MECHANIKA
TEORETYCZNA
I STOSOWANA
3/4, 20 (1982)

A MINIMUM-PRINCIPLE FOR STRESS-STATE IN ELASTIC-PLASTIC PLATES AND THE SYSTEMATICAL GENERATION OF APPROPRIATE PLATE-MODELS[1)]

DIETER WEICHERT

Institut für Mechanik, Ruhr-Universität
4630 Bochum-1, West-Germany

1. Introduction

In this paper we treat the initial boundary-value problem of elastic-plastic plates subjected to arbitrary dead-loadtype loading histories. This problem differs from analogous purely elastic problems by the fact that even under the assumption of certain shape of strain-distribution over the thickness of the plate, such as Kirchhoff-Love-hypothesis[1], no prediction about stress-distribution over the thickness of the plate can be made as no one-to-one correspondance between strains and stresses holds. So here we treat this problem genuinely as three-dimensional problem constrained by certain geometrical and statical conditions which have a distinct meaning in theory of plates and in theory of plasticity, respectively. We show, that well known Kirchhoff plate-theory is a special case of the herein presented concept. For the construction of a minimum-principle for the state of stress in the plate we make use of a recently derived minimum-principle for general three-dimensional body [2], based on the formulation of constitutive relations by means of convex analysis [3, 4] and internal parameters [5] in order to describe elastic-perfectly plastic and elastic-linear hardening material behaviour by the same mathematical model.

In the last chapter a numerical illustration of the presented method is given for the case of a proportionally loaded elastic-perfectly plastic square plate.

2. The three-dimensional initial boundary-value problem. Local formulation of the problem

A body of volume Ω as subregion of product-space of three-dimensional Euclidean space R^3 and space T of time t, defined on the intervall $T = [0, \infty)$, with sufficiently regular boundary $\partial\Omega$, is subjected to external agencies $a = a(x)$, described by the set $[f^*(x) \in \Omega,\ P^*(x) \in \partial\Omega_s,\ u^*(x) \in \partial\Omega_k$, where $f(x)$, $u(x)$ and $p(x)$ denote three-dimensional vectors of volume-forces, displacements and surface-forces, respectively. $\partial\Omega_k$ and $\partial\Omega_s$,

[1)] This paper was elaborated during a stay at the Institut of Mechanics, University of Warsaw and the author wishes to express his gratitude to Professor dr hab. Czesław Woźniak for invitation and permanent support.

denote disjoint parts of $\partial\Omega$ where kinematical and statical boundary conditions are prescribed, resp., Considering quasi-static deformation processes in the range of small deformations for conservative external agencies $a(x)$, statical and kinematical field-equations are given by:

$$\begin{aligned} \operatorname{Div}\sigma+f^* &= 0 \quad \text{in } \Omega, \\ n\cdot\sigma-P^* &= 0 \quad \text{on } \partial\Omega_s, \end{aligned} \tag{2.1}$$

$$\begin{aligned} \varepsilon-\operatorname{Grad}_s u &= 0 \quad \text{in } \Omega \\ u = u^* &= 0 \quad \text{on } \partial\Omega_k \end{aligned} \tag{2.2}$$

Div and Grad_s denote divergence-operator and symmetric part of gradient-operator, $\sigma(x)$, $\varepsilon(x)$ are elements of space τ_6^2 of symmetric, two-dimensional tensors with 6 independent components, n denotes outer normal unit-vector on $\partial\Omega$. Prescribed quantities are indicated by upper star. The problem consists of determining $\sigma\varepsilon$ and u for the entire deformation-process. Constitutive relations are described by use of internal parameters [5], such that elastic-perfectly plastic and elastic-linear hardening material behaviour can be treated by the same mathematical methods. Assuming, that entire strain $\varepsilon(x)$ can be additively decomposed into purely elastic part $\varepsilon^e(x)$ and purely plastic part $\varepsilon^p(x)$, generalized stress-, generalized elastic strain- and generalized plastic strain-tensors are defined, respectively, by the sets $s(x) = [\sigma, \pi]$, $e^e(x) = [\varepsilon^e, \omega]$, $e^p(x) = [\varepsilon^p, k]$, where internal statical, elastic and plastic parameters $\pi(x)$, $\omega(x)$ and $k(x)$ are elements of vector-space τ_r^1 with r independent components. It can be shown [2], that for vanishing $\omega(x)$ and $k(x)$ at time $t = 0$, entire generalized strain e is given by $e = [\varepsilon^e+\varepsilon^p, 0]$, defined on Ω. Assuming the existence of a convex, lower semi-continuous elastic strain energy-density $\psi(e^e)$ and introducing bilinear form (s, e^e) as inner product $s\,..\,e^e$ defined by

$$s\,..\,e^e = \sigma_{ij}\varepsilon_{ij}^e+\pi_n\omega_n \qquad i,j\in[1,2,3],\ n\in[1,2,\ldots,r] \tag{2.3}$$

where ψ and $(.,.)$ are mappings of product-space $\tau_6^2\times\tau_r^1$ onto R^1, defined on Ω, the following three relations are equivalent conditions for s and e^e to satisfy elastic material behaviour:

$$e^e \in \partial\psi^*(s), \tag{2.4}$$

$$s \in \partial\psi(e^e), \tag{2.5}$$

$$\psi(e^e)+\psi^*(s)-(s,e^e) \geqslant 0, \qquad \text{in } \Omega \tag{2.6}$$

with polar elastic energy-density $\psi^*(s)$ defined by:

$$\psi^*(s) = \sup_{e^{e*}\in\tau_6^2} [(s, e^{e*})-\psi(e^{e*})] \qquad \text{in } \Omega, \tag{2.7}$$

„$\partial(.)$" denotes subdifferential of the considered quantity. In the herein treated case of linear-elastic material behaviour (2.4 - 2.6) degenerate to

$$e^e = G\,..\,s \mathrel{\hat{=}} [\varepsilon_{ji}^e, \omega_n] = [\sigma_{kl}L_{ijkl}, \pi_m\dot{Z}_{mn}), \qquad i,j,k,l\in[1,2,3], \tag{2.8}$$

$$s = G^{-1}\,..\,e^e \mathrel{\hat{=}} [\sigma_{ij}, \pi_n] = [\varepsilon_{kl}^e L_{ijkl}^{-1}, \omega_m\dot{Z}_{mn}^{-1}] \qquad m,n\in[1,2,\ldots,r], \tag{2.9}$$

$$\frac{1}{2}e^e\,..\,G^{-1}\,..\,e^e+\frac{1}{2}s\,..\,G\,..\,s-s\,..\,e^e = 0. \tag{2.10}$$

L and $\dot{Z}$ denote here positive definit matrices with known constant coefficients of elastic and hardening-coefficients, respectively, G is defined as the set $[L, \dot{Z}]$, upper index „ -1" denotes inverse of the considered matrix.

Analogously plastic part of constitutive relations is formulated: If $\varphi(s)$ denotes plastic potential, defined as convex and lower semi-continuous indicator-function of convex closed region E_t in space $\tau_6^2 \times \tau_r^1$ of generalized stresses s, normality-rule for rate of generalized plastic strain $\dot{e}^p$, used in this paper as plastic flow-law, may be expressed by the following relations, each equivalent to the other:

$$\left.\begin{array}{ll} (2.11) & \dot{e}^p \in \partial\varphi(s) \\ (2.12) & s \in \partial\varphi^*(\dot{e}^p) \\ (2.13) & \varphi(s) + \varphi^*(\dot{e}^p) - (\dot{e}^p, s) \geqslant 0 \end{array}\right\} \quad \text{in } \Omega$$

where in (2.13) equality holds if plastic flow-law and yield-condition, demanding that every admissible state of stress s is in the interior or on the boundary of E_t, are fulfilled. Here, superposed dot denotes time-derivative, (e^p, s) denotes according to elastic part of constitutive relations, bilinear form $\dot{\varepsilon}_{ij}^p \sigma_{ij} + k_n \pi_n$, $i, j \in [1, 2, 3]$, $n \in [1, 2, \ldots, r]$. $\varphi^*(\dot{e}^p)$ is polar plastic potential, defined by:

$$(2.14) \qquad \varphi^*(\dot{e}^p) = \sup_{s^* \in \tau_6^2} [(\dot{e}^p, s^*) - \varphi(s^*)] \quad \text{in } \Omega$$

Reformulation of the problem, minimum principle for stresses. Assumption: External agencies $a(x)$ are represented by the given field-quantities σ^0, ε^0 and u^0 such that:

$$(2.15) \qquad \begin{array}{rll} \operatorname{Div}\sigma^0 + f^* &= 0 & \text{in } \Omega, \\ n.\sigma^0 - p^* &= 0 & \text{on } \partial\Omega_s, \end{array}$$

$$(2.16) \qquad \begin{array}{rll} \varepsilon^0 - \operatorname{Grad}_s u^0 &= 0 & \text{in } \Omega, \\ u^0 - u^* &= 0 & \text{on } \partial\Omega_k, \end{array}$$

$$(2.17) \qquad \varepsilon^0 - L..\sigma^0 = 0 \quad \text{in } \Omega.$$

Physically, σ^0 and u^0 represent the solution of an analogous purely elastic problem. Defining statically and kinematically admissible generalized stresses s^s and s^k, resp., by the definitions

$$(2.18) \qquad \begin{array}{l} s^k = [\sigma^k, o] := \{s \in \tau_6^2 / \sigma = L_{..}^{-1}\operatorname{Grad}_s u \quad \text{in } \Omega,\ u = 0 \text{ on } \partial\Omega_k\} \\ s^s = [\sigma^s, \pi] := \{s \in \tau_6^2 / \operatorname{Div}\sigma = 0 \quad \text{in } \Omega,\ n.\sigma = 0 \text{ on } \partial\Omega_s\} \end{array}$$

the entire problem is reformulated by: Determine s, e and e^p such that:

$$(2.19) \qquad \left.\begin{array}{l} s = s^0 - s^s \\ e = G..(s^0 + s^k) \\ \dot{e}^p = G..(\dot{s}^s + \dot{s}^k) \in \partial\varphi(s^0 - s^s) \end{array}\right\} \quad \text{in } \Omega$$

This, however, is equivalent [2] to the minimization of the functional $\lambda(s^s, s^k)$, defined by

$$(2.20) \qquad \lambda(s^s, s^k) = \varphi(s^0 - s^s) + \varphi^*(G..(\dot{s}^s + \dot{s}^k)) - (G..(\dot{s}^s + \dot{s}^k), s^0 - s^s) \geqslant 0$$

according to ((2.11) - (2.13), (2.20)).

By completion of space c^{∞}_{6+r} of smooth tensorfields of generalized stresses s with respect to the scalar-product

$$(2.21)\qquad \langle s, s^2\rangle_G = \int_{\Omega} s^{(1)} \cdot\cdot G \cdot\cdot s^{(2)} e^{-t} dx, \qquad s^1, s^2 \in c^{\infty}_{6+r}$$

in [2] Hilbert-space H of generalized stress-fields s is constructed. Global formulation of plastic part of constitutive relations is then given by

$$(2.22)\qquad \Phi(s)+\Phi^*(G^{-1}_{\cdot\cdot}e^p)-\langle G^{-1}_{\cdot\cdot}\dot{e}^p, s\rangle_G \geqslant 0,$$

where global plastic potential Φ and polar potential Φ^* are defined by

$$(2.23)\qquad \Phi(s) = \lim_{c\to\infty} \int_{\Omega} \varphi_c(s) e^{-t} dx, \qquad s \in H$$

$$\varphi_c(s) = \begin{cases} 0 & \text{if } s \in E_t \quad E_t \subset H, \\ +c & \text{if } s \notin E_t \quad c \in R^1, \quad c > 0 \end{cases}$$

$$(2.24)\qquad \Phi^*(G^{-1}_{\cdot\cdot}\dot{e}^p) = \sup_{s^*\in H} [\langle G^{-1}_{\cdot\cdot}\dot{e}^p, s^*\rangle_G - \Phi(s^*)] \quad \text{in } \Omega$$

As G is a constant positiv multiplier, here and in the following space of generalized strains is identified with Hilbert-space of generalized stress by use of the isomorphism $e = G \cdot\cdot s$. Analogously to (2.11) - (2.13), (2.22) is equivalent to:

$$(2.25)\qquad \left.\begin{aligned} G^{-1}\cdot\cdot\dot{e}^p = \dot{s}^s+\dot{s}^k \in \partial\Phi(s) \\ s \in \partial\Phi^*(\dot{s}^s+\dot{s}^k) \end{aligned}\right\} \quad \text{in } \Omega$$

Making use of the assumption of given purely elastic solution σ^0, u^0 and of orthogonality of kinematically and statically admissible stresses s^k, s^s with respect to scalar-product (2.21), stated by

$$(2.26)\qquad \langle s^k, s^s\rangle_G = 0 \quad \text{in } \Omega; \quad s^k \in H_k \subset H \supset H_s \in S^s; \quad H_k \perp H_s,$$

minimum-principle (2.20) is now stated globally: The convex functional Λ defined by

$$(2.27)\qquad \Lambda(s^k, s^s) = \Phi(s^0-s^s)+\Phi^*(\dot{s}^s+\dot{s}^k)-\langle s^0-s^s, \dot{s}^s+\dot{s}^k\rangle_G \quad \text{in } \Omega$$

assumes the minimum equal to zero for the solution $[\hat{s}^k, \hat{s}^s]$. However, as $\Lambda(s^k, s^s)$ is not strictly convex, solution may be not unique. If we resign from determination of $\hat{s}^k$, such that rate of plastic strain $\dot{e}^p = G\cdot\cdot(\dot{s}^s+\dot{s}^k)$ and as consequence entire state of strain cannot be determined, strictly convex functional $\Lambda_0(s^s)$, defined by

$$(2.28)\qquad \Lambda_0(s^s) = \Phi(s^0-s^s)+\Phi^*_0(\dot{s}^s)-\langle s^0-s^s, \dot{s}\rangle_G \geqslant 0 \quad \text{in } \Omega$$

can be constructed [2]. Lower index „0" denotes restriction of the domains of Λ and Φ^* to elements of H_s. Solution $\hat{s}^s$ of the problem is then uniquely obtained by minimization of Λ_0, if any solution exists. As in case of elastic-linear hardening material behaviour region E_t of admissible generalized stresses is constant, (2.28) can be reduced to the minimization of

$$(2.29)\qquad \Lambda_0(s^s) = \sup_{s^s\in s^0-E_t\cap H_s} \langle s^s-s^{s*}, \dot{s}^s\rangle_G;$$

3. The initial boundary-value problem of elastic-plastic plates. Systematical generation of plate-models from three-dimensional theory

A three-dimensional body of volume Ω, given by midsurface Γ as subregion in $R^2 \times T$, parametrized by rectangular coordinates x_1, x_2 and time-coordinate $t \in T = [0, \infty)$, with sufficciently regular boundary $\partial\Gamma$ and constant extension in x_3-direction with $x_3 \in [-h, +h]$, is called „plate", if $2h$ is much smaller than characteristic length L as measure of extension of Γ in $x_1 - x_2$-plane. $\partial\Gamma$ consists of parts $\partial\Gamma_k$ and $\partial\Gamma_s$, where kinematical and statical boundary-conditions are prescribed. For the moment we assume $\partial\Gamma_s \cap \partial\Gamma_k = \emptyset$, though in the sequel of the paper we shall weaken this assumption. Forces acting on upper and lower planes Γ^+, Γ^-, resp., parallel to Γ at distance h, will be treated as forces acting on Γ, kinematical conditions will only be prescribed on $\partial\Gamma_k$, not on Γ.

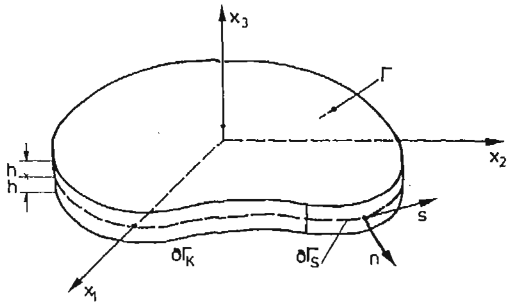

Fig. 1

In order to obtain a two-dimensional minimum-principle for state of stress in the plate according to (2.29), we define two-dimensional representatives of all three-dimensional quantities used in chapter 2. In general, they may be introduced in several manners: By use of multilayer-model, where the three-dimensional body is represented by a finite number of layers, such that to each three-dimensional field-quantity in the body for each layer a two-dimensional representative of the considered quantity is assigned [6, 7]. Here we use polynomial representatives defined in the following way: Be $f(x)$ an arbitrary smooth scalar-, vector- or tensorvalued function defined on Ω. We expand $f(x)$ into a Taylor-series with respect to midsurface Γ up to order q such that two-dimensional coefficients $F(x_1, x_2, t)$ of Taylor-expansion are defined by:

$$(3.1) \qquad F^{(k)}(x_1, x_2, t) = \frac{1}{(k-1)!} \frac{\partial^{k-1} f(x)}{(\partial x_3)^{k-1}}\bigg|_{x_3=0} \qquad k \in [1, 2, \dots, q)]$$

This represents a mapping of the domain $\mathscr{A}_3(f) \subset C_3^\infty$, where C_3^∞ denotes the space of smooth three-dimensional functions f onto the domain $\mathscr{A}_2(F) \subset (C_2^\infty)^q$, where $(C_2^\infty)^q$ denotes the product-space of smooth two-dimensional functions of power q. The inverse relation, given by

$$(3.2) \qquad f(x) = \sum_{k=1}^{q} F^{(k)}(x_1, x_2, t) x_3^{k-1} \qquad k \in [1, 2, \dots, q],$$

however maps $\mathscr{A}_2(F)$ only onto a subdomain $\mathscr{A}'_3 \subset \mathscr{A}_3$. In our approach we take only elements of $\mathscr{A}'_3$ into account and interprete this restriction as an imposure of constraints according to [6] on the three-dimensional body. This restriction is the starting-point for the construction of plate-theories characterized by the parameter q.
Here we introduce namely two-dimensional representatives $\boldsymbol{n}$, $\boldsymbol{q}$, $\boldsymbol{u}$ of three-dimensional generalized stresses s, generalized strains e and displacements u, defined by the sets:

$$\boldsymbol{n} = [N^q, \pi^q]; \qquad q^e = [Q^q, \Theta^q]; \qquad q^p = [P^q, K^q], \qquad U = [u^q]$$

with the definitions:

$$(3.3)\qquad \begin{aligned} N^q &= [N_{ij}^{(1)}, N_{ij}^{(2)}, \dots N_{ij}^{(q)}), & \Pi^q &= [\Pi_n^{(1)}, \Pi_n^{(2)}, \dots \Pi_n^{(q)}] \\ Q^q &= (Q_{ij}^{(1)}, Q_{ij}^{(2)}, \dots, Q_{ij}^{(q)}], & \Theta^q &= [\Theta_n^{(1)}, \Theta_n^{(2)}, \dots, \Theta_n^{(q)}] \\ P^q &= [P_{ij}^{(1)}, P_{ij}^{(2)}, \dots, P_{ij}^{(q)}], & K^q &= [K_n^{(1)}, K_n^{(2)}, \dots, K_n^{(q)}] \\ U^q &= (u_i^{(1)}, u_i^{(2)}, \dots u_i^{(q)}] \end{aligned}$$

with $i, j \in [1, 2, 3]$, $n \in [1, 2, \dots, r]$; q: order of Taylor-expansion.

The two-dimensional minimum-principle. Inserting so defined two-dimensional quantities into scalar-product (2.21) and using the multiplier G such that $G^{-1}_{..}\,e(q^q) \in \mathrm{H}$, we obtain:

$$(3.4)\qquad \langle\langle \boldsymbol{n}, \bar{\boldsymbol{n}} \rangle\rangle_G = \langle\langle \boldsymbol{n}, \boldsymbol{q} \rangle\rangle = \int_\Gamma \boldsymbol{n} m \boldsymbol{q} \mathrm{e}^{-t} dx_1\, dx_2\, dt,$$

with the definitions

$$(3.5)\qquad \begin{aligned} \boldsymbol{n} m \boldsymbol{q} &= \sum_{k=1}^{q} \sum_{l=1}^{q} (N_{ij}^{(k)} m_{kl} Q_{ij}^{(l)} + \Pi_n^{(k)} m_{kl} \Theta_n^{(lk)}), \\ m = m_{kl} &= \int_{-h}^{+h} x_3^{k+l-2} dx_3. \end{aligned}$$

Splitting up (3.4) into parts containing solely vector- and tensor-components in $x_1 - x_2$ — direction and those containing components in x_3-direction, we obtain:

$$(3.6)\qquad \langle\langle \boldsymbol{n}, \boldsymbol{q} \rangle\rangle = \langle\langle \boldsymbol{n}_{\alpha\beta}\, \boldsymbol{q}_{\alpha\beta} \rangle\rangle + 2\langle\langle \boldsymbol{n}_{\alpha 3}\, \boldsymbol{q}_{\alpha 3} \rangle\rangle + \langle\langle \boldsymbol{n}_{33}, \boldsymbol{q}_{33} \rangle\rangle + \langle\langle \pi_n, \Theta_n \rangle\rangle, \qquad \alpha, \beta \in [1, 2]$$

defined by:

$$(3.7)\qquad \langle\langle \boldsymbol{n}_{\alpha\beta}, \boldsymbol{q}_{\alpha\beta} \rangle\rangle = \int_\Gamma \left[\sum_{k=1}^{q} \sum_{l=1}^{q} N_{\alpha\beta}^{(k)} m_{kl} Q_{\alpha\beta}^{(l)} \right] \mathrm{e}^{-t} dx_1\, dx_2\, dt,$$

$$(3.8)\qquad \langle\langle \boldsymbol{n}_{\alpha 3}, \boldsymbol{q}_{\alpha 3} \rangle\rangle = \int_\Gamma \left[\sum_{k=1}^{q} \sum_{l=1}^{q} N_{\alpha 3}^{(k)} m_{kl} Q_{\alpha 3}^{(l)} \right] \mathrm{e}^{-t} dx_1\, dx_2\, dt,$$

$$(3.9)\qquad \langle\langle \boldsymbol{n}_{33}, \boldsymbol{q}_{33} \rangle\rangle = \int_\Gamma \sum_{k=1}^{q} \sum_{l=1}^{q} N_{33}^{(k)} m_{kl} Q_{33}^{(l)} \Big] \mathrm{e}^{-t} dx_1\, dx_2\, dt,$$

$$(3.10)\qquad \langle\langle \Pi_n, \Theta_n \rangle\rangle = \int_\Gamma \left[\sum_{k=1}^{q} \sum_{l=1}^{q} \Pi_n^{(k)} m_{kl} \Theta_n^{(l)} \right] \mathrm{e}^{-t} dx_1\, dx_2\, dt.$$

In accordance with the physical definition of „plates", given in the beginning of this chapter, we now precise that plates in general are characterized by the vanishing of (3.9) and thin plate by additionally vanishing of (3.8). In the following we shall deal exclusively with so defined thin plates. In minimum-principle (2.29) statically admissible generalized stresses were used for the construction of the solution of the problem. If now we use two-dimensional representatives for the stresses we also need a criterion for statical admissibility of these quantities. Here we use condition of orthogonality with respect to scalar-product (3.4), analoguous to orthogonality-condition (2.26). Statically admissible stress-representatives are then defined by:

$$n^s = \{n / \langle\langle n, q^k \rangle\rangle = 0 \quad \text{on } \Gamma\} \tag{3.11}$$

with kinematically admissible generalized strain-representatives $q^k = [Q^{kq}, 0]$ defined by the set

$$Q^{kq} := \{Q^q / Q^{(l)} = Q^{(l)}_{\alpha\beta} = \mathrm{Grad}_s u^{(l)}_\alpha \quad \text{in } \Gamma,\ u^{(l)}_\alpha = 0 \text{ on } \partial\Gamma_k\} \tag{3.12}$$

with $\alpha, \beta \in [1, 2]$; $l \in [1, 2, \ldots, q]$; q: order of Taylor-expansion. In order to identify Kirchhoff plate-theory lateron directly as special case of the herein presented generalized theory we impose on u^q the constraint

$$\bar{u}^{(2)}_\alpha = -u^{(2)}_\alpha, \quad \bar{u}^{(k)}_\alpha = u^{(k)}_{,\alpha} \quad k = 1, 3, \ldots, q, \quad k \neq 2 \tag{3.13}$$

By twice application of divergence-theorem (3.11) delivers immediately conditions for statical admissible two-dimensional representatives of generalized stresses.

Example for $q = 4$. If we insert into (3.11) two-dimensional representatives of order $q = 4$, we obtain:

$$\begin{aligned} \langle\langle n, q^k \rangle\rangle = & \int_\Gamma \Big[2hN^{(1)}_{\alpha\beta} u^{(1)}_{\alpha,\beta} + \frac{2}{3} h^3 (-N^{(2)}_{\alpha\beta} u^{(2)}_{|\alpha p} + N^{(1)}_{\alpha\beta} u^{(3)}_{\alpha,\beta} + N^{(3)}_{\alpha\beta} u^{(1)}_{\alpha,\beta}) + \\ & \frac{2}{5} h^5 (N^{(3)}_{\alpha\beta} u^{(3)}_{\alpha,\beta} + N^{(2)}_{\alpha\beta} u^{(4)}_{\alpha,\beta} - N^{(4)}_{\alpha\beta} u^{(2)}_{|\alpha\beta}) + \frac{2}{7} h^7 N^{(4)}_{\alpha\beta} u^{(4)}_{\alpha,\beta} \Big] e^{-t} dx_1 dx_2 dt = \\ = & \int_\Gamma \Big[\Big(2hN^{(1)}_{\alpha\beta} + \frac{2}{3} h^3 N^{(3)}_{\alpha\beta}\Big), \Big(-\frac{2}{3} h^3 N^{(2)}_{\alpha\beta} - \frac{2}{5} h^5 N^{(4)}_{\alpha\beta}\Big), \Big(\frac{2}{3} h^3 N^{(1)}_{\alpha\beta} + \frac{2}{5} h^5 N^{(3)}_{\alpha\beta}\Big), \\ & \Big(\frac{2}{5} h^5 N^{(2)}_{\alpha\beta} + \frac{2}{7} h^7 N^{(4)}_{\alpha p}\Big) \Big] \cdot [u^{(1)}_{\alpha,\beta}, u^{(2)}_{|\alpha\beta}, u^{(3)}_{\alpha,\beta}, u^{(4)}_{\alpha,\beta}]^T e^{-t} dx_1 dx_2 dt = 0. \end{aligned} \tag{3.14}$$

Where square brackets denote supervectors and superposed „T" indicates transported supervector. Twice application of divergence-theorem then delivers:

$$\begin{aligned} \langle\langle n, q^k \rangle\rangle = & -\int_\Gamma \Big[\Big(2hN^{(1)}_{\alpha\beta,\beta} + \frac{2}{3} h^3 N^{(3)}_{\alpha\beta,\beta}\Big), \Big(\frac{2}{3} h^3 N^{(2)}_{\alpha\beta,\beta\alpha} + \frac{2}{5} h^5 N^{(4)}_{\alpha\beta,\beta\alpha}\Big), \\ & \Big(\frac{2}{3} h^3 N^{(1)}_{\alpha\beta,\beta} + \frac{2}{5} h^5 N^{(3)}_{\alpha\beta,\beta}\Big), \Big(\frac{2}{5} h^5 N^{(2)}_{\alpha\beta,\beta} + \frac{2}{7} h^7 N^{(4)}_{\alpha\beta,\beta}\Big) \Big] \cdot [u^{(1)}_\alpha, u^{(2)}, u^{(3)}_\alpha, u^{(4)}_\alpha]^T \cdot \\ & \cdot e^{-t} dx_1 dx_2 dt + \int_{\partial\Gamma} \Big[\Big(2hN^{(1)}_{\alpha n} + \frac{2}{3} h^3 N^{(3)}_{\alpha n}\Big), (v + M_{ns,s}), M_{nn}, \\ & \Big(\frac{2}{3} h^3 N^{(1)}_{\alpha n} + \frac{2}{5} h^5 N^{(2)}_{\alpha n}\Big), \Big(\frac{2}{5} h^5 N^{(2)}_{\alpha n} + \frac{2}{7} h^7 N^{(4)}_{\alpha n}\Big) \Big] \cdot [u^{(1)}_\alpha, u^{(2)}, u^{(2)}_n, u^{(3)}_\alpha, u^{(4)}_\alpha]^T \cdot \\ & e^{-t} ds\, dt + \int_T [M_{ns}]^{C+}_{C-} u^{(2)} e^{-t} dt = 0, \end{aligned} \tag{3.15}$$

where last term indicates difference of lefthand and righthand limit of the square bracket at a certain point $c \in \partial\Gamma$. Here we use the definitions:

$$\frac{\partial}{\partial x_\alpha} = n_\alpha \frac{\partial}{\partial n} - n_\beta \frac{\partial}{\partial s}; \quad n = \cos(x_\alpha, n) \quad \alpha \neq \beta,$$

$$(3.16) \quad \begin{aligned} V &= n_\alpha \left(\frac{2}{3} h^3 N^{(2)}_{\alpha\beta} + \frac{2}{5} h^5 N^{(4)}_{\alpha\beta}\right)_{|\beta}; \quad N^{(l)}_{\alpha n} = N^{(l)}_{\alpha\beta} n_\beta, \\ M_{ns} &= \epsilon_{\alpha\beta} n_\alpha n_\delta \left(\frac{2}{3} h^3 N^{(2)}_{\delta\beta} + \frac{2}{5} h^5 N^{(4)}_{\alpha\beta}\right), \quad \alpha, \beta, \delta \in [1, 2] \\ M_{nn} &= n_\alpha n_\beta \left(\frac{2}{3} h^3 N^{(2)}_{\alpha\beta} + \frac{2}{3} h^5 N^{(4)}_{\alpha p}\right); \quad (.)_{|n} = \frac{\partial(.)}{\partial n}; \ (.)_{|s} = \frac{\partial(.)}{\partial s} \end{aligned}$$

n and s denote coordinates of normal and tangent direction to $\partial\Gamma$, resp., n denotes outer normal-vector on $\partial\Gamma$ and $\epsilon_{\alpha\beta}$ is permutation-symbol: $\epsilon_{12} = -\epsilon_{21} = 1$, $\epsilon_{11} = \epsilon_{22} = 0$.

Conclusion from (3.15) is, that for the chosen model all (vector-or scalarvalued) elements of supervector containing statical quantities have to be equal to zero for arbitrary admissible conjugate displacement-representatives in the integral over Γ. On $\partial\Gamma$ conditions of statical admissibility depend on the support of the plate. Necessary for the vanishing of the integral over $\partial\Gamma$ is, that the product of conjugate statical and kinematical quantities vanishes, what permits, as weaking of the introductory assumptions, mixed boundary-conditions.

Imposure of constraints to deformations is quite arbitrary as long as physically motivated. For example, in order to obtain from (3.14) a plate-model fulfilling Kirchhoff-Love-hypothesis, we impose on deformation-representative u^q the constraint:

$$(3.17) \qquad \bar{u}^{(2)}_\alpha = \bar{u}^{(l)}_\alpha = -u^{(2)}_{|\alpha}, \quad \bar{u}^{(1)}_\alpha = u^{(1)}_\alpha, \quad l = 2, 3, \dots q; \quad l \neq 1.$$

Then, after performing the same calculation as previously, we obtain instead of (3.15) the expression:

$$(3.18) \quad \begin{aligned} \langle\langle \boldsymbol{n}, \boldsymbol{q}^k \rangle\rangle = &- \int_T \left[\left(2hN^{(1)}_{\alpha\beta,\beta} + \frac{2}{3} h^3 N^{(3)}_{\alpha\beta,\beta}\right), \left(\frac{2}{3} h^3 N^{(3)}_{\alpha\beta,\beta\alpha} + \frac{2}{5} h^5 N^{(4)}_{\alpha\beta,\beta\alpha}\right)\right] \cdot \\ &[u^{(1)}_\alpha, u^{(2)}]^T e^{-t} dx_1 dx_2 dt + \int_{\partial\Gamma} \left[\left(2hN^{(1)}_{\alpha n} + \frac{2}{3} h^3 N^{(3)}_{\alpha n}\right), \right. \\ &\left.(V + m_{ns,s}), M_{nn}\right] \cdot [u^{(1)}_\alpha, u^{(2)}, u^{(2)}_{|n}]^T e^{-t} ds dt + \int_T [M_{ns}]^{C+}_{C-} u^{(2)} e^{-t} dt = 0, \end{aligned}$$

with definitions according to (3.16).

Inserting statically admissible stress-tensors determined in this way according to the chosen plate-model into the two-dimensional functional

$$(3.19) \qquad \tilde{\Lambda}_0(\boldsymbol{n}^s) = \sup_{n^s \in n^0 - \tilde{E}_t \cap H_s} \langle\langle \boldsymbol{n}^s - \boldsymbol{n}^{s*}, \boldsymbol{G} \cdot \dot{\boldsymbol{n}}^s \rangle\rangle \quad \boldsymbol{n}^s \in \boldsymbol{n}^0 - \tilde{E}_t \cap \boldsymbol{H}_s$$

where E_t denotes convex region of admissible generalized stresses s, expressed by two-dimensional representatives and $\hat{\boldsymbol{n}}^0$ denotes given purely elastic solution of the problem.

Stress-representative $\boldsymbol{n}$ of the researched state of stress in the elastic-plastic plate is then given by the superposition

$$\hat{\boldsymbol{n}} = \boldsymbol{n}^0 - \hat{\boldsymbol{n}}^s \quad \text{on } \Gamma, \tag{3.20}$$

where functional $\tilde{A}(\boldsymbol{n}^s)$ attains uniquely the minimum of value zero for the function $\hat{\boldsymbol{n}}^s$. If such function does not exist, also the solution of the problem does not exist.

4. Numerical example

A quadratic, homogeneous, on entire boundary $\partial\Gamma$ simply supported, elastic-perfectly plastic plate is proportionally loaded by a distributed force $\tilde{q}$ acting orthogonally on the midspan of the plate. Load is given by the function

$$\tilde{q}(\tilde{x}_1, \tilde{x}_2, t) = q_0(t)\cdot \cos\left(\frac{\pi}{2a}\tilde{x}_1\right)\cos\frac{\pi}{2a}\tilde{x}_2\Big), \tag{4.2}$$

where q_0 is the controlling parameter of the loading.

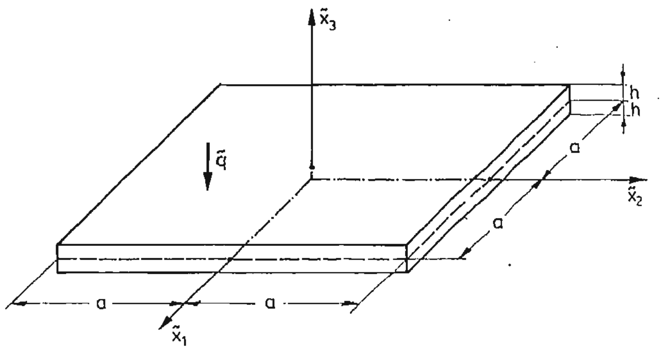

Fig. 2

In the following we use dimensionless quantities

$$x_\alpha = \frac{\tilde{x}_\alpha}{2a}; \quad x_3 = \frac{\tilde{x}_3}{2h},$$

$$q = \frac{\tilde{q}}{E}\left(\frac{a}{2h}\right)^4; \quad \sigma_s = \frac{\tilde{\sigma}_s}{E}, \quad N_{\alpha\beta} = \frac{\tilde{N}_{\alpha\beta}}{Ea^2}\left(\frac{a}{2h}\right)^4.$$

For this problem purely elastic solution is given by [1]:

$$\begin{aligned} N^{0(2)}_{\alpha\beta} &= \frac{a^2 q_0}{\pi^2}(1+\nu)\sin\left[\frac{\pi}{2}(x_1+1)\right]\sin\left[\frac{\pi}{2}(x_2+1)\right] \quad && \alpha = \beta \\ N^{0(2)}_{\alpha\beta} &= \frac{a^2 q_0}{\pi^2}(1-\nu)\cos\left[\frac{\pi}{2}(x_1+1)\right]\cos\left[\frac{\pi}{2}(x_2+1)\right] \quad && \alpha \neq \beta \end{aligned} \tag{4.2}$$

with ν as Poissons ratio. Here we use stress-representatives N up to order $q = 2$ and choose as test-functions:

$$
(4.3)\qquad
\begin{aligned}
N_{11}^{(2)} &= c_1(1-x_1^2)(1-x_2^2)+c_2(1-x_1^2)(1-x_2^4),\\
N_{22}^{(2)} &= c_3(1-x_2^2)(1-x_1^2)+c_4(1-x_2^2)(1-x_1^4),\\
N_{12}^{(2)} = N_{21}^{(2)} &= c_5\left[2x_1x_2-\frac{1}{3}(x_1^3x_2+x_1x_2^3)\right]+c_6\left[2x_1x_2-\frac{1}{5}(x_1^5x_2+x_1x_2^5)\right].
\end{aligned}
$$

with the set $[c_1, c_2, \ldots, c_6]$ as free parameters. After fulfilling conditions of symmetry and condition (3.11) of statical admissibility (4.3) reduces to:

$$
(4.4)\qquad
\begin{aligned}
N_{\alpha\beta}^{s(2)} &= c_1(1-x_\alpha^2)(1-x_\beta^2)-c_2(1-x_\alpha^2)(1-x_\beta^4) \qquad \alpha=\beta\\
N_{\alpha\beta}^{s(2)} &= c_1\left(x_\alpha-\frac{1}{3}x_\alpha^3\right)\left(x_\beta-\frac{1}{3}x_\beta^3\right)+c_2\left(x_\alpha-\frac{1}{5}x_\alpha^5\right)\left(x_\beta-\frac{1}{5}x_\beta^5\right) \qquad \alpha\neq\beta
\end{aligned}
$$

with only two free parameters c_1 and c_2, which are subjected to the minimization-process of functional $\tilde{\Lambda}_0$, which reduces now to a function of parameters c_1 and c_2:

$$
(4.5)\qquad
\begin{aligned}
\tilde{\Lambda}_0(c_1, c_2) = \sup_{n^s(c_1^*c_2^*)\in n^0-\tilde{E}_t\cap H_s} &[(c_1^2-c_1c_1^*)\cdot 4{,}01468+(c_2^2-c_2c_2^*)\cdot 6{,}01351+\\
&+(2c_1c_2-c_1c_2^*-c_2c_1^*)\cdot 5{,}21133]; \qquad n^s(c_1, c_2)\in n^0-\tilde{E}_t\cap H_s
\end{aligned}
$$

Here ν was chosen $\nu = 0.3$.

We describe region $\tilde{E}_t\cap_s H$ by Tresca-and von-Mises-yield conditions:
Tresca-yield-condition:

$$
(4.6)\qquad
\begin{aligned}
&(N_{11}^{s(2)}-N_{22}^{s(2)})^2+4N_{12}^{s(2)2} \leqslant N_\sigma^{(2)2} \qquad \text{if} \quad N_{11}^{s(2)}N_{22}^{s(2)} \leqslant N_{12}^{s(2)2}\\
&\frac{1}{2}(N_{11}^{s(2)}+N_{22}^{s(2)})\pm\left[\frac{1}{4}(N_{11}^{s(2)}-N_{22}^{s(2)})^2+N_{12}^{s(2)2}\right]^{1/2} \leqslant N_\sigma^{(2)} \quad \text{if} \quad N_{11}^{s(2)}N_{22}^{s(2)} > N_{12}^{s(2)2}
\end{aligned}
$$

von-Mises-yield-condition:

$$
(4.7)\qquad N_{11}^{s(2)2}+N_{22}^{s(2)2}-N_{11}^{s(2)}N_{22}^{s(2)}+3N_{12}^{s(2)2} \leqslant N_\sigma^{(2)2}
$$

where $N_\sigma^{(2)}$ is defined by $\frac{2h^2}{3}\sigma_\sigma$, with σ_s as stress-limit of uniaxial tension-test. Practically this means, that limit for two-dimensional stress-representative is reached (in uniaxial case), when yieldings in upper and lower planes Γ^+, Γ^- starts. From the minimization of function $\tilde{\Lambda}_0$ we obtain numerically results for different loading-parameters q_0, namely:

q_0	c_1	c_2	
1.5	0.1933	−0.2708	V. Mises
2.5	−0.3327	0.1234	,,
1.5	0.1575	−0.2351	Tresca
2.5	−0.3673	0.1580	,,

Fig. 3

In figure 5 the shape of regions of admissible parameters c_1, c_2 are drawn in c_1-c_2-plane for two values of loading-parameter q_0. The inner domain is in both cases related to Tresca yield-criterion and the outer domain to von Mises-yield-criterion. The vectors C indicate the position of minimizing parameters $\hat{c}_1$, $\hat{c}_2$. For increasing load-parameter q_0 the region of admissible parameters c_1 and c_2 becomes smaller and vanishes beyond a critical value q_0^* such that no solution of the problem in the chosen space of test-functions

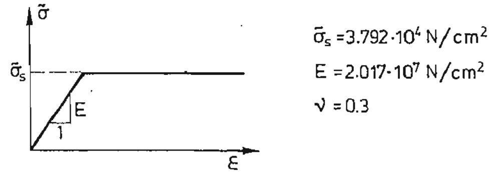

Fig. 4. Uniaxial stress-strain diagram of the considered material.

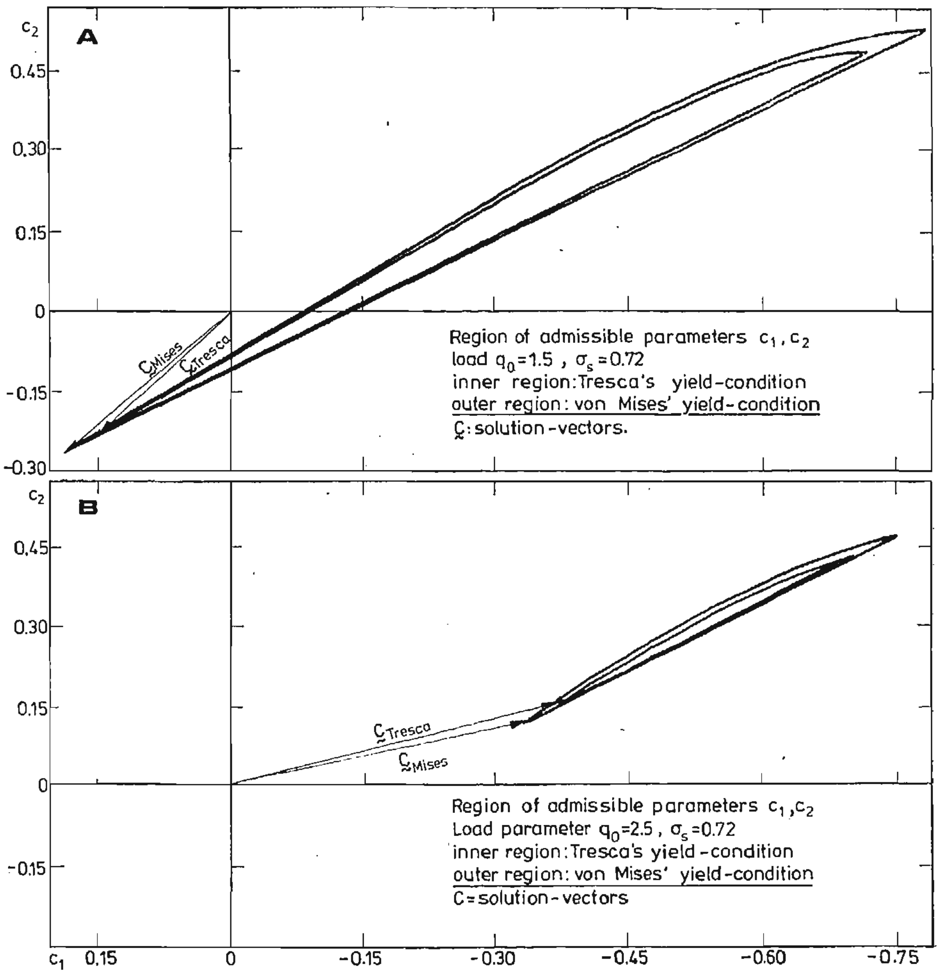

Fig. 5

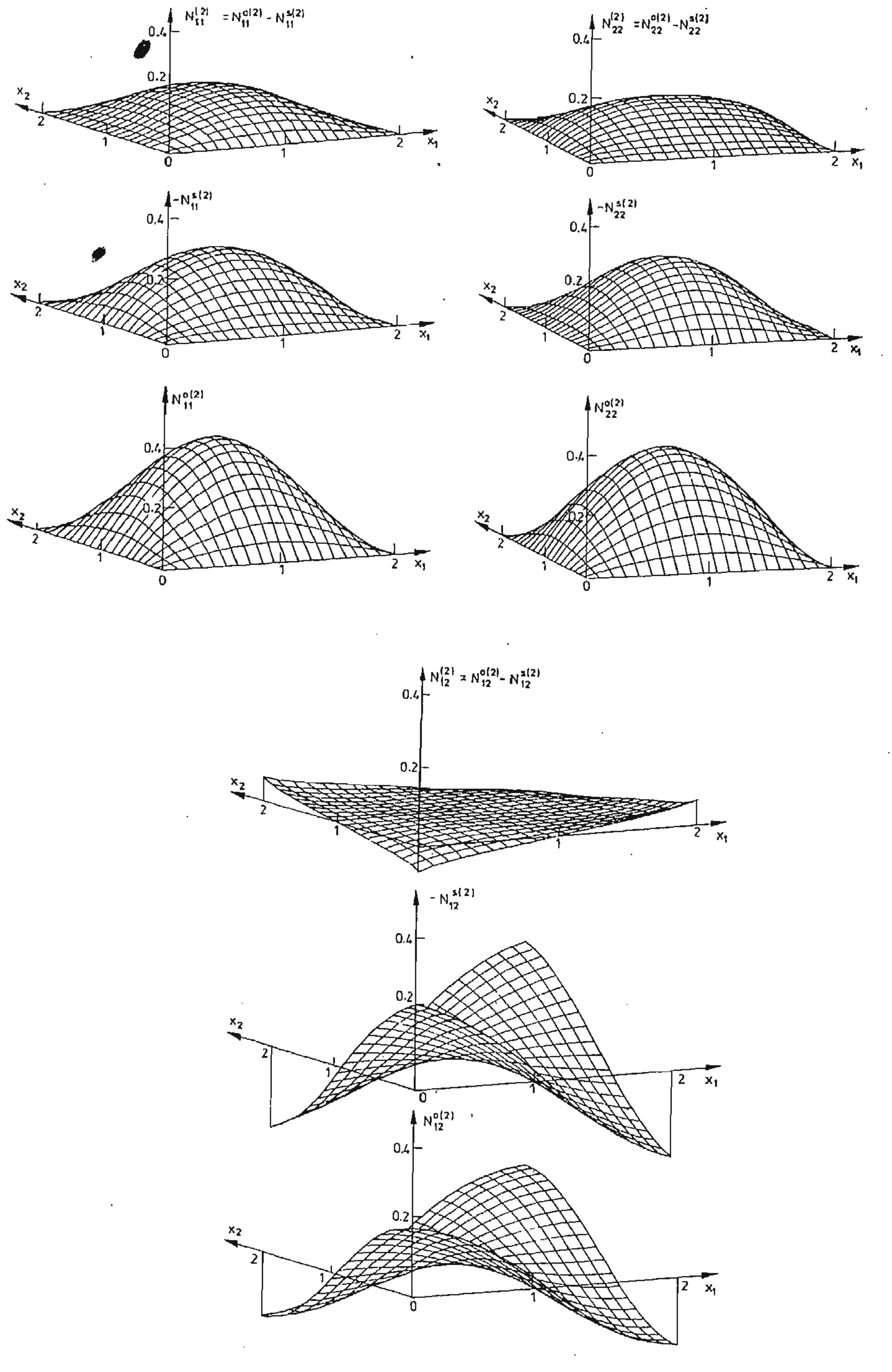
$N_{11}^{(2)} = N_{11}^{o(2)} - N_{11}^{s(2)}$
$N_{22}^{(2)} = N_{22}^{o(2)} - N_{22}^{s(2)}$
$-N_{11}^{s(2)}$
$-N_{22}^{s(2)}$
$N_{11}^{o(2)}$
$N_{22}^{o(2)}$
$N_{12}^{(2)} = N_{12}^{o(2)} - N_{12}^{s(2)}$
$-N_{12}^{s(2)}$
$N_{12}^{o(2)}$
0.4
0.2
x_1
x_2
0
1
2

Fig. 6

exists for $q_0 \geqslant q_*^0$. In figure 6 the distribution of purely elastic solution N^0, of the minimizing statical admissible stress-representative $\hat{N}^s$ and of solution $\hat{N}$ of the problem as superposition of N^0 and $\hat{N}^s$ are sketched qualitatively in $x_1 - x_2$-plane.

Literature

1. Timoschenko, Woinowski-Krieger, *Theorie of plates and shells*, Mc Graw-Hill Book company, New-York, 1959.
2. P. Rafalski, *Solution of the elastic-viscoplastic boundary-value problem*, Int. J. Engng. Sci., 15,193, 1977.
3. J. J. Moreau, *Fonctionelles convexes*, Séminaire sur les équations aux derivées partielles, 1966 - 67, Collège de France, Paris.
4. B. Nayroles, *Quelques applications variationelles de la théorie des fonctions duales à la mécanique des solides*, J. de Mécanique, Vol 10, N° 2, 1971.
5. B. Halphen, Nguyen Quoc Son, *Sur les materiaux standard generalisés*, J. de Mécanique, 14, pp. 39 - 63, 1965.
6. Cz. Woźniak, *Large deformations of elastic and non-elastic plates*, shells and rods, Mitteilungen aus dem Institut fur Mechanik 20, Bochum 1980.
7. D. Weichert, *Variational formulation and solution of boundary-value problems in the theory of plasticity and application to plate-problems*, Mitteilungen aus dem Institut fur Mechanik 25, Bochum, 1981.

Р е з ю м е

ЗАКОН МИНИМУМ ДЛЯ НЕПРАЖЕННОГО СОСТОЯНИЯ В УПРУГО-ПЛАСТИЧЕСКИХ ПЛИТАХ И СООТВЕТСТВУЮЩИЙ МОДЕЛЬ ПЛИТЫ

Напряженное состояние в плитах под действием истории нагрузки определяется применяя экспериментальные теоремы к задачам в рамках геометрически линейной теории. Предлагается, что плиты трехмерные с геометрическими связями обоснованными физически. Работа иллюстрирована численным примером.

S t r e s z c z e n i e

ZASADA MINIMUM DLA STANU NAPRĘŻENIA W PŁYTACH SPRĘŻYSTO-PLASTYCZNYCH ORAZ DYSKUSJA STOSOWNEGO MODELU PŁYT

Stan naprężenia w płycie pod działaniem dowolnych historii obciążenia wyznaczono przez zastosowanie twierdzeń ekstremalnych do zagadnień w ramach teorii geometrycznej liniowej. Przyjęto, że płyty są trójwymiarowe z nałożonymi fizycznie uzasadnionymi więzami geometrycznymi. Pracę uzupełnia przykład liczbowy.

Praca została złożona w Redakcji dnia 15 października 1981 roku