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# NUMERICAL ANALYSIS OF LARGE DEFLECTION BEHAVIOUR OF ELASTIC-PLASTIC SHELLS OF REVOLUTION

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## 1. Introductory remarks and assumptions

Plastic deformations are admitted in many cases in order to exploit maximally the load carrying capacity of structures. Collapse of thin shells often occurs as a result of the instability before the large yielding zones are developed. This fact motivates the interest in analysis of the elastic-plastic behaviour of shell structures. A limit load of shells at which the snap-through phenomenon occurs, is investigated. This kind of critical load is typical for shallow ideal shells; in high shells such an instability occurs due to the initial imperfections.

Analysis of the lost of stability in the sense of snap-through and the post-critical behaviour is possible only on the basis of nonlinear formulation. Simultaneous considerations of the physical and geometrical nonlinearities complicate the problem, therefore the only way to achieve a solution is the application of numerical methods.

A number of papers on large deflections of elastic and elastic-plastic shells have been written. With respect to elastic-plastic shells different methods were applied. The multi--segment method of numerical integration was used in [1], finite differences were explored in [2] and finite element method was used in [3].

The presented paper is a generalization of the paper [4], where the shooting method was applied to the ideal sandwich cross-section. Comparing with the mentioned paper [4] a modyfication of the algorithm and improved subroutines have been given.

The following assumptions have been introduced:

1. The shell is thin (Kirchhoff-Love hypotheses are valid); constant thichness and ideal sandwich or full-walled cross-section, approximated by an equivalent multipoint cross-section are assumed.

2. The displacement field is rotationally symmetric.

3. The theory of small strains and large displacements is assumed.

4. The material of shall is isotropic, compressible, homogeneous, elastic-plastic of a general type of strain-hardening (yield surface can translate and/or extend at the same time).

5. The load is quasi-static.

In the present paper the semi-inverse method of numerical forward integration is applied for solving differential equations. This method (shooting method) changes the two-point boundary value problem (BVP) into an initial value problem (IVP).

Assuming rotational symmetry one can confine the considerations to a geometric one-dimensional problem with independent variable $\lambda$ measured along the meridian of the shell (Fig. 1a). Application of the plastic flow theory requires introduction of quasi--time variable $\tau$.

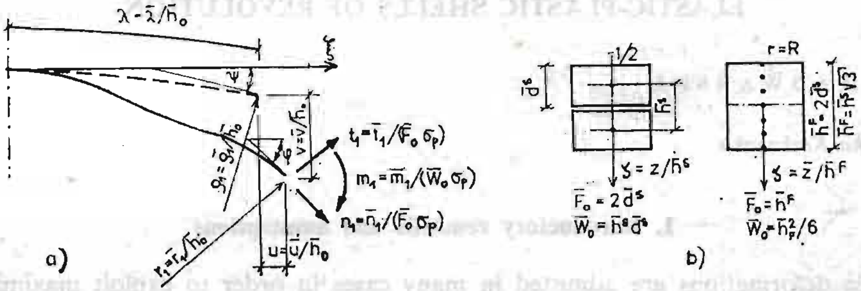


Fig. 1

From of a comprehensive numerical analysis only these examples will be mentioned in which the upper and lower critical loads for shallow spherical caps with various boundary conditions will be calculated.

## 2. Basic equations

Dimensionless quantities are used according to the notations in [4, 5]; components of the state vector $y$ are shown in Fig. 1a.

Axisymmetric equilibrium state of elastic-plastic shell is described by the nonlinear set of partial differential equations;

$$(1)\qquad \begin{aligned} y' &= f(y, p, \lambda), \\ \dot{Q} &= D(Q, q)\dot{q} \end{aligned}$$

and algebraic relations:

$$(2)\qquad z = h(w, \lambda),$$

where the following vectors are used: the state vector $y$, the vector of generalized stresses $Q$ and generalized strains $q$:

$$(3)\qquad \begin{aligned} y &= \{u, v, \varphi, n_1, t, m_1\}, \\ Q &= \{n_1, n_2, m_1, m_2\}, \qquad q = \{\vartheta_1, \vartheta_2, k_1, k_2\}. \end{aligned}$$

Additionally the vector displacement $w$ and strains $z$ are introduced:

$$(4)\qquad w = \{u, \varphi\}, \qquad z = \{\vartheta_2, k_2\},$$

which are not related to each other by differential relations.

The basic set of equations consists of the equilibrium equations, the geometrical and the physical relations. Detailed description of these equations for elastic-plastic shells of revolution is given in Appendix A2 in [5]. The initial boundary value problem must be supplemented with appropriate boundary and initial conditions.

The set of equations is separated with respect to the spatial $(\ )' = \partial(\ )/\partial\lambda$ and quasi--time derivatives $(\ )^{\cdot} = \partial(\ )/\partial\tau$. Incremental physical relations require inversion of physical equations at each step of numerical integration at nodes $l = 1, \ldots, L$.

In elastic-plastic shells the main problem lies in calculation of generalized stresses by numerical integration along the shell thickness. One of the possibilities to overcome difficulties is the application of the numerical integration concentrating shell properties at integration nodes. In this way we obtain full-walled cross-section, which will be called multipoint cross-section. Application of the Simpson quadrature formulae can correspond to the assumption of R-equally spaced points along thickness. As a special case we shell consider an ideal sandwich cross-section, when $R = 2$.

The cross-section stiffness matrices in terms of the physical relations

(5) $$\dot{n}_j = D_{ojk}\dot{\vartheta}_k + D_{1jk}\cdot\dot{k}_k, \quad \dot{m}_j = D_{1jk}\dot{\vartheta}_k + D_{2jk}\cdot\dot{k}_k, \quad j, k = 1, 2,$$

assume the form

(6) $$D_{ijk} = \int\limits_{-1/2}^{1/2} E_{jk}\zeta^i d\zeta = \sum_{r=1}^{R} Z_r\cdot(E_{jk}\cdot\zeta^i)_r,$$

are calculated by using quadrature formulae, taking into account a current state of stresses and strains, history of the loading process and type of cross-section.

Incremental physical relations, given in detail in [4, 5] are referred to a material with the combined kinematic-isotropic strain hardening, where $A$ and $C$ stend for the coefficients of isotropic and kinematic strain-hardening. Thus, the equations enable us to describe the instantaneous motion of the centre of the yield locus (the interaction curve $F$) as well as the development of the locus. Let us assume that $E_{jk}$ are elements of the local stiffness matrices, occurring in incremental physical equations. These matrices are definite at each point of the shell under assumption of the plane stress state. Elements $E_{jk}$ can be computed on the ground of information on the type of process [4, 5].

### 3. Algorithm of the shooting method in elastic-plastic shells of revolution

Due to the separation of spatial $(\ )' = \partial/\partial\lambda$ and quasi-time $(\ )^{\cdot} = \partial/\partial\tau$ derivatives in the basic equations and to the approximation:

(7) $$\dot{Q}_m \approx (Q_m - Q_{m-1})/\Delta\tau_m, \quad \dot{q} \approx (q_m - q_{m-1})/\Delta\tau_m,$$

the two-point symmetric BVP of range $r = 3$ can be computed at each time $\tau = \tau_m$. For the following values $\tau \geqslant \tau_0$ we can find points on the path of equilibrium states in the load-displacement space. As a quasi-time independent variable $\tau$ one of the monotonically increasing parameter of the BVP can be adopted. Dependence on the shape of the path, the load parameter or one of the components of the input vector $y_0$ is applied as the time $\tau$.

The BVP can be solved iteratively using the shooting method by means of the multiple numerical forward integration and choice of free initial values $X_0$ at point $l = 1$, in order to satisfy the boundary conditions $Y$ at the other boundary point $l = L$, $(X_0, Y \in R^3)$.

The incremental formulation needs storing in the computer memory vector $Q_{m-1} = Q^*$, $q_{m-1} = q^*$ and information about the history of the loading process (development of elastic-plastic zones) for the whole structures.

With respect to the convergence and accuracy of the Newton-Raphson method as well as the time consumption and computer storage the choice of spatial $\Delta\lambda$ and time $\Delta\tau$ steps is essential. In many numerical experiments the density of spatial step near the shell ends is fixed. The step $\Delta\tau$ is automatically computed on the basis of the criterion, that the value of Odquist's parameter increment is within defined limit at each point of the structures [6, 7].

## 4. Inversion of physical equations

Considering elastic-plastic properties of the material the problem of inversion of physical relations appears at each step of the numerical integration $\lambda_l$, $l = 1, \ldots, L$ at each time $\tau_m$, $m = 1, \ldots, M$. The knowledge of values $n_1, m_1, \vartheta_2, k_2$ at each point $\lambda_l$ results from the form the canonic set of equations. Appropriate procedure makes it possible to calculate the values of generalized strains $\vartheta_1(n_1, m_1, \vartheta_2, k_2)$ and $k_1(n_1, m_1, \vartheta_2, k_2)$ beside generalized stresses $n_2(q)$ and $m_2(q)$. Apart from the type of cross-section, material properties, actual type or process should be taken into account in every quasi-time increment $\Delta\tau_m$. Information stored in the computer memory concerning the values $\vartheta_2^*, k_2^*, n_1^*, m_1^*$ permits to calculate suitable increments of these quantities.

The passage from increments of strains $\dot{\vartheta}_j, \dot{k}_j (j = 1, 2)$ at a point of the mid-surface to increments of strains $(\dot{e}_j)_r$ for each point $r = 1, \ldots, R$ along the thickness is possible under the assumption of Kirchhoff-Love hypotheses:

$$(\dot{e}_j)_r = \dot{\vartheta}_j + \dot{k}_j \zeta_r, \quad \zeta_r = z/\overline{h}_0, \quad \zeta \epsilon [-1/2, 1/2]. \tag{8}$$

In this way increments $(\Delta e_j)_r$ are obtained, and thus the calculation of increments stresses $(\Delta s_j)_r$ is feasible. The local cross-section stiffness matrices require knowledge of the most actual values $(E_{ij})_r$ for every point $r = 1, \ldots, R$. Unlike the sandwich in the full-walled cross-section case we cannot easily pass from the increments of generalized quantities $\{\Delta n_j, \Delta m_j, \Delta\vartheta_j, \Delta k_j\}$ to the increments of stresses $(\Delta s_j)_r$. In the case of full-walled cross-section approximated by the multipoint section, the stresses cannot be analysed separately for each point $r$, as it is possible for two layers of the sandwich cross-section. Only in an ideal sandwich there is an immediate passage from increments of generalized stresses $\Delta n_j$, $\Delta m_j$ to increments stresses $\Delta s_j$ in the particular layer $(+, -)$:

$$\begin{aligned} \Delta n_j &= (\Delta s_j^+ + \Delta s_j^-)/2, \quad & n_j &= \overline{n}_j/(\overline{h}^s \sigma_p), \\ \Delta m_j &= (\Delta s_j^+ - \Delta s_j^-)/2, \quad & m_j &= \overline{m}_j/(\underline{d}^s \overline{h}^s \sigma_p). \end{aligned} \tag{9}$$

Values $n_1$, $m_1$ are known in the numerical integration nodes in the mid-surface on the basis of the solution of the basic set of equations. Therefore in full-walled cross-section the increments of stresses $(\Delta s_1)_r$, $r = 1, \ldots, R$, calculated iteratively, added to stresses $(s_1^*)_r$, stored in the computer memory must agree with values $n_1$ i $m_1$ in point $\lambda_l$:

$$\bar{n}_1 = \sigma_p \bar{h}^F \int_{-1/2}^{1/2} (s_1^* + \Delta s_1) d\zeta = \sigma_p \bar{h}^F \sum_{r=1}^{R} Z_r (s_1^* + \Delta s_1)_r, \tag{10}$$
$$\bar{m}_1 = \sigma_p [(\bar{h}^F)^2/6] \cdot \int_{-1/2}^{1/2} (s_1^* + \Delta s_1) \zeta d\zeta = \sigma_p [(\bar{h}^F)^2/6] \cdot \sum_{r=1}^{R} Z_r (s_1^* + \Delta s_1)_r \zeta_r .$$

Increments $(\Delta s_j)_r$ sre computed currently for each point $\zeta_r$ and $\lambda_l$ and they depend both on the type of process and on attained yielding [4, 5].

For each point r along the shell thickness and for each node $l$ of numerical integration the following four cases must be considered:

i). If the elastic material properties at the point $r$ are noted in the computer, the increment of the effective stress $\Delta s_i < 0$ indicates, that the elastic process remains. This is shown in Fig 2a, where the state for time $\tau^* + \Delta\tau$ is represented by the vector $s_0^* + \Delta s$, the end of which is at point $P$ inside yield curve $F_0$. In this case only $s_j^*$ and $e_j^*$ are corrected, coordinates of centre of yield sufrace $a_j^*$ and Odqvist's parameter remain unchanged.

ii). The curve $F_0$ is crossed by the vector $\Delta s_s$ and the vector $\Delta s_p$ leads to point $P$ inside $F_0$. Then an approximate procedure must be applied. It translates point $P_*$ into $P'$ along the path of proportional loading and a further increment $\Delta s''$ is calculated, as for an active process (Fig. 2b).

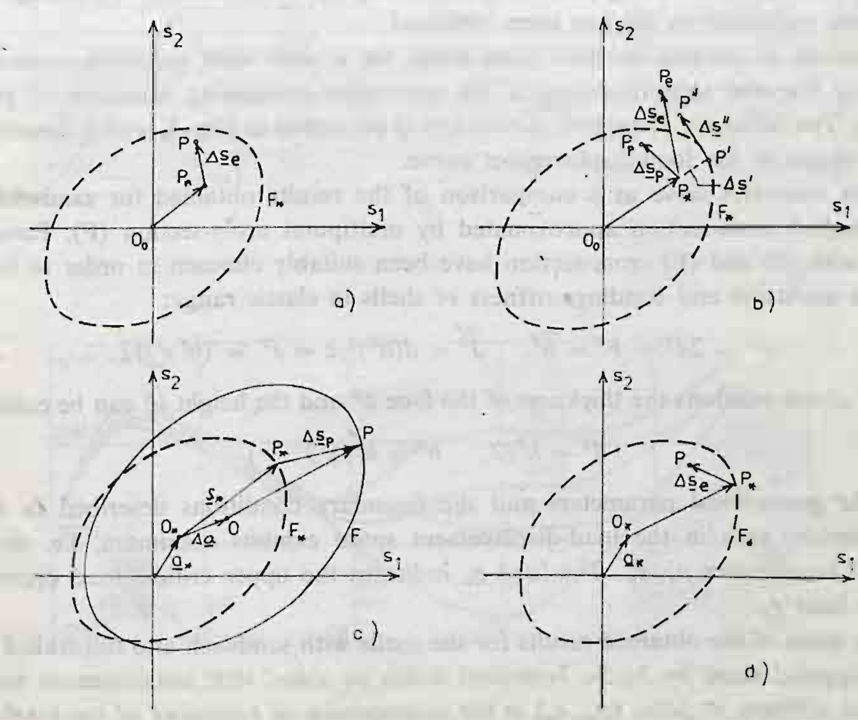



Fig. 2

iii). In the plastic zone an active process developes (Fig. 2c). Quantities $s_j^*, e_j^*, a_j^*, \bar{e}_p^*$ are corrected.

iv). The passive process occurs in the initial plastic zone if $\Delta s_i < 0$ (Fig. 2d).

If the yield curve $F_*$ is crossed at a single point $\zeta_r$ in a cross-section, it must be brought down again to the level of full cross-section. The same step $\Delta\tau$ is divided into subincrements to take into account at first the elastic and later the plastic properties at point $r$.

When the yield surface is crossed at several points, the most intensive point is looked for. It determines the way of division $\Delta\tau$ into subincrements. The number of iterations grows up when in the second stage it occurs, that curve $F$ is crossed at the next point or simultaneously unloading can take place at another point $r$ in the same cross-section.

If the boundary-value problem has been solved for time $\tau_m$ the information stored in the computer is repesated for the next time $\tau_{m+1} = \tau_m + \Delta\tau_{m+1}$.

## 5. Numerical analysis

The displacement field in a shell depends strongly on various parameters. Geometric shell parameters, boundary conditions, material properties have to be taken into account. A series of numerical examples for shells differing from one another in values of the mentioned parameters should be made with respect to the analysis of response of shells under applied load in the elastic-plastic range.

Elastic full-walled circular plate and spherical cap have been tested. A good agreement with results published in [8] has benn obtained.

First series of calculation have been made for a shell with sandwich cross-section considering essential simplifications in the subroutine containing inversion of physical equations. The influence of various parameters is presented in Fig. 3, which demonstrates different shape of the load-displacement curve.

Further examples serve as a comparison of the results obtained for sandwich (S) and full-walled cross-section approximated by multipoint cross-section (F). Parameters of shells with (S) and (F) cross-section have been suitably choosen in order to have an equivalent extension and bending stiffness of shells in elastic range:

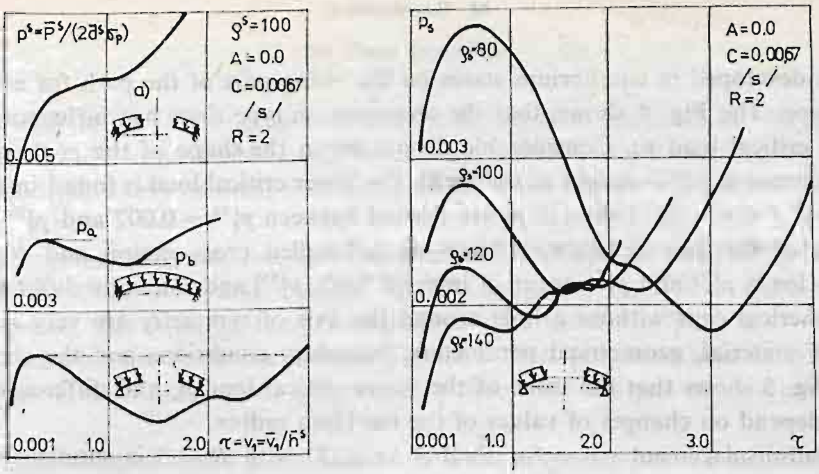
$$\bar{F}^S = 2\bar{d}^S = \bar{F}^F = \bar{h}^F, \quad \bar{J}^S = \bar{d}(\bar{h}^S)^3/2 = \bar{J}^F = (\bar{h}^F)^3/12. \tag{11}$$

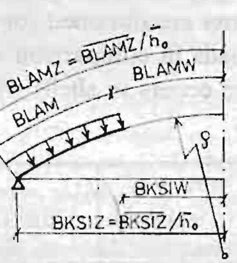
From the above relations the thickness of the face $\bar{d}^S$ and the height $\bar{h}^S$ can be calculated:

$$\bar{d}^S = \bar{h}^F/2, \quad \bar{h}^S = \bar{h}^F/\sqrt{3}. \tag{12}$$

For the geometrical parameters and the boundary conditions described in Fig. 4. the equilibrium path in the load-displacement space exhibits maximum, i.e. the limit point with coordinates $p_u$, $\tau_u$. The load $p_u$ indicates the upper critical load opposite to the lower load $p_l$.

On the basis of the obtained results for the shells with sandwich and full-walled cross-section (approximated by 3-, 5-, 7- points) it can be stated that instantaneous reducing of the shell stiffness at point $(p_u, \tau_u)$ is the consequence of vanishing of the total elastic cross-section. Passive processes appear between $p_u$ and $p_l$ loads. Secondary yielding zones




| | /s/ | /F/ |
|---|---|---|
| $\bar{h}_0$ | $\bar{h}^s$ | $\bar{h}^F$ |
| $\bar{d}$ | 0.1 | — |
| $\bar{h}$ | 0.1155 | 0.20 |
| $\varrho=\bar{\varrho}/\bar{h}_0$ | 100 | 57.75 |
| BKSIZ | 20 | 11.55 |
| BKSIW | 10 | 5.77 |
| BLAMZ | 20.136 | 11.628 |
| BLAMW | 10.017 | 0.844 |

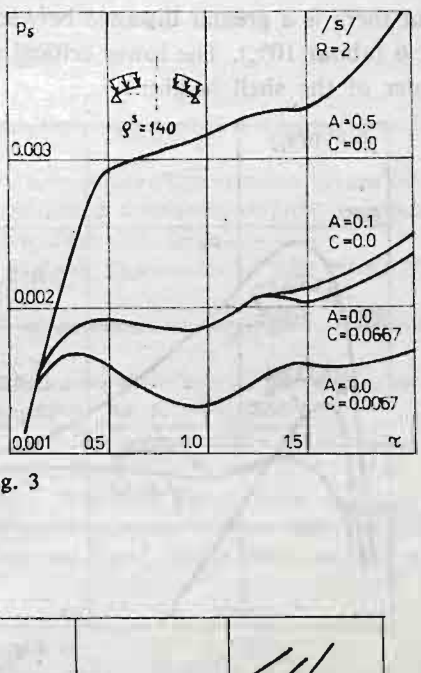



Fig. 3

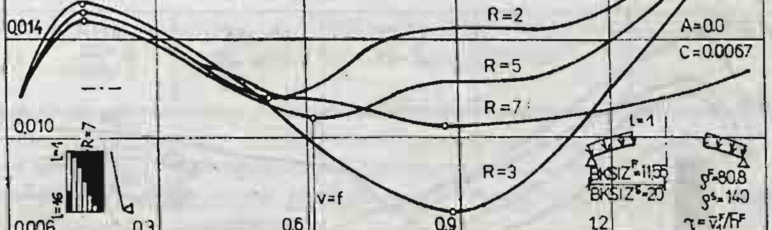



Fig. 4

have been developed in equilibrium states on the rising part of the path for every cross--section type. The Fig. 4 shows, that the cross-section type does not influence values of the upper critical load $p_u$. Considerable diversions in the shape of the $p(\tau)$ curve begin for deflections $v \approx f$ ($f$ — height of the shell). The lower critical load is found in the deflection interval $f < v < 2f$. Values of $p_l$ are limited between $p_l^{(3)} = 0.007$ and $p_l^{(S)} = 0.0115$, i.e. calculated for 3-point approximation of full-walled cross-section and ideal sandwich. The loads $p_l^{(5)}$ and $p_l^{(7)}$ occur in interval $[p_l^{(3)}, p_l^{(S)}]$ and mutually differ slightly.

The spherical caps without a hole around the axis of symmetry are very sensitive to changes of material, geometrical parameters, boundary conditions and the character of loading. Fig. 5 shows that the value of the upper critical load $p_u$ and difference between $p_u$ and $p_l$ depend on changes of values of the meridian radius.

The load-displacement curve for shell $A$ (Fig. 5) with $R = 5$ is supplemented with the equilibrium path calculated for 2, 3, 7 — point approximation of full-walled cross--section. The next diagram shows that the limit points are obtained for the same displacement but there is a greater distance between $p_u$ loads in comparison with the example from Fig. 6 (about 10%). The lower critical pressure occurs in shells with displacements of the order of the shell height.

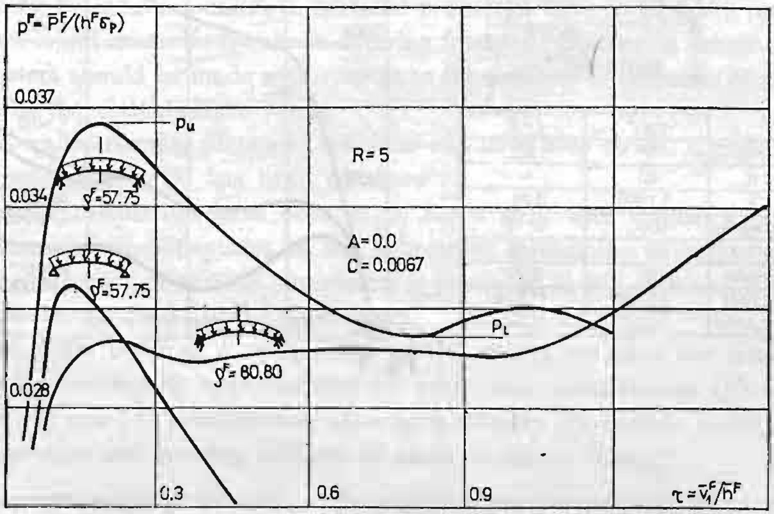



Fig. 5

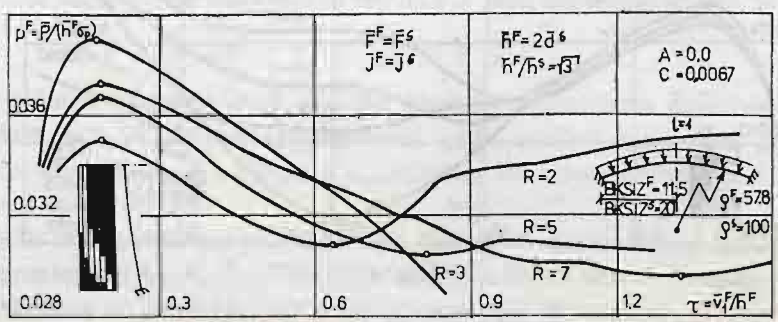



Fig. 6

## 5. Final remarks

On the basis of the shooting method and the incremental approach the algorithm and the computer program have been carried out. The program enables us to compute either idealized sandwich shells or shells with a full-walled cross-section with different boundary conditions.

The appropriate program has been written in FORTRAN-EXTENDED and implemented on the CDC-Computer CYBER-72. The program is efficient since only CPU memory is used, this shortens the computational time significantly.

Compared with the FEM the method applied in this program (semi-inverse method of numerical forward integration) is especially suitable for analysis of axisymmetric problems.

Numerical examples verify the 5-point equivalent cross-section as a good approximation of the full-walled cross-section.

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Р е з ю м е

НУМЕРИЧЕСКИЙ АНАЛИЗ БОЛЬШИХ ПРОГИБОВ УПРУГО-ПЛАСТИЧЕСКИХ ОБОЛОЧЕК ВРАЩЕНИЯ

Уравнения больших прогибов и малых деформаций применяются к анализу упруго пластических оболочек. Предполагается двухслойное поперечное сечение типа ,,сэндвич'' или апроксимацию сплошного сечения эквивалентным многоточечным сечением.

Применяются уравнения теории пластического течения к материалу с комбинированным кинематическо-изотропным упрочнением.

К интегрированию уравнений записанных в кваси-линейной форме и раздельных к отношению к пространственным и временным производным, применяется полуобратный метод нумерического интегрирования.

Для пологих оболочек потеря устойчивости связана с получением верхнего граничного давления. Нумерические вычисления для сферических оболочек нагруженных внешним давлением указывают, что в зависимости от параметров оболочки и упрочнения материала возможный хлопок. Послекритические прогибы зависят сильно от расширения пластических зон, внутренних упругих разгрузок и вторичных пластических деформаций.

## Streszczenie

NUMERYCZNA ANALIZA DUŻYCH PRZEMIESZCZEŃ SPRĘŻYSTO-PLASTYCZNYCH POWŁOK OBROTOWO-SYMETRYCZNYCH

Równania dużych ugięć i małych odkształceń są przyjęte do analizy sprężysto-plastycznych powłok obrotowo-symetrycznych. Zakłada się dwuwarstwowy przekrój typu ,,sandwich" lub aproksymację pełnościennego przekroju równoważnym wielopunktowym przekrojem.

Przyjęto równania teorii płynięcia plastycznego dla materiału z mieszanym kinematyczno-izotropowym wzmocnieniem. Do całkowania równań, zapisanych w postaci quasi-liniowej i rozdzielonych względem przestrzennej i czasowej zmiennej przyjęto półodwrotną metodę numerycznego całkowania.

Dla małowyniosłych powłok utrata stateczności jest związana z osiągnięciem górnego obciążenia granicznego. Przykłady liczbowe dla powłok sferycznych obciążonych ciśnieniem zewnętrznym wskazują, że w zależności od parametrów powłoki i wzmocnienia materiału możliwy jest przeskok. Pozakrytyczne ugięcia silnie zależą od rozwoju stref plastycznych, lokalnych obciążeń i wtórnych uplastycznień.

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