

ON THEORY OF LATTICE-REINFORCED SHELLS

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1. Introduction

In the present paper a continuum approach for analysing elastic shells with lattice-type reinforcement is proposed.

The shell structures are widely used in engineering practice. In many cases (especially in the civil engineering) these structures require to be reinforced. Thus, from the theoretical point of view material of the shell ought to be treated as nonhomogeneous mixture of two components: reinforcement and matrix. Even if additional, simplifying assumptions of homogeneity and isotropy of both components are being utilised, the known composite and mixture theories lead to the complex mathematical models, which cannot be recommended for engineering practice (analysis). That is why in the majority of papers, authors do not apply the theories mentioned above and assume stronger simplifications: in most cases material of the shell is supposed to be homogeneous and anisotropic (or even isotropic); the crucial point is to determine effective moduli for the hypothetical material of the shell.

The purpose of this work is to generalize the energy functional for continuum shell by adding a term concerning elastic reinforcement energy and then deriving the equations of equilibrium as well as appropriate (in particular: natural) boundary conditions.

In the course of the paper, materials of both components are supposed to be linear-elastic, homogeneous and isotropic. Considerations are confined to the case of small strains and displacements. The state of strain of the reinforcement is described according to Woźniak's lattice-type shell theory [1]. The state of strain of the matrix is assumed according to generalised Reissner's hypothesis. Thus, both models belong to the six-parameter classes of surface structures theories and no additional constraints on the reinforcement are imposed.

Considerations concern the lattice reinforcement constructed of two or three families of intersecting bars, lying on a surface parallel to the middle surface of the shell. Compatibility conditions of matrix displacements and approximated displacements of lattice nodes are supposed to be satisfied.

2. Geometry of shell

The region of the shell is parametrised by two convected normal coordinate systems $\{x\}$ and $\{x^i\}$. At every point x_a^a of the fundamental surface π , $x^3 = 0$ a natural reference triplet (g_a) is fixed. Similarly, at all points x^i in the shell region marks (g_i) can be determined. Particularly, the surface $\tilde{\pi}$, $x^3 = \tilde{x}^3 = \text{const}$, which includes the axes of the reinforcement bars. The bases (\tilde{g}_i) refer to the point of this surface.

Base vectors q_i can be expressed by means of q_a vectors; the same can be stated about the reciprocal bases q^i and q^a . The mentioned relations have the forms

$$(2.1) \quad g_i = V_i^a g_a, \quad g^i = A_a^i g^a.$$

Eqs (2.1) yield from Weingarten formulae and make it possible to shift an arbitrary tensor object from the point x^a on $\tilde{\pi}$ along the normal to the point x^a on the reference fundamental surface.

The shifters V_i^a, A_a^i (cf. eg [2] [3]) are defined as follows

$$(2.2) \quad V_i^a = \delta_i^a + \delta_i^b g_b^a x^3, \quad V_i^a A_a^j = \delta_i^j, \quad V_i^b A_a^i = \delta_a^b,$$

hence

$$(2.3) \quad A_a^k = A \delta_a^k [(1 - 2Hx^3) \delta_a^b + x^3 g_a^b + (x^3)^2 \delta_a^3 \delta_a^3 K],$$

where

$$(2.4) \quad A = V^{-1}, \quad V = \det(V_a^k) = 1 - 2x^3 H + (x^3)^2 K,$$

and

$$(2.5) \quad 'g_b^a = -g_{3,b} g^a = g_{3,b} g^a,$$

$$H = \frac{1}{2} \text{tr}('g_{\beta}^{\alpha}), \quad K = \det('g_{\beta}^{\alpha}).$$

It is easy to prove, that the relation

$$(2.6) \quad d\tilde{\pi} = V d\pi,$$

hold true.

3. Tensor fields and their derivatives

Values of any vector field v are elements of linear spaces. Thus they can be represented by their components either in the basis g_i or in the basis g_a as follows:

$$(3.1) \quad v = v_i g^i = v^i g_i = v_a g^a = v^a g_a.$$

From (2.1) it yields that different components of the same object are related by the formulae

$$(3.2) \quad v_i = V_i^a v_a, \quad v^i = A_a^i v^a.$$

Per analogiam one can obtain similar relation for any tensor of (p, q) valency. The appropriate formula takes the form

$$(3.3) \quad t_{j_1 \dots j_p}^{i_1 \dots i_q} = A_{a_1}^{i_1} \dots A_{a_p}^{i_p} V_{j_1}^{b_1} \dots V_{j_q}^{b_q} t_{b_1 \dots b_q}^{a_1 \dots a_p}.$$

Applying spatial gradient operation to the tensor field t , and taking into account (2.1) one arrives at the following formulae for the components of tensor derivatives:

$$(3.4) \quad \begin{aligned} t_{j_1 \dots j_p}^{i_1 \dots i_p} \hat{}_{|k} &= \delta_k^e A_{a_1}^{i_1} \dots A_{a_p}^{i_p} V_{j_1}^{b_1} \dots V_{j_p}^{b_p} t_{b_1 \dots b_p}^{a_1 \dots a_p} |_{|c} \\ t_{j_1 \dots j_p}^{i_1 \dots i_p} \hat{}_{||k} &= \delta_k^e \delta_l^d A_{a_1}^{i_1} \dots A_{a_p}^{i_p} V_{j_1}^{b_1} \dots V_{j_p}^{b_p} \cdot (t_{b_1 \dots b_p}^{a_1 \dots a_p} |_{|cd} - \delta_{n_c}^e A_{j^m}^m V_{c||d} t_{b_1 \dots b_p}^{a_1 \dots a_p} |_{|e}), \end{aligned}$$

where

$$\overset{\circ}{V}_c^d = \delta_c^t V_i^d.$$

4. Displacements

The state of displacements in the shell region is assumed to be compatible with the generalised Reissner's kinematic hypothesis

$$(4.1) \quad u(x^*, x^3) = \overset{\circ}{u}(x^*) + 'u(x^*)x^3$$

Additionally vector functions (cf. [1]) approximating displacements $\tilde{u} = \tilde{u}(x^*)$ and rotations $\tilde{\vartheta} = \tilde{\vartheta}(x^*)$ of reinforcement lattice nodes are introduced. The meaning of \tilde{u}_k components is the same as in the continuum surface structure theory, whereas $\tilde{\vartheta}_x$ components describe the node rotations in the planes perpendicular to parametric lines $x^* = \text{const}$ and $\tilde{\vartheta}_3$ — its rotation in the plane tangent to π . The vector functions u and ϑ are supposed to be of C^1 — class of continuity. Obviously they can be interpreted as displacements only in the lattice nodes.

It is assumed, that interactions between the reinforcement and the material of the matrix (in which it is embedded) are caused by the ideal adherence; and therefore the compatibility displacement conditions are supposed to be valid. Moreover the latter relations are assumed to be weakened by substituting into them the approximated lattice displacements, instead of the real ones. Such procedure leads to the following formulae

$$(4.2) \quad \tilde{u} = u|_{x^3=\tilde{x}^3} \quad \text{and} \quad \tilde{\vartheta} = \vartheta|_{x^3=\tilde{x}^3},$$

where ϑ denotes an infinitesimal rotation vector. Hence

$$(4.3) \quad \vartheta = \frac{1}{2} r \hat{\circ} t u = \frac{1}{2} \hat{e}^{klm} u_m \hat{\circ} |_{|l} g_k$$

where \hat{e}^{klm} — Ricci tensor on $\hat{\pi}$.

By virtue of (2.1) (3.3) (3.4) and (4.1) the components of lattice displacements can be referred to the basis on the fundamental, reference surface

$$(4.4) \quad \tilde{u}_k = \tilde{V}_k^a (u_a + 'u_a x^3) = \tilde{V}_k^a \tilde{u}_a,$$

$$\tilde{\vartheta}^k = \frac{1}{2} \delta_a^i \delta_c^e \epsilon^{abc} \tilde{V}_i^d \tilde{A} \tilde{u}_{d||b},$$

$$(4.5) \quad \tilde{\vartheta}_k = \frac{1}{2} g_{da} \delta_m^e \epsilon^{abe} \tilde{V}_k^d \tilde{A}_b^m \tilde{u}_{c||e}.$$

5. The state of strain

The state of strain of the lattice can be determined by means of the functions $e_{\kappa l}(x^\mu)$, $'e_{\kappa l}(x^\mu)$ as follows

$$(5.1) \quad \tilde{e}_{\kappa l} = \tilde{u}_{l|\kappa} + \tilde{\epsilon}_{l\kappa m} \tilde{\vartheta}^m, \quad 'e_{l\kappa} = \tilde{\vartheta}_{l|\kappa}.$$

By virtue of (4.2) and (4.3) we have

$$(5.2) \quad \tilde{e}_{\kappa l} = \frac{1}{2} (\tilde{u}_{l|\kappa} + \tilde{u}_{\kappa|l}), \quad 'e_{\kappa l} = \frac{1}{2} \epsilon_l^{mn} \tilde{u}_{n|m\kappa}.$$

Taking into account (3.4) and (2.1) and making some simple rearrangements the following formulae expressing the strains (5.2) can be obtained:

$$(5.3) \quad \tilde{e}_{\kappa l} = \frac{1}{2} \Phi_{\kappa l}^{ab} \tilde{u}_{a|b},$$

$$'e = \frac{1}{2} (\Psi_{\kappa l}^{abc} \tilde{u}_{a|bc} - \Theta_{\kappa l}^{ab} \tilde{u}_{a|b}),$$

with the denotations

$$(5.4) \quad \Phi_{\kappa l}^{ab} = \tilde{V}_\kappa^a \delta_l^b + \delta_\kappa^b \tilde{V}_l^a,$$

$$\Psi_{\kappa l}^{abc} = \delta_\kappa^c \epsilon_e^{da} V_l^e \tilde{A}_a^b,$$

$$\Theta_{\kappa l}^{ab} = \Psi_{\kappa l}^{ab\gamma} \Lambda_\gamma^b V_{d|\kappa}^e.$$

6. Potential energy of shell

The total potential energy of the shell is expressed by the formula

$$(6.1) \quad J = \int_{\tilde{\pi}} \tilde{\sigma} d\tilde{\pi} + \int_{\pi} (\sigma - q) d\pi - \int_{\partial\pi} \tau d(\partial\pi),$$

where σ and $\tilde{\sigma}$ denotes surface densities of the strain energy of the matrix and reinforcement, respectively; q and τ denotes densities of potential energy due to external surface and boundary loads.

Methods of determining the quantities σ , q , and τ can be found in papers devoted to so called six-parameter shell theories (cf. [4] [5])

In this paper attention is confined to the method of the derivation of the function

$$(6.2) \quad \tilde{J} = \int_{\tilde{\pi}} \tilde{\sigma} d\tilde{\pi} = \int_{\pi} \tilde{\sigma} \tilde{V} d\pi,$$

and — the stationary condition $dJ = 0$ of the variational theorem coupled with (6.1).

The density of reinforcement strain energy can be written in the form

$$(6.3) \quad \tilde{\sigma} = \frac{1}{2} (A^{\kappa l \mu \nu} \tilde{e}_{\kappa l} \tilde{e}_{\mu \nu} + C^{\kappa l \mu \nu} 'e_{\kappa l} 'e_{\mu \nu}).$$

Where, for the lattices under consideration

$$(6.4) \quad \begin{aligned} A^{xlmn} &= \sum_{\Delta} t_{(\Delta)}^x t_{(\Delta)}^m (t_{(\Delta)}^l t_{(\Delta)}^n R_{(\Delta)}^{(0)} + n_{(\Delta)}^l n_{(\Delta)}^n R_{(\Delta)}^{(1)}) + \delta_3^l \delta_3^n R_{(\Delta)}^{(2)}, \\ C^{xlmn} &= \sum_{\Delta} t_{(\Delta)}^x t_{(\Delta)}^m (t_{(\Delta)}^l t_{(\Delta)}^n S_{(\Delta)}^{(0)} + n_{(\Delta)}^l n_{(\Delta)}^n S_{(\Delta)}^{(1)}) + \delta_3^l \delta_3^n S_{(\Delta)}^{(2)}. \end{aligned}$$

The vectors $t_{(\Delta)}$ and $n_{(\Delta)}$ are referred to the plane $\tilde{\pi}$ and are tangent and normal to the axis of the bar (belonged to the Δ — family of bars), respectively.

The quantities $R_{(\Delta)}^{(T)}$, $S_{(\Delta)}^{(T)}$, $T = 0, 1, 2$ are stiffnesses of reinforcement rods.

Applying the Gauss-Stokes theorem in the form

$$(6.5) \quad \int_{\pi} v_{||\alpha}^x d\pi = - \int_{\pi} 2Hv^3 d\pi + \int_{\partial\pi} v^{\alpha} l_{\alpha} d(\partial\pi),$$

when stationary condition $dJ = 0$ is being examined, we arrive to the following set of equilibrium equations

$$(6.6) \quad \begin{aligned} \delta^{\circ} u_d: t_{||e}^{de} + 2Ht^{d3} + \underline{\circ \varepsilon^d} - \underline{\circ f^d} &= 0, \\ \delta' u_d: \tilde{x}^3 t^{de} + (2H\tilde{x}^3 - 1) t^{d3} + \underline{\prime \varepsilon^d} - \underline{\prime f^d} &= 0, \end{aligned}$$

with boundary conditions

$$(6.7) \quad \begin{aligned} \delta \tilde{u}_{d||\alpha}: r^{d\alpha\beta} l_{\beta} &= 0, \\ \delta^{\circ} u_d: -(t_{d\beta} - \underline{\circ \lambda^{d\beta}}) l_{\beta} &= \underline{\circ} t^d, \\ \delta' u_d: (r^{d3\beta} - \tilde{x}^3 t^{d\beta} + \underline{\prime \lambda^{d\beta}}) l_{\beta} &= \underline{\prime} t^d. \end{aligned}$$

Underlined terms in (6.6) and (6.7) appear, when variation of appropriate terms in (6.1) is considered.

The vector l_{β} is tangent to π and is exteriorly normal to the boundary line $\partial\pi$.

Moreover following auxilliary quantities are introduced

$$(6.8) \quad r^{ab} = r_{||\alpha}^{ab\alpha} - s^{ab} - p^{ab},$$

and

$$\begin{aligned} p^{de} &= \frac{1}{4} \tilde{V} A^{xlmn} \Phi_{\mu n}^{de} \Phi_{xl}^{ab} \tilde{u}_{a||b}, \\ r^{de\eta} &= \frac{1}{4} \tilde{V} C^{xlmn} \Psi_{\mu n}^{de\eta} (\psi_{xl}^{ab\gamma} \tilde{u}_{a||b\gamma} - \Theta_{xl}^{ab} \tilde{u}_{a||b}), \\ r^{de3} &= 0, \\ s^{de} &= \frac{1}{4} \tilde{V} C^{xlmn} \Theta_{\mu n}^{de} (\Psi_{xl}^{ab\gamma} \tilde{u}_{a||b\gamma} - \Theta_{xl}^{ab} \tilde{u}_{a||b}). \end{aligned}$$

7. Conclusions

In the present paper the energy functional for the shells with lattice-type reinforcement is obtained. In the variational way equilibrium equations as well as natural boundary

conditions are derived. The assumed mathematical model makes it possible to consider an influence of reinforcement stiffness on resultant shell response in more systematic way than in the hitherto used approaches in which homogeneity of the structure is postulated. In the proposed model a geometry, directions, and full set of elastic features of the fibrous is taken into account.

In the case of slender reinforcement rods a formal resemblance of the proposed theory to the anisotropic model of Reissner's shell is worth mentioning.

Equations obtained in the paper can be applied in several other special cases.

Presented variational approach, in particular the energy functional (6.1) can be used for the finite element formulation of the problem considered.

References

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Резюме

О ТЕОРИИ ОБОЛОЧЕК АРМИРОВАННЫХ СЕТКАМИ

В работе используя вариационный метод, построено уравнения теории упругих оболочек армированных сетками.

Изгиб силовой оболочки (матрицы) описано, принимая общую гипотезу Рейсснера. Деформация сетчатой арматуры согласно теории сетчатых оболочек Возняка.

Представлен в работе прием определения влияния арматуры на деформацию оболочки по характеру „технический“. Это следует из принятой идеализации структуры сетчатой арматуры как и ее непрерывного описания.

Streszczenie

О ТЕОРИИ POWŁOK ZBROJONYCH SIATKAMI

W pracy na drodze wariacyjnej uzyskano równania teorii powłok sprężystych zbrojonych siatkami.

Zginanie kontynuualnej powłoki (matrycy) opisano przyjmując uogólnioną hipotezę Reissnera, a deformację zbrojenia określono zgodnie z równaniami teorii powłok siatkowych Woźniaka [1].

Przedstawiona w pracy próba uwzględnienia zbrojenia na stan deformacji dźwigara ma charakter „techniczny” co wynika zarówno z założeń odnośnie struktury zbrojenia jak i z zastosowanego kontynuualnego opisu siatki.