if

$$\langle w, v^* \rangle \geqslant 0 \quad \text{for every} \quad w \in \mathrm{con}[T_H(m) - v],$$

i.e., $K^*_{A(m)}(v) = \mathrm{con}^*[T_H(m) - v]$, $A(m) = T_H(m)$, where $\mathrm{con}^*\Omega$ stands for a closed cone conjugate to $\mathrm{con}\,\Omega$. The ultimate condition leads directly to Eq. (4.5).

Corollary 2. Under the assumptions of Proposition 2 the following equality

$$\langle v, v^* \rangle = 0, \quad v^* \in K^*_{A(m)}(v) \tag{4.6}$$

holds for every $v \in A(m)$. Hence

$$v^* \in K^*_{A(m)}(v) \quad \text{if} \quad \langle v, v^* \rangle = 0 \quad \text{and} \quad \langle u, v^* \rangle \geqslant 0 \quad \text{for every} \quad u \in A(m), \tag{4.7}$$

holds for every $v \in A(m) = T_H(m)$.

Equality (4.6) can be obtained from Eq. (4.5) by substituting $u = kv$ with $k \geqslant 0$. Then $(k-1)\langle v, v^* \rangle \geqslant 0$ for every $k \geqslant 0$ and hence we arrive at Eq. (4.6).

Now assume that M is (finite dimensional) linear space and H is convex in M. Then

$$A(m) = T_H(m) = \overline{\mathrm{con}}(H - m)$$

for every $m \in H$, where $\overline{\mathrm{con}}(\cdot)$ stands for a closure of $\mathrm{con}(\cdot)$ in M. Taking into account Eq. (4.7) we arrive at the following final

Conclusion. If a non-empty set H is convex in a finite dimensional linear space M and H determines holonomic-scleronomic TM-constraints, then

$$\begin{aligned} \langle u, v^*\rangle &\geqslant \langle m, v^*\rangle \quad \text{for every} \quad u \in H, \\ \langle v, v^*\rangle &= 0, \\ v \in \overline{\text{con}}&(H-m), \quad m \in H, \end{aligned} \tag{4.8}$$

if and only if $v^* \in K^*_{A(m)}(v)$, where $A(m) = \overline{\text{con}}(H-m)$.

From now on we shall confine ourselves to holonomic-scleronomic TM-constraints in which M is a finite dimensional linear space, $M = R^n$, and TM-constraints are determined by a non-empty convex subset H of M. Let us take into account constitutive relations given by Eqs. (2.11) or (2.12). Combining together Eqs. (2.11) and (4.8) we arrive at the following TM-constrained constitutive relations

$$\begin{aligned} \psi(t) &= F_t(\varphi^{(t)}, \varphi'(t)) \pm r(t), \\ \langle u, r(t)\rangle &\geqslant \langle \varphi(t), r(t)\rangle \quad \text{for every} \quad u \in H, \\ \langle \varphi'(t), r(t)\rangle &= 0, \\ \varphi(t) &\in H, \end{aligned}$$

for $t \in R$. Let us confine ourselves to the internal constitutive relations only, putting $F_t = F$ for every $t \in R$ and taking into account the sign „$-$" in the first from the foregoing relations. Let us also take into account Remark 1 of Sec. 2 and Remark 2 of Sec. 1, assuming that $F = F^\delta$, $\delta \in \Delta$ and $H = H_\xi$, $\xi \in \Xi$ (implicit constraints). Then we finally arrive at the following special form of TM-constrained constitutive relations[7]

$$\begin{aligned} \psi(t) &= F^\delta(\varphi^{(t)}, \varphi'(t)) - r(t), \quad \delta \in \Delta, \\ \langle u, r(t)\rangle &\geqslant \langle \varphi(t), r(t)\rangle \quad \text{for every} \quad u \in H_\xi, \\ \langle \varphi'(t), r(t)\rangle &= 0, \\ \varphi(t) &\in H_\xi, \quad \xi \in \Xi, \end{aligned} \tag{4.9}$$

which has to hold for $t \in R$ and where Δ, Ξ are the known sets. If Δ, Ξ are singletons then the indices δ, ξ, respectively, drop out from Eqs. (4.9).

TM-constrained internal constitutive relations (4.9) will be the basis in Sec. 5 for analysis of different special cases of internal constraints in different ideal materials.

5. Materials with constraints

Formulas (4.9) represent the abstract form of TM-constrained internal constitutive relations (with holonomic-scleronomic implicit constraints in which H_ξ is convex in $M = R^n$ for every $\xi \in \Xi$), i.e., the form which is independent of any special class of ideal materials. Interpretations of Eqs. (4.9) in mechanics (as well as interpretations of any other relation of Secs. 1 - 4) will be realized by assigning the physical meaning to elements of manifolds M and T^*M and to elements of sets Δ and Ξ (provided that they are not singletons). At the same time we shall specify the families of mappings F^δ and sets H_ξ.

---

[7] If $s \to \varphi(t+s)$ is differentiable in $(-\varepsilon, \varepsilon)$, then Eqs. $(4.9)_2$ imply Eq. $(4.9)_3$. Mind, that Eqs. (4.9) hold only if H_ξ is convex in $M = R^n$ for every $\xi \in \Xi$.

Let $M = R^\delta$ be interpreted as a space of all (symmetric) second Piola-Kirchhoff stress tensors and let every $T^*_m M = R^\delta$ be a space of all (symmetric) strain-rate tensors. Moreover, assume that Δ, Ξ are singletons (i.e. $F^\delta = F$, $H_\xi = H$) and

$$F(\varphi^{(t)}, \varphi'(t)) = L(\varphi(t))\varphi'(t), \tag{5.1}$$

where $L(m)$: $R^\delta \to R^\delta$ is the linear continuous operator (known for every $m \in M$). Introducing the denotations

$$\dot{e}(t) \equiv \psi(t), \quad \sigma(t) \equiv \varphi(t), \quad \dot{\sigma}(t) \equiv \varphi'(t), \quad \varepsilon(t) \equiv -r(t),$$

we shall rewrite Eqs. (4.9) to the form

$$\begin{gathered} \dot{e}(t) = L(\sigma(t))\dot{\sigma}(t) + \varepsilon(t), \\ \langle \tau, \varepsilon(t) \rangle \leqslant \langle \sigma(t), \varepsilon(t) \rangle \quad \text{for every} \quad \tau \in H, \\ \langle \dot{\sigma}(t), \varepsilon(t) \rangle = 0, \\ \sigma(t) \in H. \end{gathered} \tag{5.2}$$

Under the forementioned interpretation it can be observed that Eqs. (5.2) may represents constitutive relations of an arbitrary elastic-ideal plastic material provided that ∂H is the loading surface (yield surface) and $L(\sigma) = \partial^2\gamma(\sigma)/\partial\sigma^2$, $\sigma \in R^\delta$, where $\gamma(\cdot)$ is a potential characterizing a hyperelastic material. In this case Eq. $(5.2)_1$ are the Prandtl-Reuss equations with $\varepsilon(t)$ as a plastic and $L(\sigma(t))\dot{\sigma}(t)$ as an elastic parts of the strain rate tensor, respectively. At the same time formula $(5.2)_4$ includes the yield condition and formula $(5.2)_2$ represents Hill's principle of maximum plastic work[8]. Let us also observe that Eqs. (5.2) can be obtained from the constitutive functional relation

$$\dot{e}(t) = L(\sigma(t))\dot{\sigma}(t), \tag{5.3}$$

by the method of constraints. It can be seen that Eq. (5.3) is the constitutive relation of a certain rate-type material. Thus we shall arrive at the conclusion that the convex explicit constraints imposed on the constitutive relations of rate-type materials lead to the constitutive relations of ideal plastic materials. The character of yielding is uniquely determined by the subset H, i.e., it is due entirely to the effect of constraints.

Now let $M = R^\delta$ as above, but M be interpreted as a space of all (symmetric) strain tensors of the linear elasticity. Let every $T_m M = R^\delta$ be a space of all (symmetric) stress tensors. Let us also assume that

$$F(\varphi^{(t)}, \varphi'(t)) = L\varphi(t), \tag{5.3}$$

where $L: R^\delta \to R^\delta$ is the tensor of elastic moduli of the linear elasticity. Introducing the denotations

$$\sigma(t) \equiv \psi(t), \quad e(t) \equiv \varphi(t), \quad \tau(t) \equiv -r(t),$$

we rewrite Eqs. (4.9) (in the sequel we shall neglect Eq. $(4.9)_3$, cf. Footnotes 7) and 8) to the form

$$\begin{gathered} \sigma(t) = Le(t) + \tau(t), \\ \langle \varepsilon, \tau(t) \rangle \leqslant \langle e(t), \tau(t) \rangle \quad \text{for every} \quad \varepsilon \in H, \\ e(t) \in H. \end{gathered} \tag{5.4}$$

[8] This principle implies also Eq. $(5.2)_3$ provided that $t \to \sigma(t)$ is differentiable, cf. Footnote (7).

If $\text{int} H \neq \phi$ and $0 \in \text{int} H$, then Eqs. (5.4) can be interpreted as the constitutive relations of Prager's locking materials. Eqs. (5.4) can be also obtained from the linear stress-strain relation

$$\sigma(t) = Le(t) \tag{5.5}$$

by the method of constraints. Hence it follows that the constitutive relations of ideal locking materials can be obtained by imposing suitable constraints on stress-strain relations (5.5) of the linear elasticity theory.

Let $M = R^{\delta}$ be interpreted now as the space of all (symmetric) strain rate tensors and $T_m^* M = R^{\delta}$ be a space of all (symmetric) stress tensors. Let us also assume that Ξ is a non empty sub set in a space Δ of all right Cauchy-Green deformation tensors (strain tensors) Introducing the denotation

$$\sigma(t) \equiv \psi(t), \quad \dot{e}(t) \equiv \varphi(t), \quad e(t) \equiv \delta, \quad \tau(t) \equiv -r(t),$$

and assuming that

$$F(\varphi^{(t)}, \varphi'(t)) = F^{\delta}(\varphi^{(t)}, \varphi'(t)) = E(\delta), \tag{5.6}$$

where $E: R^{\delta} \to R^{\delta}$ is the known function [9],we obtain from Eqs. (4.9)

$$\begin{aligned} \sigma(t) &= E(e(t)) + \tau(t), \\ \langle \varepsilon, \tau(t) \rangle &\leqslant \langle \dot{e}(t), \tau(t) \rangle \quad \text{for every} \quad \varepsilon \in H_{e(t)}, \\ \dot{e}(t) &\in H_{e(t)}, \quad e(t) \in \Xi. \end{aligned} \tag{5.7}$$

The foregoing constitutive relations can be treated as obtained by the method of constraints from the constitutive relations

$$\sigma(t) = E(e(t)), \quad e(t) \in \Delta, \tag{5.8}$$

which can be postulated as stress relations of the non-linear elasticity; here Δ is the set of all symmetric strain tensors in the space R^{δ}. A set Ξ in Eqs. (5.7) can be not convex but has to be closed in Δ (but not in R^{δ})[10]. We shall also assume that

$$H_{e(t)} = T_{\Xi}(e(t)), \quad e(t) \in \Xi, \tag{5.9}$$

where $T_{\Xi}(e)$ is a convex cone of all directions tangent to Ξ at e, $e \in \Xi$ (cf. Sec. 1). Hence we see that Eqs. (5.7), (5.9) represent the constitutive relations of elastic materials with an arbitrary holonomic (scleronomic) internal constraints for the strain measures $e(t)$. Mind, that the form of these implicit constraints (cf. Remark 2 of Sec. 1) is rather general since no regularity conditions are imposed on the set Ξ apart from those that $T_{\Xi}(e)$ are convex for every $e \in \Xi$ and that Ξ is closed in the set Δ of all strain tensors.

If Ξ is a differentiable manifold embedded in R^{δ} then, by virtue of Eq. (5.8), every $H_{e(t)}$ is a linear subspace of R^{σ}. In this case we obtain

$$\begin{aligned} \sigma(t) &= E(e(t)) + \tau(t), \\ \langle \varepsilon, \tau(t) \rangle &= 0 \quad \text{for every} \quad \varepsilon \in T_{e(t)}\Xi, \\ & e(t) \in \Xi, \end{aligned} \tag{5.10}$$

[9] We have assumed here that $E(\cdot)$ is independent of the history $\varphi^{(t)}$ and the velocity $\varphi'(t)$.

[10] cf. the basic assumptions of the method of constraints in Sec. 3.

where

$$T_{e(t)}\Xi = T_{\Xi}(e(t))$$

is the space tangent to Ξ at $e(t)$, $e(t) \in \Xi$. Thus we have obtained the case of smooth bilateral internal constraints well known in the present literature.

Returning to the general case of holonomic constraints imposed on the stress relation of nonlinear elasticity (5.8), let us observe that the „maximum" principle $(5.7)_2$ can be represented by the formula (cf. Sec. 4)

$$\langle \varepsilon, \tau(t) \rangle \leqslant 0 \quad \text{for every} \quad \varepsilon \in \mathrm{con}[T_{\Xi}(e(t)) - \dot{e}(t)], \tag{5.11}$$

and hence

$$\begin{gathered} E(e(t)) - \sigma(t) \in \mathrm{con}^*[T_{\Xi}(e(t)) - \dot{e}(t)], \\ e(t) \in \Xi, \quad \dot{e}(t) \in T_{\Xi}(e(t)), \end{gathered} \tag{5.12}$$

where $\mathrm{con}^*[\cdot]$ is a cone conjugate to the cone $\mathrm{con}[\cdot]$. Eqs. (5.12) constitute an alternative form of Eqs. (5.7). From Eq. $(5.12)_1$ it follows that the elastic materials by imposing the constraints for deformations, in the general case, have lost their elastic properties; this is due to the fact that the „reaction" part $\tau(t)$ of the stress tensor can depend not only on the strain tensor $e(t)$ but also on the strain rate tensor $\dot{e}(t)$. Such situation does not take place for the smooth bilateral constraints since the strain rate tensor $\dot{e}(t)$ does not enter Eqs. (5.10).

Eqs. (5.6), (5.7), (5.10), (5.12) can be easily generalized. To this aid the assumption that Δ is a set of all strain tensors has to be replaced by the assumption that Δ is a set of all strain histories. In this case instead of Eqs. (5.7), (5.9) we obtain

$$\begin{gathered} \sigma(t) = E(e^{(t)}) + \tau(t), \\ \langle \varepsilon, \tau(t) \rangle \leqslant \langle \dot{e}(t), \tau(t) \rangle \quad \text{for every} \quad \varepsilon \in T_{\Xi}(e(t)), \\ e(t) \in \Xi, \quad \dot{e}(t) \in T_{\Xi}(e(t)). \end{gathered} \tag{5.13}$$

Hence we conclude that Eqs. (5.13) can be treated as a result of imposing constraints (determined by the TM-constraint multifunction $T_{\Xi}(e)$, $e \in \Xi$) on the constitutive relation of simple materials

$$\sigma(t) = E(e^{(t)}), \quad e^{(t)} \in \Delta, \quad t \in R. \tag{5.14}$$

The alternative form of Eqs. (5.13) will be obtained by substituting response functional $E(e^{(t)})$ in Eqs. (5.12) on the place of response function $E(e(t))$[11]. Moreover, if Ξ is a differentiable manifold we obtain the generalization of Eqs. (5.10) to the form

$$\begin{gathered} \sigma(t) = E(e^{(t)}) + \tau(t), \\ \langle \varepsilon, \tau(t) \rangle = 0 \quad \text{for every} \quad \varepsilon \in T_{e(t)}\Xi, \\ e(t) \in \Xi, \end{gathered}$$

which represents the well known constitutive relations of simple materials with smooth internal constraints.

To conclude the Section let us discuss the case in which $M = R^\sigma$ be interpreted as a space

[11] Symbol $E(\cdot)$ in Eqs. (5.13) stands for a response functional and in Eqs. (5.12) for a response function.

of all (symmetric) stress rate tensors. and $T^*M = R^\delta$ as the space of all (symetric) strain rate tensores. Moreover, let Ξ be the closed (but in general not convex) subset of R^δ. Then from Eqs. (4.9), under notation

$$\dot{e}(t) \equiv \psi(t), \quad \dot{\sigma}(t) \equiv \varphi(t), \quad \varepsilon(t) \equiv -r(t),$$

we obtain[12)]

$$\begin{gathered} \dot{e}(t) = L(\sigma(t))\dot{\sigma}(t) + \varepsilon(t), \\ \langle \tau, \varepsilon(t)\rangle \leqslant \langle \dot{\sigma}(t), \varepsilon(t)\rangle \quad \text{for every} \quad \tau \in H_{\sigma(t)}, \\ \sigma(t) \in \Xi, \quad \dot{\sigma}(t) \in H_{\sigma(t)}. \end{gathered}$$

Taking into account that every $H_{\sigma(t)}$ is a convex cone and putting

$$H_{\sigma(t)} = T_\Xi(\sigma(t)),$$

we arrive finally at the constitutive relations

$$\begin{gathered} \dot{e}(t) = L(\sigma(t))\dot{\sigma}(t) + \varepsilon(t), \\ \langle \tau, \varepsilon(t)\rangle \leqslant \langle \dot{\sigma}(t), \varepsilon(t)\rangle \quad \text{for every} \quad \tau \in T_\Xi(\sigma(t)), \\ \dot{\sigma}(t) \in T_\Xi(\sigma(t)), \quad \sigma(t) \in \Xi, \end{gathered} \tag{5.15}$$

which can be also written down in a form

$$\begin{gathered} E(\sigma(t))\dot{\sigma}(t) - \dot{e}(t) \in \text{con}^*[T_\Xi(\sigma(t)) - \dot{\sigma}(t)], \\ \sigma(t) \in \Xi, \quad \dot{\sigma}(t) \in T_\Xi(\sigma(t)). \end{gathered} \tag{5.16}$$

We deal here with the rate-type materials with the holonomic constraints (Ξ is closed in R^σ but not convex in general) for stresses. Assuming that $\Xi = H$, where H is convex, we arrive again at Eqs. (5.2) .Assuming that Ξ is a differentiable manifold in R^δ, we obtain

$$\begin{gathered} \dot{e}(t) = L(\sigma(t))\dot{\sigma}(t) + \varepsilon(t), \\ \langle \tau, \varepsilon(t)\rangle = 0 \quad \text{for every} \quad \tau \in T_{\sigma(t)}\Xi, \\ \sigma(t) \in \Xi, \quad \dot{\sigma}(t) \in T_{\sigma(t)}\Xi, \end{gathered} \tag{5.17}$$

where

$$T_{\sigma(t)}\Xi = T_\Xi(\sigma(t))$$

is a space tangent to Ξ at $\sigma(t) \in \Xi$. Thus Eqs. (5.2) and (5.17) constitute two different special cases of the constitutive relations (5.15) of the rate-type materials (of the hyperelastic materials if $L(\sigma) = \partial^2\gamma(\sigma)/\partial\sigma^2$) with holonomic (scleronomic) constraints for stresses.

Conclusions and final remarks.

Summing up we conclude that the abstract form (4.9) of TM-constrained internal constitutive relations (with convex implicit constraints) is an appropriate basis for obtaining constitutive internal relations for a large class of ideal materials. In this way we have obtained the known relations (5.13) for simple materials with internal constraints for deformations, the known relations (5.2) for elastic-ideal plastic materials and relations

12) Here $L(\sigma(t))$ has the same meaning as in Eq. (5.3).

(5.4) for ideal locking materials. Thus we have shown that the elastic-ideal plastic materials and ideal locking materials can be treated as the rate type materials with constraints for stresses and as the linear elastic materials with constraints for strains, respectively. We have also derived, using the method of constraints, new classes of ideal materials. They are simple materials with convex implicit constraints, defined by Eqs. (5.13), and rate type materials with convex implicit constraints, defined by Eqs. (5.15) or (5.16). The new classes of ideal materials, which have been obtained by the method of constraints, are also given by Eqs. (5.12) and Eqs. (5.17) (they are the subclasses of materials with internal constraints defined by Eqs. (5.13) and Eqs. (5.15), respectively).

Examples of applications of the general approach to the problem of constraints in constitutive relations of mechanics have been restricted here only to problems of ideal materials with internal constraints. However, it can be observed that the method of contraints is a useful tool of the formation of new constitutive relations of mechanics on the basis of the known constitutive relations. This method can be applied not only to the theory of ideal materials, i.e., to internal constitutive relations, but also to the problems of interactions between a body and its exterior, i.e., to the formation of external constitutive relations. The form of constraints which are described within an approach outlined in the paper is very general; as a matter of fact no restrictions of any kind are imposed on the sets of states[13)] which are admissible by constraints. Due to this fact certain new classes of constitutive internal relations have been obtained. More special classes of materials with internal constraints, obtained by the method of constraints, are discussed in [5]. Some applications of this method to the problems in structural mechanics will be given in forthcoming papers [6, 7].

References

1. J. Sławianowski, *Geometry of the Phase Space* [*Geometria przestrzeni fazowych*, in Polish [PWN, Warszawa 1965.
2. C. Truesdell; *A First Course in Rational Continuum Mechanics*, The Johns Hopkins University, Baltimore 1972.
3. E. Volterra, *Equations of motion for curved elastic bars by the use of the „method of internal constraints"*, Ing. Arch. 23, 402 - 409, (1955).
4. Cz. Woźniak, M. Kleiber, *Nonlinear Structural Mechanics* [*Nieliniowa mechanika konstrukcji*, in Polish], PWN, Warszawa 1982.
5. Cz. Woźniak, *Materials with generalized constraints*, Arch. Mech., 4/84.
6. S. Konieczny, Cz. Woźniak, *Generalized constraints in the plate theory*, Mech. Teor. i Stos., to appear.
7. A. Gałka, Cz. Woźniak, *Composite materials with unilateral constraints for deformations and stresses*, Mech. Teor. i Stos., to appear.

[13)] If $R \ni t \to \varphi(t) \in M$ is a mapping restricted by constraints, then $(\varphi(t), \varphi'(t))$, $\varphi'(t) \in T_{\varphi(t)}M$, is said to be a state at $t \in R$.

Р е з ю м е

СВЯЗИ В ОПРЕДЕЛЯЮЩИХ УРАВНЕНИЯХ МЕХАНИКИ

В статье представленный общий подход к понятию связей в определяющих уравнениях механики дискретных сплошных систем. Получено общий вид определяющих уравнений их связями и проведено метод формулирования новых определяющих уравнений при помощи связей. Таким способом получено некоторые новые классы материалов с внутренними связями.

S t r e s z c z e n i e

WIĘZY W PORÓWNANIACH KONSTYTUTYWNYCH MECHANIKI

W pracy przedstawiono ogólne podejście do pojęcia więzów w relacjach mechaniki układów dyskretnych i ośrodków ciągłych. Uzyskano ogólną postać równań konstytutywnych z więzami oraz zaproponowano metodę formułowania nowych relacji konstytutywnych za pomocą nakładania więzów na znane relacje konstytutywne. Na tej drodze otrzymano pewne nowe klasy materiałów idealnych z więzami wewnętrznymi

Praca zostala złożona w Redakcji dnia 1 lutego 1984 roku