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APPLICATION OF THE METHOD OF FUNDAMENTAL SOLUTIONS WITH THE LAPLACE TRANSFORMATION FOR THE INVERSE TRANSIENT HEAT SOURCE PROBLEM

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The paper deals with the inverse determination of heat source in an unsteady heat conduction problem. The governing equation for the unsteady Fourier heat conduction in 2D region with unknown internal heat source is known as the inverse boundary-initial-value problem. The identification of strength of the heat source is achieved by using the boundary condition, initial condition and a known value of temperature in chosen points placed inside the domain. For the solution of the inverse problem of determination of the heat source, the Laplace transformation with the method of fundamental solution and radial basis functions is proposed. Due to ill conditioning of the inverse transient heat conduction problem, the Tikhonov regularization method based on SVD and L-curve criterion was used. As the test problems, the 2D inverse boundary-initial-value problems (2D_IBIVP) in region Ω with known analytical solutions are considered.

Key words: method of fundamental solutions, radial basis functions, inverse transient heat source problem

1. Introduction

The problem of inverse heat sources presents an interesting challenge in many areas of engineering where the strength of heat sources is not exactly recognized. It belongs to a broad class of inverse heat conduction problems which are usually ill-posed because low random errors in measurements can lead to big errors in identifications. Direct heat conduction problems on the other hand are well-posed. So far, many different methods have been applied to solve the inverse heat source problem, and in some papers, the problem of uniqueness was considered, e.g. Ling *et al.* (2006). Generally speaking, the papers published in this field can be divided into two groups. The first group consists of papers in which point or line heat sources are considered, e.g. Karami and Hematiyan (2000). The second group is concerned with the sources as a continuous function in a considered region, e.g. Jin and Marin (2007). The majority of authors consider the transient heat conduction problem, e.g. Yan *et al.* (2008), but some authors analyze the steady state heat conduction problem, e.g. Kołodziej *et al.* (2010). Among numerical methods, the conjugate gradient method coupled with the finite difference method, e.g. Le Niliot and Lefevre (2001), finite difference method (FDM) for discretization of the time term and the finite element method (FEM) for discretization of the space term, e.g. Yang (1998), finite difference with the discrete mollification method, e.g. Yi and Murio (2004), finite difference coupled with other special methods, e.g. Yan *et al.* (2010), and in the last decade, boundary methods were used. Generally speaking, the boundary methods can be divided into two groups: boundary element methods (BEM), e.g. Karami and Hematiyan (2000), and the method of fundamental solutions (MFS) (Kołodziej *et al.*, 2010).

The MFS is a relatively new meshless method for solution of certain boundary value problems, and was first proposed in the 60ties by Kupradze and Aleksidze. In recent years, it has become increasingly popular because of its simplicity of implementation to problems of complicated geometries. The solution is approximated by linear combinations of fundamental solutions in terms of singularities which are placed on a fictitious boundary lying outside the considered domain. Due to this, an approximated solution is given as a continuous function with continuous derivatives which is very convenient in the case of inverse problems. These are the basic advantages of this method in comparison with BEM.

In papers by Alves *et al.* (2008), Jin and Marin (2007), and Kołodziej *et al.* (2010), the authors employed the method of fundamental solutions to recover the heat source in steady state heat conduction problems. In papers by Alves *et al.* (2008), and Jin and Marin (2007), the problem is solved using the a priori information that the source is harmonic or satisfies the Helmholtz equation, and for the reconstruction of source the Cauchy data was used. On the other hand, in the paper by Kołodziej *et al.* (2010), the identification of source in the steady state heat conduction problem was considered without restriction for the form of heat source, but some information about temperature in some inner point in the considered region was used. In the paper by Yan *et al.* (2008), a 1D transient heat conduction problem was considered in which heat source is taken to be time-dependent only. The identification of a heat source dependent only on space was considered in the paper by Yan *et al.* (2009) for 1D and 2D transient heat conduction problems. In papers by Yan *et al.* (2008, 2009), the authors used MFS with the fundamental solution to the diffusion equation. The identification of heat source dependent on space and time was considered in paper by Mierzwiczak and Kołodziej (2010) using the MFS with discretization of time derivative by θ-method.

For the inverse heat transfer problem other than the inverse source problem, this method was applied for boundary heat flux (Hon and Wei, 2004), Cauchy problem (Marin, 2005), or identification of heat transfer conductivity (Mierzwiczak and Kołodziej, 2011).

The goal of this paper is to apply the boundary collocation technique (Kołodziej and Zieliński, 2009) to the identification of 2D heat sources in a transient case. In our work, the type of heat sources considered is not limited by any restrictions with the exception that it is a continuous function of space and time. Contrary to the paper by Yan *et al.* (2008) where sources depended only on time, and the paper by Yan *et al.* (2009) where the sources were exclusively space dependent, in this paper the source is a function of space and time. Using the Laplace transform, the transient heat conduction problem can be expressed by inhomogeneous modified Helmholtz equations with adequate conditions in the Laplace transfer domain (the s-domain). The problem is then solved by solving the non-homogeneous equation for each s-parameter. The radial basis function (RBF) is used for interpolation of the right- hand side of the governing equation and obtaining a particular solution. Several numerical examples for the inverse source problem are presented to demonstrate the efficacy of the proposed method.

2. Mathematical formulation of the problem

Consider a general two-dimensional homogeneous isotropic region Ω with boundary $\partial\Omega$. We want to find the non-dimensional function $q(\xi,\eta,\tau)$ of heat sources and the non-dimensional temperature field $T(\xi,\eta,\tau)$, which is governed by

$$\begin{aligned}
&\frac{\partial T}{\partial \tau}(\xi,\eta,\tau) = \nabla^2 T(\xi,\eta,\tau) + q(\xi,\eta,\tau) && \text{for} \quad (\xi,\eta)\in\Omega\\
&T(\xi,\eta,0) = T_0(\xi,\eta) && \text{for} \quad (\xi,\eta)\in\Omega\\
&a_1 T(\xi,\eta,\tau) + a_2\frac{\partial T(\xi,\eta,\tau)}{\partial n} = a_3 && \text{for} \quad (\xi,\eta)\in\partial\Omega
\end{aligned} \tag{2.1}$$

where $\nabla^2 = \partial^2/\partial\xi^2 + \partial^2/\partial\eta^2$ is the Laplace operator, n outward normal to the boundary, a_1, a_2, a_3 are the specified functions of position, τ is non-dimensional time, $T_0(\xi,\eta)$ is the specified initial non-dimensional temperature distribution.

The MFS can be used to solve the initial-boundary value problem formulated by Eq. $(2.1)_1$, initial condition $(2.1)_2$ and boundary condition $(2.1)_3$. Because the source function $q(\xi,\eta,\tau)$ is unknown, the problem expressed by equations (2.1) is an inverse problem and requires an additional condition. This extra condition is provided by the known temperature in a few points placed inside the domain

$$T(\xi_i,\eta_i,\tau) = T_i \qquad \text{for} \qquad \{(\xi_i,\eta_i)\}_{i=1}^{M} \in \Omega \tag{2.2}$$

3. MFS with Laplace transformation

Taking the Laplace transform of equation $(2.1)_1$, we obtain a new differential equation in the Laplace transfer domain (the s-domain)

$$\nabla^2\overline{T}(\xi,\eta,s) - s\overline{T}(\xi,\eta,s) = -T_0(\xi,\eta) - \overline{q}(\xi,\eta,s) \qquad \text{for} \qquad (\xi,\eta) \in \Omega \tag{3.1}$$

where $\overline{T}(\xi,\eta,s)$ is the Laplace transform of $T(\xi,\eta,\tau)$ with respect to dimensionless time τ

$$\overline{T}(\xi,\eta,s) = \int\limits_0^\infty \mathrm{e}^{-s\tau}T(\xi,\eta,\tau)\;d\tau \tag{3.2}$$

The Laplace transform of the boundary condition in Eq. $(2.1)_3$ is

$$a_1\overline{T}(\xi,\eta,s) + a_2\frac{\partial\overline{T}(\xi,\eta,s)}{\partial n} = \frac{a_3}{s} \tag{3.3}$$

and of additional condition (2.2)

$$\overline{T}(\xi_i,\eta_i,s) = \frac{T_i}{s} \qquad\qquad \{(\xi_i,\eta_i)\}_{i=1}^{M} \in \Omega \tag{3.4}$$

Now observe that (3.1) is a sequence of the inhomogeneous modified Helmholtz equation

$$\nabla^2\overline{T}(\xi,\eta,s) - s\overline{T}(\xi,\eta,s) = f(\xi,\eta,s) \tag{3.5}$$

The right hand side in the modified Helmholtz equation $f(\xi,\eta,s) = f(T_0(\xi,\eta),\overline{q}(\xi,\eta,s))$ is unknown, because the heat source function $\overline{q}(\xi,\eta,s)$ is unknown. The solution to Eq. (3.5) is the sum $\overline{T} = \overline{T}^{(h)} + \overline{T}^{(p)}$ of the homogeneous solution $\overline{T}^{(h)}$, and the particular solution $\overline{T}^{(p)}$

$$\nabla^2\overline{T}^{(h)} - s\overline{T}^{(h)} = 0 \qquad\qquad \nabla^2\overline{T}^{(p)} - s\overline{T}^{(p)} = f(\xi,\eta,s) \tag{3.6}$$

The approximate solution to Eq. $(3.6)_1$ is a superposition of fundamental solutions to the modified Helmholtz equation with the unknown coefficients $\{W_j\}_{j=1}^{NS}$

$$\overline{T}^{(h)}(\xi,\eta,s) = \sum_{j=1}^{NS} W_j K_0(\widetilde{r}_j\sqrt{s}) \tag{3.7}$$

where $\widetilde{r}_j = \sqrt{(\xi-\widetilde{\xi}_j)^2 + (\eta-\widetilde{\eta}_j)^2}$, K_0 is the modified Bessel function of the second kind and zero order and $\{(\widetilde{\xi}_j,\widetilde{\eta}_j)\}_{j=1}^{NS}$ are coordinates of source points which are located outside the region Ω. In order to find the particular solution for non-homogeneous equation $(3.6)_2$, the right hand side of the modified Helmholtz equation $f(\xi,\eta,s)$ should be interpolated by means of the RBFs and mononomials

$$f(\xi,\eta,s)=\sum_{m=1}^{M}\alpha_m\widehat{\varphi}(r_m)+\sum_{k=1}^{K}\beta_k\widetilde{\varphi}_k(\xi,\eta) \tag{3.8}$$

where $r_m=\sqrt{(\xi-\xi_m)^2+(\eta-\eta_m)^2}$ is the argument of RBF, $\varphi(r_m)=r_m^2\ln(r_m)$ is a thin plate spline function, $\{\widetilde{\varphi}_k(\xi,\eta)\}_{k=1}^{K}$ are mononomials given in Table 1 (K is the number of polynomials) and $\{(\xi_m,\eta_m)\}_{m=1}^{M}$ are the interpolation points placed inside the region Ω (Fig. 1). The unknown coefficients $\{\alpha_m\}_{m=1}^{M}$, $\{\beta_k\}_{k=1}^{K}$ are connected to the unknown source function by a system of linear equations

$$\begin{aligned}&\sum_{m=1}^{M}\alpha_m\widehat{\varphi}(r_{mi})+\sum_{k=1}^{K}\beta_k\widetilde{\varphi}_k(\xi_i,\eta_i)=f(\xi_i,\eta_i,s) && 1\leqslant i\leqslant M\\ &\sum_{m=1}^{M}\alpha_m\widetilde{\varphi}_k(\xi_m,\eta_m)=0 && 1\leqslant k\leqslant K\end{aligned} \tag{3.9}$$

where $r_{mi}=\sqrt{(\xi_i-\xi_m)^2+(\eta_i-\eta_m)^2}$ and $f(\xi_i,\eta_i,s)$ the function of the right hand side is unknown.

Table 1. Form of mononomials and their particular solutions

k	1	2	3	4	5	6
$\widetilde{\varphi}_k(\xi,\eta)$	1	ξ	η	$\xi\eta$	ξ^2	η^2
$\widetilde{\psi}_k(\xi,\eta,s)$	$-\dfrac{1}{s}$	$-\dfrac{\xi}{s}$	$-\dfrac{\eta}{s}$	$-\dfrac{\xi\eta}{s}$	$-\dfrac{\xi^2}{s}-\dfrac{2}{s^2}$	$-\dfrac{\eta^2}{s}-\dfrac{2}{s^2}$

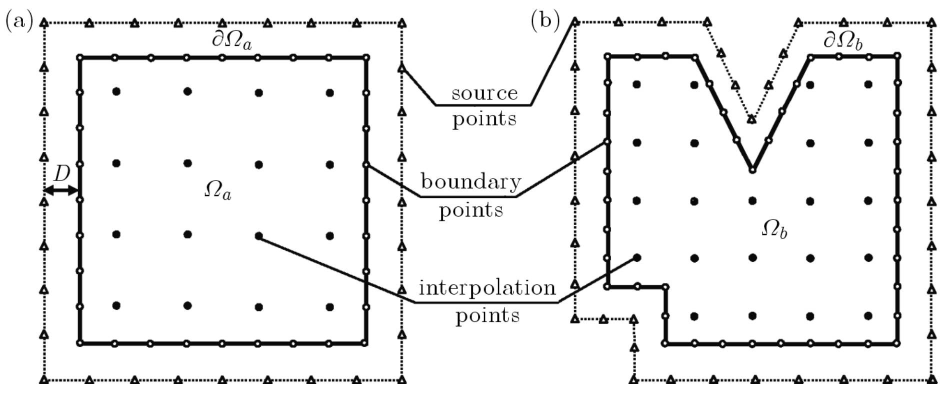

Fig. 1. Location ∘ collocation, △ sources and • interpolation points

The particular solution to the non-homogeneous modified Helmholtz equation has form

$$\overline{T}^{(p)}(\xi,\eta,s)=\sum_{m=1}^{M}\alpha_m\widehat{\psi}(r_m,s)+\sum_{k=1}^{K}\beta_k\widetilde{\psi}_k(\xi,\eta,s) \tag{3.10}$$

where functions $\widehat{\psi}(r_m,s)$ and $\widetilde{\psi}_k(\xi,\eta,s)$ are solutions to equations

$$\begin{aligned}&\nabla^2\widehat{\psi}(r_m,s)-\lambda^2\widehat{\psi}(r_m,s)=\widehat{\varphi}(r_m) && 1\leqslant m\leqslant M\\ &\nabla^2\widetilde{\psi}_k(\xi,\eta,s)-\lambda^2\widetilde{\psi}_k(\xi,\eta,s)=\widetilde{\varphi}_k(\xi,\eta) && 1\leqslant k\leqslant K\end{aligned} \tag{3.11}$$

The functions $\widetilde{\psi}_k(\xi,\eta,s)$ are given in Table 1, whereas the function

$$\widehat{\psi}(r_m,s)=\begin{cases} -\dfrac{4}{s^2}[K_0(r_m\sqrt{s})+\ln(r_m)+1]-\dfrac{r_m^2\ln(r_m)}{s} & \text{for} \quad r_m>0 \\ \dfrac{4}{s^2}\Big[\gamma_{euler}+\ln\Big(\frac{\sqrt{s}}{2}\Big)-1\Big] & \text{for} \quad r_m=0 \end{cases} \tag{3.12}$$

where $\gamma_{euler}=0.57721$ is the Euler constant.

Benefiting from homogeneous solution (3.7), and particular solution (3.10), the solution to differential equation (3.5) can now be given in the following form

$$\overline{T}(\xi,\eta,s)=\overline{T}^{(h)}+\overline{T}^{(p)}=\sum_{j=1}^{NS}W_jK_0(\widetilde{r}_j\sqrt{s})+\sum_{m=1}^{M}\alpha_m\widehat{\psi}(r_m,s)+\sum_{k=1}^{K}\beta_k\widetilde{\psi}_k(\xi,\eta,s) \tag{3.13}$$

The unknown coefficients W_j, α_m, β_k are determined from boundary conditions (3.3) and added condition (3.4) – the known value of temperature in M-points placed inside the domain Ω necessary to solve the inverse value problem.

For each value of s, the collocation condition leads to a system of linear equations:
— for $\{(\xi_i,\eta_i)\}_{i=1}^{M}\in\Omega$

$$\sum_{j=1}^{NS}W_jK_0(\widetilde{r}_{ji}\sqrt{s})+\sum_{m=1}^{M}\alpha_m\widehat{\psi}(r_{mi},s)+\sum_{k=1}^{K}\beta_k\widetilde{\psi}_k(\xi_i,\eta_i,s)=\frac{T_i}{s} \tag{3.14}$$

— for $\{(\xi_i,\eta_i)\}_{i=M+1}^{M+NB}\in\partial\Omega$

$$\begin{aligned} a_{1i}\Big[&\sum_{j=1}^{NS}W_jK_0(\widetilde{r}_{ji}\sqrt{s})+\sum_{m=1}^{M}\alpha_m\widehat{\psi}(r_{mi},s)+\sum_{k=1}^{K}\beta_k\widetilde{\psi}_k(\xi_i,\eta_i,s)\Big]+\\ &+a_{2i}\Big[\sum_{j=1}^{NS}W_j\frac{\partial K_0(\widetilde{r}_{ji}\sqrt{s})}{\partial n}+\sum_{m=1}^{M}\alpha_m\frac{\partial\widehat{\psi}(r_{mi},s)}{\partial n}+\sum_{k=1}^{K}\beta_k\frac{\partial\widetilde{\psi}_k(\xi_i,\eta_i,s)}{\partial n}\Big]=\frac{a_{3i}}{s} \end{aligned} \tag{3.15}$$

— for $\{(\xi_m,\eta_m)\}_{m=1}^{M}\in\Omega$

$$\sum_{m=1}^{M}\alpha_m\widetilde{\varphi}_i(\xi_m,\eta_m)=0 \qquad i=1,2,\dots,K \tag{3.16}$$

where $N1=NB+M+K$ create an algebraic system of linear equations with $N2=NS+M+K$ unknown coefficients W_j, α_m, β_k, which can be written in matrix form as

$$\mathbf{A}\boldsymbol{\omega}=\mathbf{r} \tag{3.17}$$

where $\mathbf{A}$ is the $N1\times N2$ influence matrix, $\boldsymbol{\omega}$ is a column vector of unknown coefficients W_j, α_m, β_k, and $\mathbf{r}$ is a column vector containing the prescribed boundary, additional and interpolation conditions.

For each parameter s, the temperature field and the source function in the transformed domain (s-domain) in the NF field points at which we want the solution, can be calculated from equation (3.13) and from formula

$$\overline{q}(\xi,\eta,s)=-\sum_{m=1}^{M}\alpha_m\widehat{\varphi}(r_m)-\sum_{k=1}^{K}\beta_k\widetilde{\varphi}_k(\xi,\eta)-T_0(\xi,\eta) \tag{3.18}$$

For each NF inner point, the transformed solution of temperature and of sources must be inverted back to the time domain. We utilize a relatively accurate method which states that a function of dimensionless time $p(\tau)$ may be approximated by the following formula

$$p(\tau) = A + B\tau + \sum_{j=1}^{N} C_j \exp(-D_j\tau) \tag{3.19}$$

where A, B, C_j, D_j, $j = 1, \dots, N$ coefficients are evaluated by taking the Laplace transform of Eq. (3.19) and multiplying the result by the Laplace parameter s

$$sp(s) = A + \frac{B}{s} + \sum_{j=1}^{N} \frac{C_j}{1 + \frac{D_j}{s}} \tag{3.20}$$

We select a set of $N + 2$ linear algebraic equations in $N + 2$ coefficients A, B, and C_j, D_j, $j = 1, 2, \dots, N$. These equations are obtained by input of the value of transformed temperature or transformed strength of source for the field point to equation (3.20) for different values of the parameter s. Solving this system of linear equations for each inner point, we obtain A, B, C_j, D_j coefficients which allow us to calculate the value of temperature or strength of the source in s chosen point at different time increments using (3.19).

4. Numerical experiments

In order to validate the proposed numerical method, 2D numerical examples given in Table 2 are carried out. The accuracy of the method is verified by calculating the maximal and the root mean squared relative error

$$\begin{aligned} \delta_{MAX} &= \frac{\max_i\{|T_{i,ANAL} - T_{i,NUM}|\}}{\max_i\{T_{i,ANAL}\}} \\ \delta_{RMSE} &= \frac{\sqrt{\sum_{i=1}^{NN}(T_{i,ANAL} - T_{i,NUM})^2}}{\sqrt{\sum_{i=1}^{NN} T_{i,ANAL}^2}} \end{aligned} \tag{4.1}$$

for the source function and the temperature field in $NN = 200$ inner control points. The temperature $T_{i,ANAL}$ is the exact temperature in the i-th point estimated from the analytical solution (Table 2), and $T_{i,NUM}$ is the approximated temperature in the i-th point calculated using the numerical algorithm.

Table 2. The source function and the analytical solution of test examples

Example	$T(\xi,\eta,\tau) =$	$q(\xi,\eta,\tau) =$
1st	$[1 - \exp(-4\tau)][\cos(2\xi) + \cos(2\eta)]$	$4[\cos(2\xi) + \cos(2\eta)]$
2nd	$\tau \sin[\pi(\xi + \eta)/5]$	$(2\pi^2\tau/25 + 1)\sin[\pi(\xi + Y)/5]$
3rd	$\tau[(\xi - 6)^3 + (\eta - 6)^3]/6$	$(\xi^3 + \eta^3)/6 - 3(\xi^2 + \eta^2) + 18(\xi + \eta - 4)$ $- \tau(\xi + \eta - 12)$
4th	$\tau(\xi + \eta)/2$	$(\xi + \eta)/2$

All these examples are formulated in the unit square $\Omega_a = [0, 1] \times [0, 1]$ (Fig. 1a) with the exception of the fourth example, which is formulated in a polygonal area Ω_b, shown in Fig. 1b.

The form of the source function as well as the exact solutions are known. In the numerical solution, the boundary condition as well as the values of temperature in the chosen points of the region result from the exact solution. Inside the area Ω_a, $M = 25$ and the area Ω_b, $M = 22$ uniformly distributed interpolation points $\{(\xi_m, \mu_m)\}_{m=1}^{M}$ are chosen (see Fig. 1). The number of collocation points on the boundary $\partial\Omega_a$ or on the boundary $\partial\Omega_b$ is $NB = 44$. The number of source points equals the number of collocation points $NS = NB$, and they are placed on a contour geometrically similar to the contour of the boundary (see Fig. 1). The distance between the contour of the boundary and the contour of sources D equals 0.2.

If the exact data are considered, and if the number of equations resulting from collocation conditions (3.3) and (3.4) is greater than the number of unknowns $N1 > N2$, we obtain an overdetermined system of equations which can be solved in the least squared sense. If $N1 = N2$, the Gauss elimination method is used.

The results for the undisturbed data for all examples in the form of the root mean squared relative error of identification of the temperature field and the source function are shown in Fig. 2a and 2b, respectively.

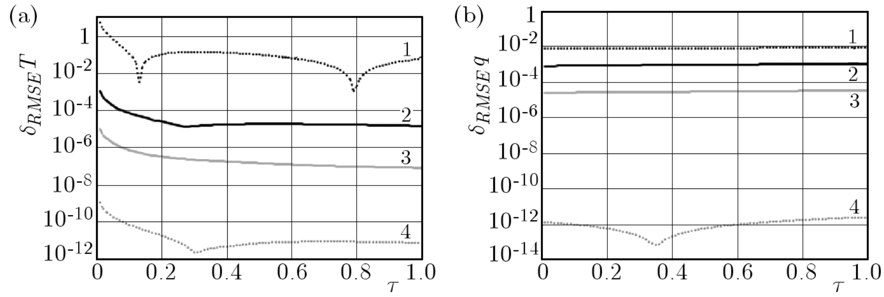

Fig. 2. Values of the root mean squared relative error of identification of (a) temperature field and (b) sources function for $M = 25$, $NB = 44$, $\Delta T = 0\%$

For the undisturbed data, the proposed algorithm is quite accurate. The analysis of the results shows that the unknown source function is identified much more accurately than the temperature field.

The worst results of identification were obtained for the first test example. The important advantage of the proposed method is that it allows one to determine the value of temperature of the source function at any point in the concerned area. This is possible because the temperature field and the source functions are approximated using the continuous functions.

The test calculations carried out for data from the known exact solutions are a certain idealization of reality. While solving the inverse heat conduction problem, the input data will come from the measurement, which is as we know, not exactly accurate, but disrupted by some error. In order to consider the measuring error, the exact data (from the analytical solution) are loaded with some random disturbance value $T_{MEAS} = T_{ANAL}(1 + RN\Delta T)$, where $\Delta T = 1\%$ is the disturbance coefficient, and $RN \in [-1, 1]$ is a random number. The results for the disturbed data $\Delta T = 1\%$ for the all examples in form of the root mean squared relative error of identification of the temperature field and the source function are shown in Fig. 3a and 3b, respectively.

For the disturbance data, the collocation matrix $\mathbf{A}$ is highly ill-conditioned, and some regularization techniques are needed. In this paper, the Tikhonov regularization method based on the singular value decomposition (SVD) is used. The Tikhonov regularization solution $\boldsymbol{\omega}_\lambda$ for problem (3.14)-(3.16) is defined as the solution to the following least-square problem

$$\min_{\omega_\lambda}\{\|\mathbf{A}\boldsymbol{\omega}_\lambda - \mathbf{r}\| + \lambda^2\|\boldsymbol{\omega}_\lambda\|\}$$

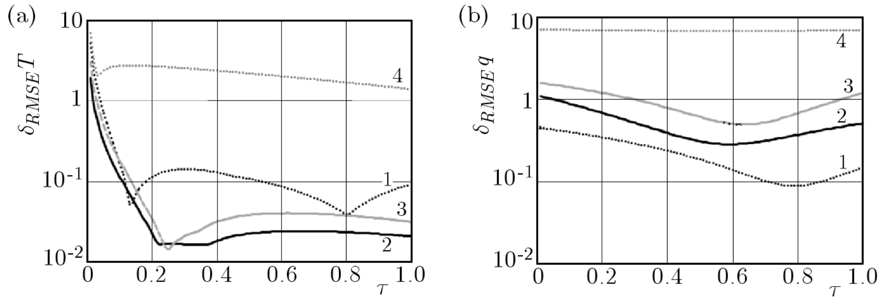

Fig. 3. Values of the root mean squared relative error of identification of (a) temperature field and (b) source function for $M = 25$, $NB = 44$, $\Delta T = 1\%$ (one regularization)

where $\|\cdot\|$ denotes the 2-norm and λ is a regularization parameter. The regularization solution can be then expressed as

$$\boldsymbol{\omega}_\lambda = \sum_{i=1}^{L} f_i \frac{w_i^{\mathrm{T}}\mathbf{r}}{\sigma_i} v_i$$

where

$$\mathbf{A} = \mathbf{W}\boldsymbol{\Sigma}\mathbf{V}^{\mathrm{T}} = \sum_{i=1}^{L} w_i \sigma_i v_i^{\mathrm{T}}$$

and the filter factors are $f_i = \sigma_i^2/(\sigma_i^2 + \lambda^2)$. In our computations, the L-curve criterion given by Hansen and O'Leary (1993) for choosing the regularization parameter λ was used. The suitable regularization parameter λ is the one that corresponds to the regularized solution near the corner of the L-curve: $\{(\log \|\mathbf{A}\boldsymbol{\omega}_\lambda - \mathbf{r}\|, \log \|\boldsymbol{\omega}_\lambda\|), \lambda > 0\}$. The coefficients W_j, α_m, β_k obtained for each s-parameter for the optimal regularization parameter enable determination of the value of temperature field (3.13), and the source function given by equation (3.18) in the s-domain.

Then, using formula (3.19), we can determine the value of the temperature field or strength of sources at any point in the considered area.

The results of identification of the temperature field and the source function for the disturbed data $\Delta T = 1\%$ and for the Tikhonov regularization for all the test examples are shown in Fig. 4a and 4b, respectively.

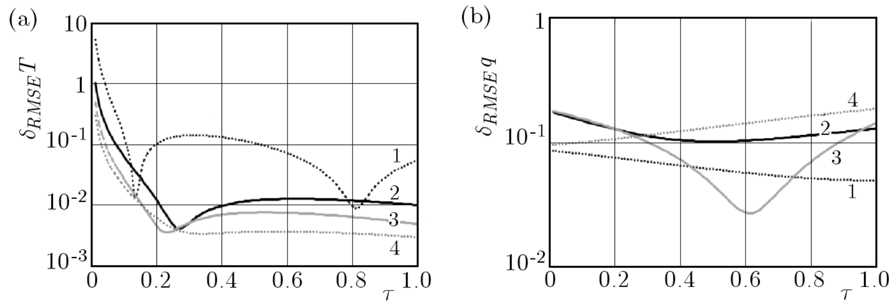

Fig. 4. Values of the root mean squared relative error of identification of (a) temperature field and (b) source function for $M = 25$, $NB = 44$, $\Delta T = 1\%$ (with Tikhonov regularization)

Figure 5 presents examples of L-curves with the designated regularization parameter located in the corner of this curve for all the test examples.

Figure 6 shows the impact of the disturbance coefficient ΔT on the value of the maximum and root mean squared relative error of the identification of the temperature fields and heat sources for the third test example, when $\tau = 0.8$.

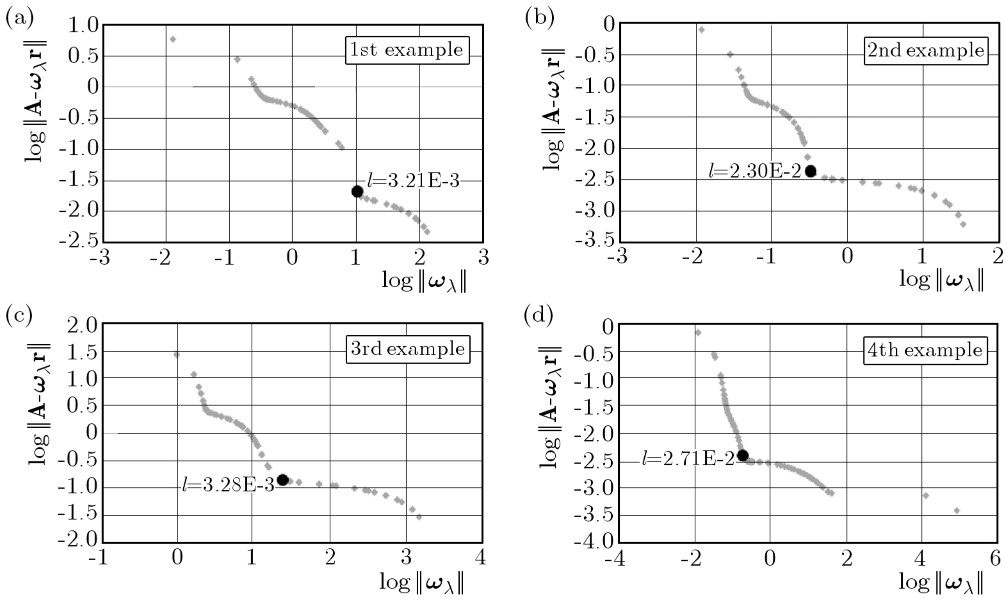

Fig. 5. L-curve with the regularization parameter λ located near the corner for all test examples and for $\varDelta T = 1\%$

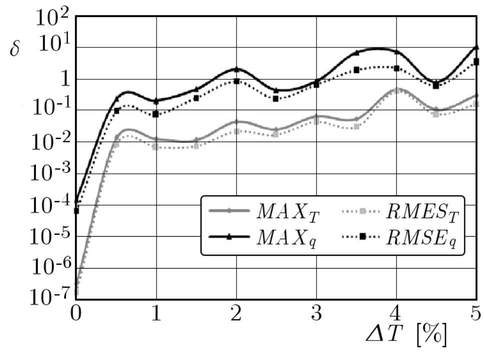

Fig. 6. Maximal and root mean squared relative error of the identification of the temperature field and the source function for 3rd test example as a function of the number of the disturbance coefficient $\varDelta T$ [%]

As it was expected, with increasing values of the disturbance coefficient $\varDelta T$, the relative error of identification increases.

In order to demonstrate the impact of the values of disturbance coefficient $\varDelta T$ on the quality of the results, two test examples were considered. Figure 7 presents the results of the test comparison for the second example and Fig. 8 for the fourth example. Both figures clearly show an increase in the error identification, taking into account the measurement error. For the disturbance coefficient of data $\varDelta T = \{1\%, 2\%, 3\%\}$, the results are comparable.

The MFS used in the computing algorithm depends on the number of interpolation points, collocation and source points and the distance between the contour of the boundary and the contour of sources D. Therefore, we decided to present the effect of these parameters on the quality of results of identification of the temperature field and the heat source function for the third test example at the time $\tau = 0.8$.

The dependence of the maximum error and root mean square error on the number of collocation points is shown in Fig. 10 and on the interpolation points in Fig. 9. In the case of the interpolation points, a regular distribution of these points in the region $\varOmega$ and the random distribution was considered.

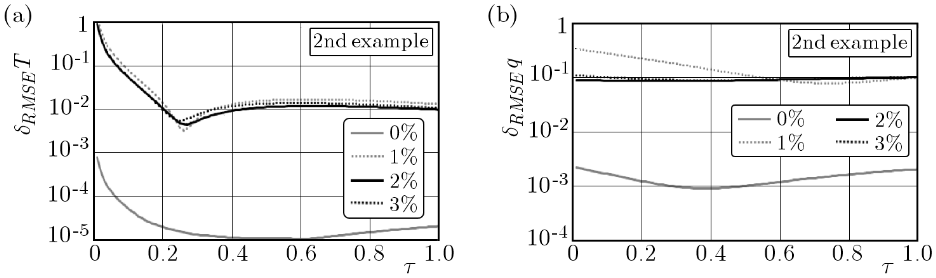

Fig. 7. Root mean squared relative error of the identification of (a) temperature field and (b) source function for the 2nd test different value of the disturbance coefficient $\Delta T = \{1\%, 2\%, 3\%\}$

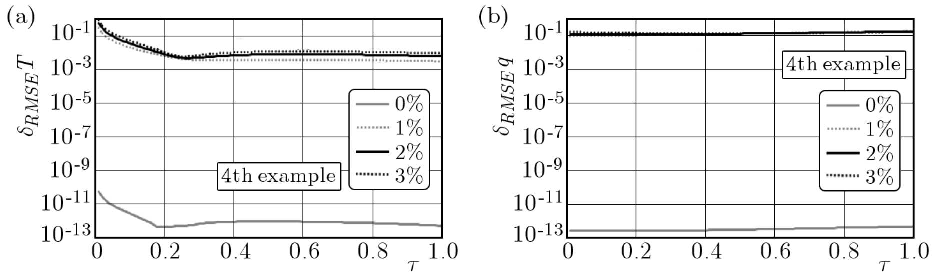

Fig. 8. Root mean squared relative error of the identification of (a) temperature field and (b) source function for the 4th test different value of the disturbance coefficient $\Delta T = \{1\%, 2\%, 3\%\}$

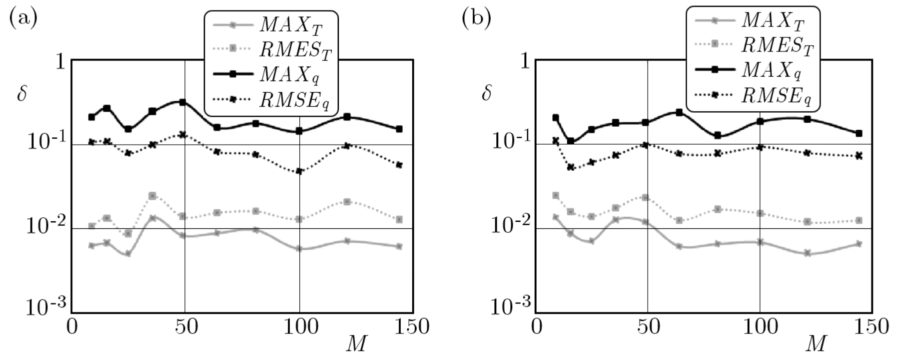

Fig. 9. Maximal and root mean squared relative error of the identification of the temperature field and source function for the 3rd test example $(\Delta T = 1\%, \tau = 0.8)$ as a function of the number of (a) uniformly distributed and (b) randomly distributed interpolation points M

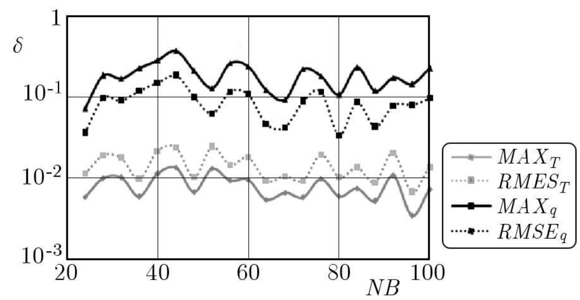

Fig. 10. Maximal and root mean squared relative error of the identification of the temperature field and source function for the 3rd test example $(\Delta T = 1\%, \tau = 0.8)$ as a function of the number of collocation points NB

One of the important parameters of the MFS is the location of source points outside the area Ω. In the presented work, the source points, equal in the number the collocation points, lay on the contour similar to the boundary at a distance D.

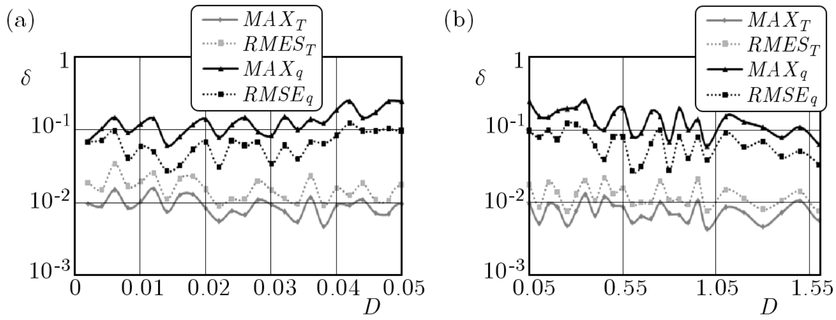

Fig. 11. Maximal and root mean squared relative error of the identification of the temperature field and the source function for the 3rd test example $(\Delta T = 1\%, \tau = 0.8)$ as a function of distance between the contour of the boundary and the contour of sources D

Changing the value of the parameter D in the range from 10^{-3} to 1.55 for the inverse transient heat conduction problem of the identification of source function does not affect significantly the quality of the results (see Fig. 11) for the identification of source function and temperature field.

Presented in this work MFS with Laplace Transformation (MFS & LT) for the identification of the temperature field and heat source function was compared to the MFS combined with the finite differential (MFS & FD). A detailed description of this method (MFS & FD) to the transient inverse source problem can be found in Mierzwiczak and Kołodziej (2010).

The comparison of the methods was conducted for the second and third test examples (Table 2) for both the exact $(\Delta T = 0\%)$ and perturbed data $(\Delta T = 1\%)$.

The results of the comparison of MFS with the Laplace Transform (MFS & LT) with MFS with the finite of differential (MFS & FD) in the form of root mean square relative error of the identification of the temperature field and heat source function are shown in Fig. 12 and Fig. 13 respectively, for the exact $(\Delta T = 0\%)$ and disturbed case $(\Delta T = 1\%)$.

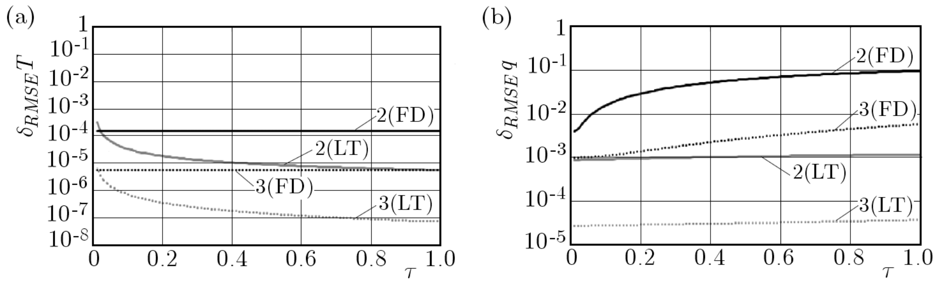

Fig. 12. Comparison of MFS & LT with MFS & FD with respect to values of the root mean squared relative error of the identification of (a) temperature function and (b) source function for $M = 25$, $NB = 44$, ΔT=0%

For the non-disturbed data, more accurate results of identification of both the temperature field and the source function were obtained with MFS & LT. However, for the perturbed data $(\Delta T = 1\%)$ better results of the identification of the temperature field are obtained with MFS combined with the finite differential method. It should be noted that for MFS & LT in the initial moments of time, the error is big and for MFS & FD error has a similar value for the whole computation time.

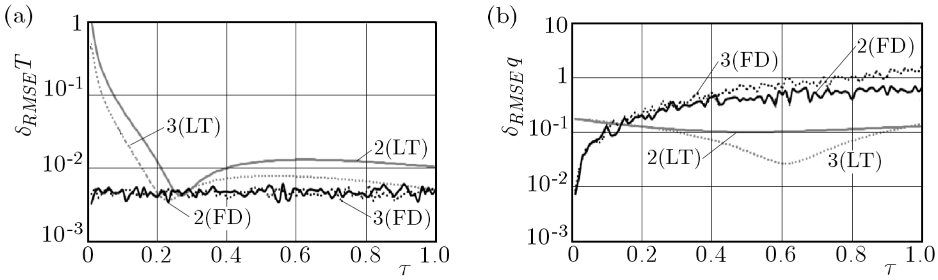

Fig. 13. Comparison of MFS & LT with MFS & FD with respect to values of the root mean squared relative error of the identification of (a) temperature function and (b) source function for $M = 25$, $NB = 44$, $\Delta T = 1\%$ (with Tikhonov regularization)

The error of identification of source function in the case of MFS & FD alarmingly increases over time. Such behavior has no place in the application of MFS & LT, hence this method should be regarded as appropriate in solving this type of heat conduction problem.

5. Conclusions

The paper presents a version of the MFS with Laplace transformation for solving the inverse transient heat source problem in a 2D domain. The advantage of the presented method is its meshless character and, consequently, the temperature is a continuous function at each time contrary to traditional mesh methods such as FDM, FEM, and BEM. Due to ill-conditioning of the inverse transient heat conduction problem, the Tikhonov regularization method based on SVD was used. In order to determine the optimum value of the regularization parameter λ, the L-curve criterion was used. Based on our numerical experiments, it was shown that the proposed method is easy to implement and accurate for undisturbed data. For the distorted data, the proposed algorithm is not accurate, and its computation is time-consuming due to the use of regularization based on SVD. The unsatisfactory accuracy of the results for the disturbed data indicate a need for further work on the application of MFS for solving the inverse transient heat conduction problems.

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References

1. Alves C.J.S., Colaco M.J., Leitao V.M.A., Martins N.F.M., Orlande H.R.B., Roberty N.C., 2008, Recovering the source term in a linear diffusion problem by the method of fundamental solutions, *Inverse Problems in Science and Engineering*, **16**, 8, 1005-1021

2. Hansen P.Ch., O'Leary D.P., 1993, The use of the L-curve in the regularization of discrete ill-posed problems, *SIAM Journal of Computing*, **14**, 1487-1503

3. Hon Y.C., Wei T., 2004, A fundamental solution method for inverse heat conduction problem, *Engineering Analysis with Boundary Elements*, **28**, 5, 489-495

4. Jin B., Marin L., 2007, The method of fundamental solutions for inverse source problems associated with the steady-state heat conduction, *International Journal for Numerical Methods in Engineering*, **69**, 8, 1570-1589

5. Karami G., Hematiyan M.R., 2000, A boundary element method of inverse non-linear heat conduction analysis with point and line heat sources, *Communications in Numerical Methods in Engineering*, **16**, 2, 191-203
6. Kołodziej J.A., Mierzwiczak M., Ciałkowski M., 2010, Application of the method of fundamental solutions and radial basis functions for inverse heat source problem in case of steady-state, *International Communications in Heat and Mass Transfer*, **37**, 2, 121-124
7. Kołodziej J.A., Zieliński A.P., 2009, *Boundary Collocation Techniques and their Application in Engineering*, pp. 15-17, WIT Press, Southampton
8. Le Niliot C., Lefevre F., 2001, Multiple transient point heat sources identification in heat diffusion: application to numerical 2D and 3D problems, *Numerical Heat Transfer, Part B*, **39**, 3, 277-303
9. Ling L., Yamamoto M., Hon Y.C., Takeuchi T., 2006, Identification of source locations in two-dimensional heat equations, *Inverse Problem*, **22**, 4, 1289-1305
10. Marin L., 2005, Numerical solution of the Cauchy problem for steady-state heat transfer in two-dimensional functionally graded materials, *International Journal of Solids and Structures*, **42**, 15, 4338-4351
11. Mierzwiczak M., Kołodziej J.A., 2010, Application of the method of fundamental solutions and radial basis functions for inverse transient heat source problem, *Computer Physics Communications*, **181**, 2035-2043
12. Mierzwiczak M., Kołodziej J.A., 2011, The determination temperature dependent thermal conductivity as inverse steady heat conduction problem, *International Journal of Heat and Mass Transfer*, **54**, 790-796
13. Yan L., Fu C.-L., Yang F.-L., 2008, The method of fundamental solutions for the inverse heat source problem, *Engineering Analysis with Boundary Elements*, **32**, 3, 216-222
14. Yan L., Yang F.-L., Fu C.-L., 2009, A meshless method for solving an inverse spacewise-dependent heat source problem, *Journal of Computational Physics*, **228**, 1, 123-136
15. Yan L., Fu C.-L., Dou F.-F., 2010, A computational method for identifying a spacewise-dependent heat source, *International Journal for Numerical Methods in Biomedical Engineering*, **26**, 5, 597-608
16. Yang C.-Y., 1998, Solving the two-dimensional inverse heat source problem through the linear least-squares error method, *International Journal of Heat and Mass Transfer*, **41**, 2, 393-398
17. Yi Zh., Murio D.A., 2004, Source term identification in 2-D IHCP, *Computers Mathematical Applied*, **47**, 10/11, 1517-1533

Zastosowanie metody rozwiązań podstawowych i transformacji Laplace'a do odwrotnego niestacjonarnego problemu źródeł ciepła

Streszczenie

Artykuł dotyczy odwrotnego problemu określenia mocy źródeł ciepła dla nieustalonego zagadnienia przewodzenia ciepła. Równanie różniczkowe nieustalonego przewodzenia ciepła Fouriera w obszarze dwuwymiarowym z nieznanymi wewnętrznymi źródłami ciepła jest znane jako odwrotny problem brzegowo-początkowy. Przy określeniu mocy źródeł ciepła korzysta się z warunku brzegowego, warunku początkowego oraz znanej wartości temperatury w wybranych punktach rozmieszczonych wewnątrz rozważanego obszaru. Do rozwiązania odwrotnego problemu źródeł ciepła została zaproponowana transformacja Laplace'a połączona z metodą rozwiązań podstawowych i promieniowymi funkcjami bazowymi. Ze względu na złe uwarunkowanie problemu, w pracy zastosowano metodę regularyzacji Tichonowa dla rozkładu SVD oraz kryterium L-krzywej. Jako przykłady testowe rozważono dwuwymiarowe odwrotne problemy brzegowo-początkowe (2D_IBIVP) ze znanym rozwiązaniem analitycznymi.

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