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INQUIRE INTO THE MARVELLOUSNESS OF AUTOFRETTAGE FOR MONO-LAYERED CYLINDERS

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With the help of the equation of optimum overstrain or depth of the plastic zone, a set of concise and accurate equations for residual stresses and their equivalent stress as well as the total stress and their equivalent stresses are obtained, and features of these stresses are discussed, thereupon the law of distribution and the varying tendency of these stresses become clearer. Safe and optimum load-bearing conditions for a cylinder are presented.

Key words: thick-wall cylinder, autofrettage, load-bearing capacity, overstrain, residual stress

1. Introduction

Much of mechanical problems is involved in the autofrettage of cylinders. Currently, researches on autofrettage have been concentrated mostly on specific engineering problems, while a general theoretical study is rare. Due to their structural geometric features and the load bearing pattern, we feel that there must be a mathematical mystery behind autofrettage theory of cylinders, which theoretically describes the physical meaning of autofrettaged cylinders. To discover the general law contained in autofrettage theory, the autofrettage of a cylinder is investigated based on the fourth strength theory by theoretical analysis and the image method.

The autofrettage technology is a clever and effective measure to obtain a favorable stress pattern inside the wall of a cylinder and raise of load-bearing capacity for (ultra-)high pressure vessels. Much of mechanical problems is concerned in the autofrettage of cylinders. Lots of researches concentrated upon specific engineering problems in the autofrettage have been done (Gao *et al.*, 2008; Hameed *et al.*, 2004; Huang *et al.*, 2009, 2011; Levy *et al.*, 2003; Lin *et al.*, 2009; Zheng *et al.*, 2010; Zheng and Xuan, 2010; etc.), nevertheless, many theoretical problems in the autofrettage remain unsolved. No doubt it would be necessary to solve specific engineering problems about the autofrettage, but theoretical studies are more penetrating and can probe deeply into the essence of things, thus have universality and generality. Moreover, because of their structural geometric features and load bearing pattern, we feel that there must be more to it than meets the eye in autofrettage theory of cylinders, which theoretically describes the physical meaning of autofrettaged cylinders. Therefore, we dismiss specific engineering problems and do general research about the autofrettage in this paper.

For an autofrettaged cylinder, depth of the plastic zone k_j or overstrain ε is key, which affects residual stresses and load-bearing capacity. For determination of k_j, previous researchers presented various methods. A repeated trial calculation method to determine the radius of elastic-plastic juncture r_j was presented by Yu (1990), which is too tedious and inaccurate, and this method is based on limiting only the hoop stress and is essentially based on the first strength theory which is in agreement with brittle materials, while pressure vessels are made usually from ductile materials which are in excellent agreement with the third or fourth strength theory (Yu, 1990). Another method for determination of r_j to ensure the equivalent stress of total stress at elastic-plastic juncture σ_{ej} to be minimum was also suggested by Yu (1990).

However, to ensure σ_{ej} to be minimum is not ideal and optimal, for it cannot ensure that a cylinder is not yielded compressively (reversed yielding) when processed with autofrettage, and its load-bearing capacity cannot be raised as high as possible. Thus, Zhu (2008) advanced an expression to calculate the depth of the plastic zone k_j for a cylinder with the radius ratio k not to be yielded compressively when it is autofrettaged in his previous research, which is $k^2 \ln k_j^2 - k^2 - k_j^2 + 2 = 0$. If k_j of a cylinder is determined by this equation, its ultimate load-bearing capacity can reach two times the initial yield pressure (the maximum elastic load-bearing capability of an unautofrettaged cylinder), $2p_e$ (Zhu, 2008). By use of $k^2 \ln k_j^2 - k^2 - k_j^2 + 2 = 0$, Zhu and Zhu (2013a) simplified the equations for the residual stresses and the total stress, thus the laws of distribution and varying tendency of these stresses were discovered and relations among various parameters were revealed. By limiting the hoop residual stress, Zhu and Zhu (2013b) studied load-bearing capacity and depth of the plastic zone of an autofrettaged cylinder, where load-bearing capacity and depth of the plastic zone are both fixed for a certain k, and there is a sole corresponding depth of the plastic zone $k_{j\theta}$ for a certain k. We found that the research should be carried on in a more extensive and more general sense, and it is well known that the greater the k_j, the greater the load-bearing capacity, while the more different to perform the autofrettage technology. Therefore, if it is not necessary for a cylinder to bear $2p_e$ or $(\sqrt{3}+2)p_e/2$ (Zhu and Zhu, 2013b), k_j can be lowered to be beneficial to the performance of autofrettage technology. Then, how to determine k_j of a cylinder for a certain load-bearing capacity and the radius ratio k? Or what is the relation between k_j and k for a certain load-bearing capacity when $p < 2p_e$? What results will be brought about by this relation? How to determine the load-bearing capacity of a cylinder for a certain k_j and k? What is the characteristics of residual stresses and their equivalent stress as well as total stress and their equivalent stress under new conditions? Therefore, on the basis of the author's previous work, this paper is intended to resolve more general theoretical problems in the autofrettage and bring to light essential relations and laws contained in the current theory on the autofrettage according to the fourth strength theory (Mises yield criterion).

Because we deal with the ideal case, and problems about the autofrettage under specific engineering conditions can be resolved by reference to the results of this paper on the basis of the specific engineering conditions, we bypass specific engineering conditions which vary in thousands of ways and do our research based on the following ideal conditions as in our previous works: (1) the material of a cylinder is perfectly elastic-plastic and Bauschinger's effect is neglected, the compressive yield limit is equal to the tensile one; (2) strain hardening is ignored; (3) there is not any defect in the material.

It is hoped that the obtained theoretical results are of academic value and are referential as well as applicable to the design of (ultra-)high pressure apparatus.

2. Residual stresses under ordinary condition

At a general location (relative location, r/r_i) within the plastic zone, the residual stresses are as follows(ZHU, 2008)

$$
\begin{aligned}
\frac{\sigma'_z}{\sigma_y} &= \frac{1}{\sqrt{3}}\Big[\frac{k_j^2}{k^2} + \ln\frac{(r/r_i)^2}{k_j^2} - \Big(1 - \frac{k_j^2}{k^2} + \ln k_j^2\Big)\frac{1}{k^2-1}\Big] \\
\frac{\sigma'_r}{\sigma_y} &= \frac{1}{\sqrt{3}}\Big[\frac{k_j^2}{k^2} - 1 + \ln\frac{(r/r_i)^2}{k_j^2} - \Big(1 - \frac{k_j^2}{k^2} + \ln k_j^2\Big)\frac{1}{k^2-1}\Big(1 - \frac{k^2}{(r/r_i)^2}\Big)\Big] \\
\frac{\sigma'_\theta}{\sigma_y} &= \frac{1}{\sqrt{3}}\Big[\frac{k_j^2}{k^2} + 1 + \ln\frac{(r/r_i)^2}{k_j^2} - \Big(1 - \frac{k_j^2}{k^2} + \ln k_j^2\Big)\frac{1}{k^2-1}\Big(1 + \frac{k^2}{(r/r_i)^2}\Big)\Big]
\end{aligned}
\tag{2.1}
$$

Therefore, the equivalent residual stress at a general radius location within the plastic zone is (Yu, 1990)

$$\frac{\sigma'_e}{\sigma_y}=\frac{\sqrt{3}}{2}\Big(\frac{\sigma'_\theta}{\sigma_y}-\frac{\sigma'_r}{\sigma_y}\Big)=1-\frac{k^2-k_j^2+k^2\ln k_j^2}{(k^2-1)(r/r_i)^2} \tag{2.2}$$

where $\sigma'_z, \sigma'_r, \sigma'_\theta$ are axial, radial and hoop residual stress, respectively; r_i, r_j, r_o are inside radius, elastic-plastic juncture radius, outside radius, respectively; k is the radius ratio or ratio of the outside to inside radius, $k=r_o/r_i$; k_j is depth of the plastic zone, or plastic depth, $k_j=r_j/r_i$; σ_y is yield strength; σ'_e is equivalent residual stress; σ'_e/σ_y is relative equivalent residual stress; subscript i represents the internal surface, subscript j represents the elastic-plastic juncture.

The residual stresses at a general location within the elastic zone are as follows (Zhu, 2008)

$$\begin{aligned}
\frac{\sigma'_z}{\sigma_y}&=\frac{1}{\sqrt{3}}\Big[\frac{k_j^2}{k^2}-\Big(1-\frac{k_j^2}{k^2}+\ln k_j^2\Big)\frac{1}{k^2-1}\Big]\\
\frac{\sigma'_r}{\sigma_y}&=\frac{1}{\sqrt{3}}\Big(1-\frac{k^2}{(r/r_i)^2}\Big)\Big[\frac{k_j^2}{k^2}-\Big(1-\frac{k_j^2}{k^2}+\ln k_j^2\Big)\frac{1}{k^2-1}\Big]=\Big(1-\frac{k^2}{(r/r_i)^2}\Big)\frac{\sigma'_z}{\sigma_y}\\
\frac{\sigma'_\theta}{\sigma_y}&=\frac{1}{\sqrt{3}}\Big(1+\frac{k^2}{(r/r_i)^2}\Big)\Big[\frac{k_j^2}{k^2}-\Big(1-\frac{k_j^2}{k^2}+\ln k_j^2\Big)\frac{1}{k^2-1}\Big]=\Big(1+\frac{k^2}{(r/r_i)^2}\Big)\frac{\sigma'_z}{\sigma_y}
\end{aligned} \tag{2.3}$$

Therefore, the equivalent residual stress at a general radius location within the elastic zone is (Yu, 1990)

$$\frac{\sigma'_e}{\sigma_y}=\frac{\sqrt{3}}{2}\Big(\frac{\sigma'_\theta}{\sigma_y}-\frac{\sigma'_r}{\sigma_y}\Big)=\frac{k^2(k_j^2-1-\ln k_j^2)}{(k^2-1)(r/r_i)^2} \tag{2.4}$$

3. Discussion about plastic depth or overstrain

When the equivalent stress of total stress (residual stress plus the stresses caused by the operation pressure p) at the elastic-plastic juncture reaches the yield strength, or $\sigma_{ej}=\sigma_y$, the relation for p, the pressure a cylinder can contain, σ_y, k and k_j is as follows(Yu, 1990)

$$\frac{p}{\sigma_y}=\frac{k^2-k_j^2+k^2\ln k_j^2}{\sqrt{3}k^2} \tag{3.1}$$

Zhu (2008) showed that when radius ratio is greater than critical radius ratio, or $k>k_c=2.218\,457\,489\,916\,7\ldots$, if $k_j\leqslant k_{j^*}$, where k_{j^*} is determined by $k^2\ln k_{j^*}^2-k^2-k_{j^*}^2+2=0$, the absolute value of equivalent stress of residual stress at the internal surface $|\sigma'_{ei}|\leqslant\sigma_y$, when $k_j=k_{j^*}$, $|\sigma'_{ei}|=\sigma_y$ and

$$\frac{p}{\sigma_y}=2\frac{k^2-1}{\sqrt{3}k^2}=2\frac{p_e}{\sigma_y}$$

when $k<k_c$, $k_j=k$ (entire yielded), $|\sigma'_{ei}|<\sigma_y$ and a cylinder can bear the entire yield loading, $p_y/\sigma_y=(\ln k^2)/\sqrt{3}$. Then

$$\frac{p}{\sigma_y}=\frac{p_y}{\sigma_y}=\frac{\ln k^2}{\sqrt{3}}=\frac{k^2\ln k^2}{k^2-1}\frac{p_e}{\sigma_y}=\eta\frac{p_e}{\sigma_y} \qquad (k<k_c) \tag{3.2}$$

where $\eta=k^2\ln k^2/(k^2-1)$, p_e is the initial yield pressure of an unautofrettaged cylinder, $p_e/\sigma_y=(k^2-1)/(\sqrt{3}k^2)$; η is called the reinforcing coefficient and reflects the level of increase in the load-bearing capacity. When $k > k_c$, to reflect the level of increase in the load-bearing capacity, letting $p = \lambda p_e$, then

$$\frac{p}{\sigma_y} = \lambda \frac{k^2 - 1}{\sqrt{3}k^2} = \lambda \frac{p_e}{\sigma_y} \qquad (k > k_c) \tag{3.3}$$

where λ is also called the reinforcing coefficient. Substituting Eq. (3.3) into Eq. (3.1), one obtains

$$k^2 \ln k_{j\lambda}^2 - (\lambda - 1)k^2 - k_{j\lambda}^2 + \lambda = 0 \tag{3.4}$$

where k_j is written as $k_{j\lambda}$ to indicate that the safe plastic depth k_j is related with λ.

The overstrain is defined as

$$\varepsilon = \frac{r_j - r_i}{r_o - r_i} = \frac{k_j - 1}{k - 1} \tag{3.5}$$

Substituting k_j from Eq. (3.5) into Eq. (3.4), one obtains

$$k^2 \ln[\varepsilon_\lambda(k-1)+1]^2 - (\lambda - 1)k^2 - [\varepsilon_\lambda(k-1)+1]^2 + \lambda = 0 \tag{3.6}$$

If ε and k meet Eq. (3.6), where ε is written as ε_λ, and a cylinder contains pressure determined by Eq. (3.3), $\sigma_{ej}/\sigma_y = 1$, $|\sigma'_{ei}| \leqslant \sigma_y$. When $\lambda = 1$, $\sigma'_{ei} = -\sigma_y$. ε_λ determined by Eq. (3.6) is called the optimum overstrain, and $k_{j\lambda}$ determined by Eq. (3.4) is called the optimum plastic depth. They are plotted in Fig. 1.

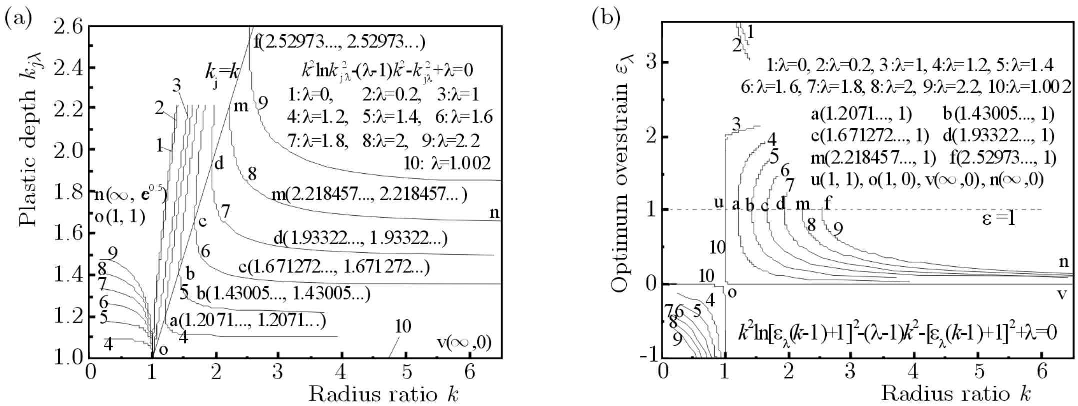

Fig. 1. The optimum plastic depth and optimum overstrain; (a) the optimum plastic depth, (b) the optimum overstrain

It can be known that from Eq. (3.4), Eq. (3.6) and Fig. 1 that:

(1) If $\lambda \leqslant 1$ (curves 1-3), $k < k_j$ and $\varepsilon_\lambda \geqslant 1$. This is meaningless in the engineering.

(2) If $\lambda \geqslant 1$ (curves 4-9), the curves for Eq. (3.4) or (3.6) are divided into two branches, for the left of which, $k < 1$ and $\varepsilon_\lambda < 0$, which is meaningless in the engineering. For the right of the two branches, the line $k = k_j$ or $\varepsilon_\lambda = 1$ divides the curves into two parts: above the line, $k < k_j$ or $\varepsilon_\lambda > 1$, this is meaningless in application; below the line, $k > k_j$ and $\varepsilon_\lambda < 1$. So, it is the part below the line $k = k_j$ or $\varepsilon_\lambda = 1$ that is of significance.

When $k < k_c$, letting $k_{j\lambda} = k$ $(= k_{c\lambda})$ in Eq. (3.4) or $\varepsilon_\lambda = 1$ in Eq. (3.6) one obtains the critical radius ratio $k_{c\lambda}$ or the radius ratio when the whole wall is yielded while $|\sigma'_{ei}| \leqslant \sigma_y$ under a certain λ, which is

$$k_{c\lambda}^2 \ln k_{c\lambda}^2 - \lambda(k_{c\lambda}^2 - 1) = 0 \qquad \text{or} \qquad \lambda = \frac{1}{k_{c\lambda}^2 - 1} k_{c\lambda}^2 \ln k_{c\lambda}^2 \tag{3.7}$$

When $k < k_{c\lambda}$, k_j can be k, i.e. the entire yield autofrettage. When $k > k_{c\lambda}$, k_j should be determined by Eq. (3.4). Equation (3.7) is the same as the above η in form. The prerequisite to $k_j = k$ is $k < k_c$. Thus we obtain the reinforcing coefficient η for $k < k_c$ by a different method. η in Eq. (3.2) is the greatest reinforcing coefficient when $k_j = k$ (entire yielded autofrettage cylinder) in the case that $k < k_c$. In the case that $k < k_c$, $k_j = k$ is not always required, if a shallower plastic zone is feasible, a lesser reinforcing coefficient λ can be determined by Eq. (3.4). Then, Eq. (3.4) can be regarded as embodying Eq. (3.2) and Eq. (3.7). In the case that $k > k_c$, the greatest reinforcing coefficient is $\lambda \equiv 2$. So, integrating Eq. (3.2), we obtain the greatest reinforcing coefficient for any k ($1 \leqslant k \leqslant \infty$), as shown in Fig. 2.

(3) k_j and ε_λ decrease with k increasing on the right of the two branches.

Thus, discussion about the autofrettage is not significant unless $\lambda > 1$. Since compressive yield occurs when $\lambda > 2$ (when $\lambda > 2$, ε_λ is higher than the value on curve 8 on which $\sigma'_{ei} = -\sigma_y$), curve 9 in Fig. 1a is meaningless. Besides, curves 1-3 and the left of curves 4-8 in Fig. 1 are meaningless. The abscissa can be taken as the curve with $\lambda = 1$ ($\varepsilon = 0$). Thus, significant and possible plastic depth lies in a trapezoid surrounded by the abscissa ($\lambda = 1$), the slanting straight line $k = k_j$ and the curve $k^2 \ln k^2 - k^2 - k_{j\lambda}^2 + 2 = 0$ (curve 8 in Fig. 1a for $\lambda = 2$). The coordinates of four vertexes of this quasi-infinite area (m, o, v, n) are shown in Fig. 1a. The significant and possible overstrain lies in a trapezoid surrounded by the horizontal line $\varepsilon = 0$ ($\lambda = 1$), the vertical line ($k = 1$), the horizontal line $\varepsilon = 1$ and the curve $k^2 \ln[\varepsilon_\lambda(k-1)+1]^2 - k^2 - [\varepsilon_\lambda(k-1)+1]^2 + 2 = 0$ (curve 8 in Fig. 1b for $\lambda = 2$). The coordinates of five vertexes of this quasi-infinite area (m, u, o, v, n) are shown in Fig. 1b. When $k \to \infty$, points n and v coincide.

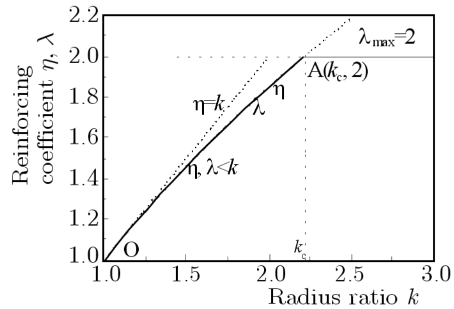

Fig. 2. The greatest reinforcing coefficient

4. Discussion about residual stresses and their equivalent stress under $k_{j\lambda}$

If k_j is determined by Eq. (3.4) or $k_j = k_{j\lambda}$ and $k \geqslant k_{c\lambda}$, with the help of Eq. (3.4), Eqs. (2.1)-(2.4) become

$$
\begin{aligned}
\frac{\sigma'_z}{\sigma_y} &= \frac{1}{\sqrt{3}}(\ln x^2 - \lambda + 1) & \qquad \frac{\sigma'_r}{\sigma_y} &= \frac{1}{\sqrt{3}}\Big(\ln x^2 + \frac{\lambda}{x^2} - \lambda\Big) \\
\frac{\sigma'_\theta}{\sigma_y} &= \frac{1}{\sqrt{3}}\Big(\ln x^2 - \frac{\lambda}{x^2} - \lambda + 2\Big) & \qquad \frac{\sigma'_e}{\sigma_y} &= 1 - \frac{\lambda}{x^2}
\end{aligned}
\tag{4.1}
$$

where $x = r/r_i$, the same below. When $\lambda \leqslant 1 + \sqrt{3}/2$, $|\sigma'_{\theta i}/\sigma_y| \leqslant 1$; when $\lambda \leqslant 2$, $|\sigma'_{ei}/\sigma_y| \leqslant 1$.

σ'_z, σ'_r, σ'_θ and σ'_e have nothing to do with k_j and k within the plastic zone, which means that for some λ, the curves of residual stress in whichever direction (axial, radial and hoop direction), and the equivalent residual stress (i.e. σ'_z, σ'_r, σ'_θ and σ'_e) for various plastic depth $k_{j\lambda}$ and the radius ratio k coincide

$$\begin{aligned} &\frac{\sigma'_z}{\sigma_y} \equiv \frac{1}{\sqrt{3}k^2}(k_{j\lambda}^2 - \lambda) && \frac{\sigma'_r}{\sigma_y} = \Big(1 - \frac{k^2}{x^2}\Big)\frac{\sigma'_z}{\sigma_y} \\ &\frac{\sigma'_\theta}{\sigma_y} = \Big(1 + \frac{k^2}{x^2}\Big)\frac{\sigma'_z}{\sigma_y} && \frac{\sigma'_e}{\sigma_y} = \frac{1}{x^2}(k_{j\lambda}^2 - \lambda) \end{aligned} \tag{4.2}$$

The curves of residual stresses at a general location for $k = 3$, $\lambda = 1.2$ and 1.8 when $k_j = k_{j\lambda}$ are plotted in Fig. 3. In this three cases, $|\sigma'_{ei}| < \sigma_y$ for $k_j = k_{j\lambda}$ and $\lambda < 2$.

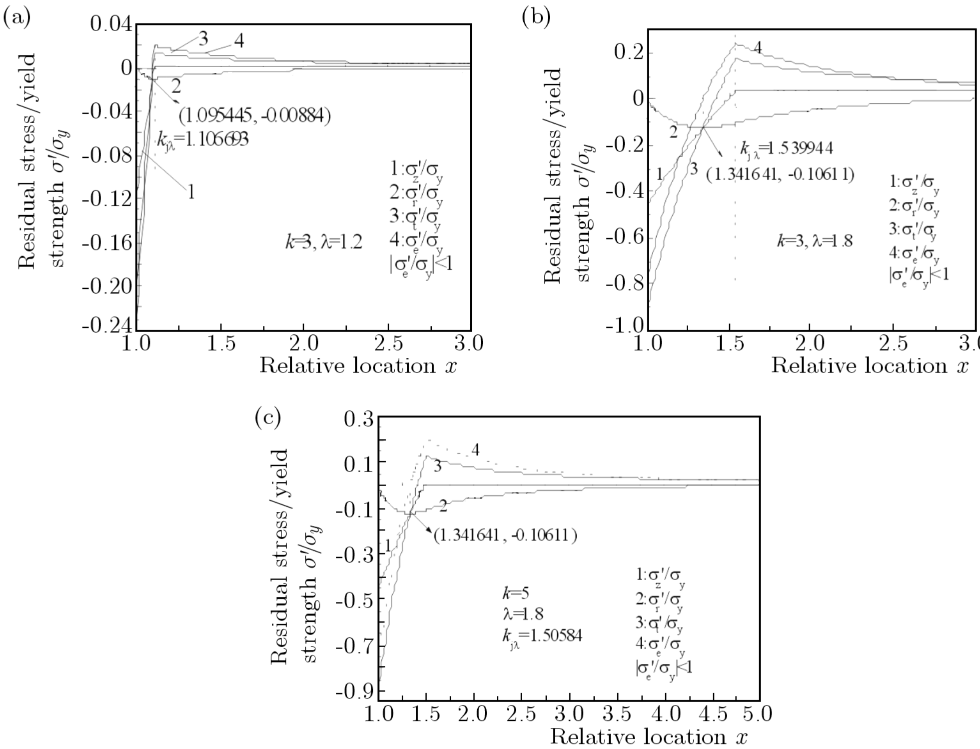

Fig. 3. Curves of residual stresses and their equivalent stress at a general location; (a) $k = 3$, $\lambda = 1.2$, $k_{j\lambda} = 1.106693$, (b) $k = 3$, $\lambda = 1.8$, $k_{j\lambda} = 1.539944$, (c) $k = 5$, $\lambda = 1.8$, $k_{j\lambda} = 1.50584$

The three curves of residual stress (σ'_z/σ_y, σ'_r/σ_y and σ'_θ/σ_y) at a general location collect at a fixed point within plastic zone: $[\sqrt{\lambda}, (\ln\lambda + 1 - \lambda)/2]$ for any k, k_j and λ, and the coordinate of the intersection is not related with k and k_j but only with λ. If $k_j \neq k_{j\lambda}$, this situation does not happen, the coordinate of the intersection is related not only with λ but also with k and k_j.

The equivalent stress of residual stress at the internal surface is the most dangerous, and when $\lambda \geqslant 1$, $\sigma'_{ei} \leqslant 0$, which implies compressive stress; when $\lambda \leqslant 2$, $\sigma'_{ei} \geqslant -\sigma_y$. Since $k_j \geqslant \sqrt{\lambda}$ ($k_j \geqslant \mathrm{e}^{(\lambda-1)/2}$ at the same time), then, when $\lambda \leqslant 1$, $k_j^2 - \lambda > 0$ within the elastic zone, or equivalent stress of residual stress $\sigma'_e > 0$ (tension) within the elastic zone. At the elastic-plastic juncture, where $x = k_{j\lambda}$, the equivalent stress of residual stress is the maximum (algebraic value, not absolute value) within the whole elastic zone, or $\sigma'_{ej}/\sigma_y = (k_{j\lambda}^2 - \lambda)/k_{j\lambda}^2 = 1 - \lambda/k_{j\lambda}^2$. Obviously, $0 < \sigma'_{ej}/\sigma_y < 1$.

From Eq. (3.4), when $k = \infty$, $k_{j\lambda} = \mathrm{e}^{(\lambda-1)/2} = k_{j\lambda}^\infty$, then from Eqs. (4.2), within the whole elastic zone

$$\frac{\sigma'_z}{\sigma_y} \equiv \frac{\mathrm{e}^{\lambda-1} - \lambda}{\sqrt{3}k^2} \qquad \frac{\sigma'_r}{\sigma_y} = \Big(1 - \frac{k^2}{x^2}\Big)\frac{\sigma'_z}{\sigma_y} \qquad \frac{\sigma'_\theta}{\sigma_y} = \Big(1 + \frac{k^2}{x^2}\Big)\frac{\sigma'_z}{\sigma_y} \qquad \frac{\sigma'_e}{\sigma_y} = \frac{\mathrm{e}^{\lambda-1} - \lambda}{x^2}$$

$x = \mathrm{e}^{(\lambda-1)/2} \sim \infty$ within the elastic zone, therefore

$$
\frac{\sigma'_z}{\sigma_y} \equiv \frac{\mathrm{e}^{\lambda-1} - \lambda}{\sqrt{3}k^2} \qquad\qquad \frac{\sigma'_r}{\sigma_y} = \Big(1 - \frac{k^2}{\mathrm{e}^{\lambda-1}}\Big)\frac{\sigma'_z}{\sigma_y} \sim 0
$$

$$
\frac{\sigma'_\theta}{\sigma_y} = \Big(1 + \frac{k^2}{\mathrm{e}^{\lambda-1}}\Big)\frac{\sigma'_z}{\sigma_y} \sim \frac{\mathrm{e}^{\lambda-1} - 2\lambda}{k^2} \qquad\qquad \frac{\sigma'_e}{\sigma_y} = 1 - \frac{\lambda}{\mathrm{e}^{\lambda-1}} \sim 0
$$

The distribution of σ'_e/σ_y within the whole wall for $\lambda = 1.8$ and different k and $k_{j\lambda}$ is shown in Fig. 4.

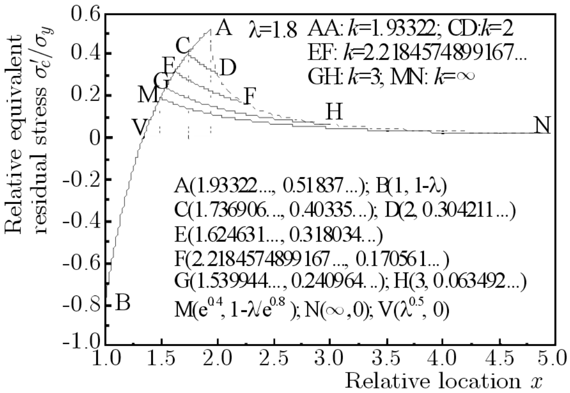

Fig. 4. Distribution of σ'_e/σ_y within the whole wall for $\lambda = 1.8$ and various k and $k_{j\lambda}$

Figure 4 is explained as follows:

- Curve BAA: $k = 1.93322\ldots$, $k_{j\lambda} = k = k_{c\lambda} = 1.93322\ldots$. Within the plastic zone or point B to A, x varies from 1 to $k_{j\lambda}$, σ'_e/σ_y varies from -0.8 to $0.51837\ldots$; within elastic zone (no elastic zone) or point A to A, x varies from $1.93322\ldots$ ($k_{j\lambda}$) to $1.93322\ldots$ (k), σ'_e/σ_y varies from $0.51837\ldots$ to $0.51837\ldots$.
- Curve BCD: $k = 2$, $k_{j\lambda} = 1.736906\ldots$. Within the plastic zone or point B to C, x varies from 1 to $k_{j\lambda}$, σ'_e/σ_y varies from -0.8 to $0.40335\ldots$; within the elastic zone or point C to D, x varies from $k_{j\lambda}$ to k, σ'_e/σ_y varies from $0.40335\ldots$ to $0.304211\ldots$.
- Curve BEF: $k = k_c$ or $k_{c\lambda}$ when $\lambda = 2$, $k_{j\lambda} = 1.624631\ldots$. Within the plastic zone or point B to E, x varies from 1 to $k_{j\lambda}$, σ'_e/σ_y varies from -0.8 to $0.318043\ldots$; within the elastic zone or point E to F, x varies from $k_{j\lambda}$ to k, σ'_e/σ_y varies from $0.318043\ldots$ to $0.170561\ldots$.
- Curve BGH: $k = 3$, $k_{j\lambda} = 1.539944\ldots$. Within the plastic zone or point B to G, x varies from 1 to $k_{j\lambda}$, σ'_e/σ_y varies from -0.8 to $0.240964\ldots$; within the elastic zone or point G to H, x varies from $k_{j\lambda}$ to $k = 3$, σ'_e/σ_y varies from $0.240964\ldots$ to $0.063492\ldots$.
- Curve BMN: $k = \infty$, $k_{j\lambda} = \mathrm{e}^{0.4}$. Within the plastic zone or point B to M, x varies from 1 to $k_{j\lambda} = \mathrm{e}^{0.4} = 1.491825\ldots$, σ'_e/σ_y varies from -0.8 to $1 - \lambda/\mathrm{e}^{0.8} = 0.191208\ldots$; within the elastic zone or point M to N (far infinitely), x varies from $k_{j\lambda} = \mathrm{e}^{0.4}$ to $k = \infty$, σ'_e/σ_y varies from $1 - \lambda/\mathrm{e}^{0.8} = 0.191208\ldots$ to 0.

From Fig. 4 and Eq. $(4.1)_4$, it is known that all curves of equivalent residual stresses for any k and $k_{j\lambda}$ within the plastic zone are located on the identical curve AB and pass through the same point $(1.8^{0.5}, 0)$, except that a different curve for different k and $k_{j\lambda}$ is located on a different section of curve AB. Saying, the above curves for the plastic zone, BA, BC, BE, BG, BM, are all on curve BA, or they coincide with each other. However, if $k_j \neq k_{j\lambda}$, or relation between k_j and k does not satisfy Eq. (3.4), the above conclusion is untenable, even $|\sigma'_{ei}/\sigma_y| > 1$. This case is illustrated in Fig. 5 ($\lambda = 1.8$), where curve 1 and 2 coincide with each other in the plastic zone and both pass through the point $(1.8^{0.5}, 0)$ for $k_j = k_{j\lambda}$, but curve 3 and 4 do not coincide with each other in the plastic zone and neither pass through the point $(1.8^{0.5}, 0)$, and they do not coincide with curve 1 and 2 for $k_j \neq k_{j\lambda}$. When $k = 3$, $k_j = 1.9 > k_{j\lambda}$, $\sigma'_{ei}/\sigma_y = -1.11792 < -1$; when $k = 3$, $k_j = 1.4 < k_{j\lambda}$, $\sigma'_{ei}/\sigma_y = -0.63706 > -1$, but the equivalent stress of total stress σ_{ei} may exceed σ_y.

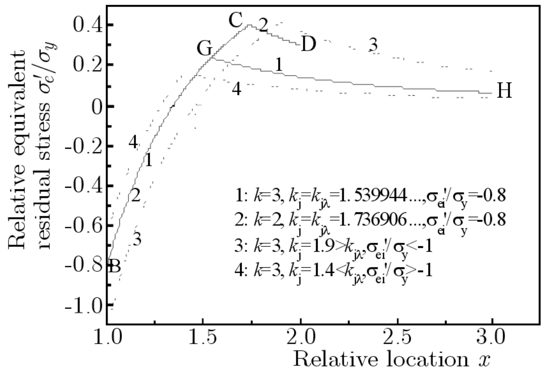

Fig. 5. A comparison of the equivalent residual stress

To know these phenomena and laws about the autofrettage of cylinders well is beneficial to design, manufacturing and academic research on pressure vessels.

When $k_j = k_{j\lambda} = k_{c\lambda}$ (entire yielded), distributions of σ'_e/σ_y and ,σ'_θ/σ_y within the plastic zone for various λ are shown in Figs. 6a and 6b, respectively.

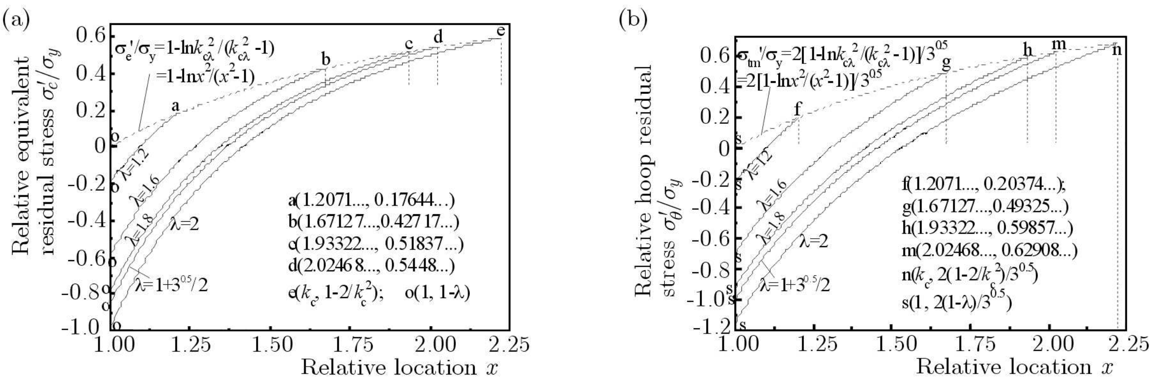

Fig. 6. Distribution of σ'_e/σ_y (a) and σ'_θ/σ_y (b) for various λ

The top dash curve in Fig. 6b is the maximum hoop residual stress $\sigma'_{\theta m}/\sigma_y$ under the critical radius ratio $k_{c\lambda}$ and different λ, the equation of which is

$$\frac{\sigma'_{\theta m}}{\sigma_y} = \frac{2}{\sqrt{3}}\Big(1 - \frac{\ln k^2_{c\lambda}}{k^2_{c\lambda} - 1}\Big) \tag{4.3}$$

$\frac{d(\sigma'_{\theta m}/\sigma_y)}{dk_{c\lambda}} = \frac{4}{\sqrt{3}}\frac{\lambda-1}{k_{c\lambda}(k^2_{c\lambda}-1)} \geqslant 0$ for $\lambda \geqslant 1$, and $k_{c\lambda}$ increases with λ increasing, when $\lambda = 2$, $k_{c\lambda}$ gets the maximum k_c, thereby $\sigma'_{\theta m}/\sigma_y$ gets the maximum

$$\frac{\sigma'_{\theta m}}{\sigma_y} = \frac{2}{\sqrt{3}}\Big(1 - \frac{\ln k^2_c}{k^2_c - 1}\Big) \tag{4.4}$$

From Zhu (2008), it is known that $\frac{k^2_c \ln k_c}{k^2_c - 1} = 1$, then, $\frac{\sigma'_{\theta m}}{\sigma_y} = \frac{2}{\sqrt{3}}\Big(1 - \frac{2}{k^2_c}\Big)$. From Eq. (4.4), when $k_c < 2\sqrt{2+\sqrt{3}} = 3.86$, $\sigma'_{\theta m}/\sigma_y < 1$. So the hoop residual tension is safe. However, when

$\lambda > 1 + \sqrt{3}/2$, the hoop residual compressive stress is not safe. When $\lambda \leqslant 2$, the equivalent residual stress σ'_e is invariably safe.

$\sigma'_e = 0$ at $x = \sqrt{\lambda}$, which is just the abscissa of intersection of the three curves of residual stress at a general location. Generally, in Eq. (2.2), letting $1 - \frac{k^2 - k_j^2 + k^2 \ln k_j^2}{(k^2-1)(r/r_i)^2} = 0$, one obtains

$$x = \sqrt{\frac{k^2 - k_j^2 + k^2 \ln k_j^2}{k^2 - 1}} = \sqrt{\frac{\sigma'_{ei}}{\sigma_y} + 1} < k_j \tag{4.5}$$

On the other hand, the solution of letting $\sigma'_z = \sigma'_r$, $\sigma'_r = \sigma'_\theta$ and $\sigma'_\theta = \sigma'_z$ within the plastic zone (Eqs (2.1)) is also Eq. (4.5). This shows that under the general condition ($k_j = k_{j\lambda}$ is not required), the three curves of the residual stress at a general radial location also collect at one point within the plastic zone and the abscissa of intersection is just Eq. (4.5), where $\sigma'_e/\sigma_y = 0$. If $k_j = k_{j\lambda}$, or k_j and k are in conformity with Eq. (3.4), Eq. (4.5) just becomes $x = \sqrt{\lambda}$.

5. Discussion about stresses caused by internal pressure p and total stresses

At a general location, the stresses caused by internal pressure p are

$$\frac{\sigma_z^p}{\sigma_y} = \frac{1}{k^2-1}\frac{p}{\sigma_y} \qquad \frac{\sigma_r^p}{\sigma_y} = \Big(1 - \frac{k^2}{(r/r_i)^2}\Big)\frac{\sigma_z^p}{\sigma_y} \qquad \frac{\sigma_\theta^p}{\sigma_y} = \Big(1 + \frac{k^2}{(r/r_i)^2}\Big)\frac{\sigma_z^p}{\sigma_y} \tag{5.1}$$

The equivalent stress of the stresses caused by p is

$$\frac{\sigma_e^p}{\sigma_y} = \frac{\sqrt{3}}{2}\Big(\frac{\sigma_\theta^p}{\sigma_y} - \frac{\sigma_r^p}{\sigma_y}\Big) = \frac{\sqrt{3}k^2}{k^2-1}\frac{p}{\sigma_y}\Big(\frac{r}{r_i}\Big)^{-2} \tag{5.2}$$

If $p = \lambda p_e$, Eqs (5.1), (5.2) become

$$\begin{aligned} &\frac{\sigma_z^p}{\sigma_y} = \frac{1}{\sqrt{3}}\lambda\frac{1}{k^2} \qquad \frac{\sigma_r^p}{\sigma_y} = \frac{1}{\sqrt{3}}\lambda\Big(\frac{1}{k^2} - \frac{1}{x^2}\Big) \qquad \frac{\sigma_\theta^p}{\sigma_y} = \frac{1}{\sqrt{3}}\lambda\Big(\frac{1}{k^2} + \frac{1}{x^2}\Big) \\ &\frac{\sigma_e^p}{\sigma_y} = \lambda\frac{1}{x^2} \end{aligned} \tag{5.3}$$

Equations (5.3) are plotted in Fig. 7 for $k = 3$ and $\lambda = 1.8$. Clearly, at the internal surface, $\sigma_{ei}^p/\sigma_y > 1$ when $\lambda > 1$ if a cylinder is not treated with the autofrettage and $p > p_e$.

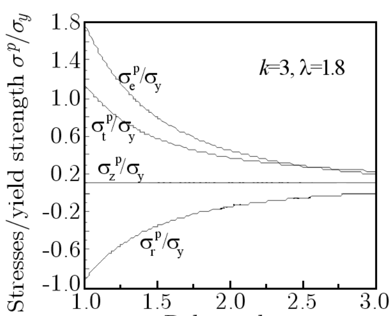

Fig. 7. Stresses caused by p at a general location

From Eqs (5.3) and Fig. 7, when $x \geqslant \sqrt{\sqrt{3} - 1}k$, $\sigma_e^p/\sigma_y \leqslant \sigma_\theta^p/\sigma_y$.

The total stresses σ/σ_y include the residual stresses and the stresses caused by p, or

$$\sigma_z = \sigma'_z + \sigma_z^p \qquad \sigma_r = \sigma'_r + \sigma_r^p \qquad \sigma_\theta = \sigma'_\theta + \sigma_\theta^p \tag{5.4}$$

The equivalent stress of total stress is

$$\sigma_e = \frac{\sqrt{3}}{2}(\sigma_\theta - \sigma_r) = \frac{\sqrt{3}}{2}[(\sigma'_\theta + \sigma^p_\theta) - (\sigma'_r + \sigma^p_r)] = \frac{\sqrt{3}}{2}(\sigma'_\theta - \sigma'_r) + \frac{\sqrt{3}}{2}(\sigma^p_\theta - \sigma^p_r) = \sigma'_e + \sigma^p_e \quad (5.5)$$

For $k = 3$, $\lambda = 1.8$, $p = \lambda p_e$, $k_j = 1.4 < k_{j\lambda}$ as mentioned above (see Fig. 5), $\sigma'_{ei}/\sigma_y = -0.63706$, $\sigma^p_{ei}/\sigma_y = 1.8$, then the equivalent stress of total stress at the internal surface $\sigma_{ei} = \sigma'_{ei} + \sigma^p_{ei} = 1.1629 > 1$. So, for $p = \lambda p_e$, k_j must be determined by Eq. (3.4), i.e. $k_j = k_{j\lambda}$ (in this example, $k_{j\lambda} = 1.539944\ldots > 1.4$). If $k_j < k_{j\lambda}$, the total stresses will be dangerous; if $k_j > k_{j\lambda}$, the residual stresses will be dangerous. The greater the λ, the higher the load-bearing capacity, but the deeper the plastic zone, leading to a more difficult autofrettage treatment; conversely, the less the λ, the lower the load-bearing capacity, but the shallower the plastic zone, leading to an easier autofrettage treatment. This finding helps us to weigh the advantages and disadvantages in the design of pressure vessels.

If $p = \lambda p_e$ and $k_j = k_{j\lambda}$, the components of total stresses are:

— Within the plastic zone

$$\frac{\sigma_z}{\sigma_y} = \frac{\sigma'_z}{\sigma_y} + \frac{\sigma^p_z}{\sigma_y} = \frac{1}{\sqrt{3}}\Big(\ln x^2 - \lambda + 1 + \lambda\frac{1}{k^2}\Big) \qquad \frac{\sigma_r}{\sigma_y} = \frac{\sigma'_r}{\sigma_y} + \frac{\sigma^p_r}{\sigma_y} = \frac{1}{\sqrt{3}}\Big(\ln x^2 - \lambda + \lambda\frac{1}{k^2}\Big)$$
$$\frac{\sigma_\theta}{\sigma_y} = \frac{\sigma'_\theta}{\sigma_y} + \frac{\sigma^p_\theta}{\sigma_y} = \frac{1}{\sqrt{3}}\Big(\ln x^2 - \lambda + 2 + \lambda\frac{1}{k^2}\Big) \qquad \frac{\sigma_e}{\sigma_y} = \frac{\sigma_\theta}{\sigma_y} - \frac{\sigma_r}{\sigma_y} \equiv 1 \quad (5.6)$$

Eqation $(5.6)_4$ means that if a cylinder is subject to $p = \lambda p_e$ and its plastic depth is determined by Eq. (3.4), the equivalent stress of total stress everywhere within the plastic zone is $\sigma_e/\sigma_y \equiv 1$.

— Within elastic zone

$$\frac{\sigma_z}{\sigma_y} = \frac{\sigma'_z}{\sigma_y} + \frac{\sigma^p_z}{\sigma_y} = \frac{k^2_{j\lambda}}{\sqrt{3}k^2} \qquad \frac{\sigma_r}{\sigma_y} = \frac{\sigma'_r}{\sigma_y} + \frac{\sigma^p_r}{\sigma_y} = \frac{k^2_{j\lambda}}{\sqrt{3}}\Big(\frac{1}{k^2} - \frac{1}{x^2}\Big)$$
$$\frac{\sigma_\theta}{\sigma_y} = \frac{\sigma'_\theta}{\sigma_y} + \frac{\sigma^p_\theta}{\sigma_y} = \frac{k^2_{j\lambda}}{\sqrt{3}}\Big(\frac{1}{k^2} + \frac{1}{x^2}\Big) \qquad \frac{\sigma_e}{\sigma_y} = \frac{\sigma_\theta}{\sigma_y} - \frac{\sigma_r}{\sigma_y} = \frac{k^2_{j\lambda}}{x^2} \quad (5.7)$$

Equations (5.6) and (5.7) are plotted in Fig. 8 for $k = 3$, $\lambda = 1.2$ and 1.8, respectively.

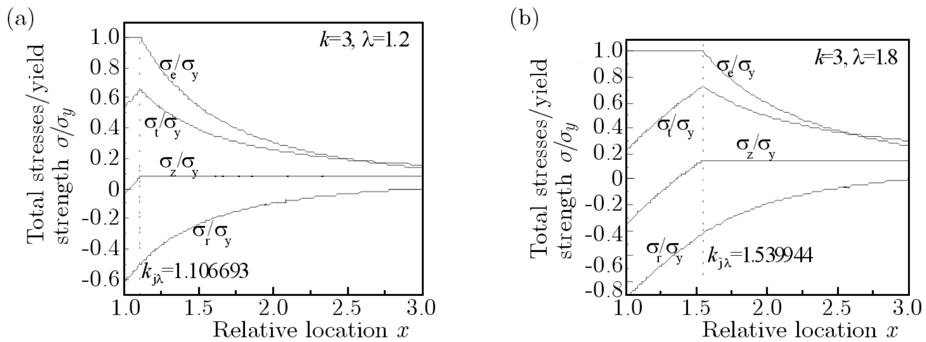

Fig. 8. Total stresses and their equivalent stress; (a) $k = 3$, $\lambda = 1.2$, (b)) $k = 3$, $\lambda = 1.8$

From Eqs. $(5.7)_{3,4}$ and Fig. 8, when $x \geqslant \sqrt{\sqrt{3} - 1}k$, $\sigma_e/\sigma_y \leqslant \sigma_\theta/\sigma_y$.

When k_j, k are related by Eq. (3.4), at $r = r_j$, the stresses determined by Eqs. (5.6) are consistent with the corresponding stresses determined by Eqs. (5.7). This testifies reliability of this paper.

According to Eqs. (5.7), seemingly the total stresses within the elastic zone are not concerned with λ. Nevertheless, $k_{j\lambda}$ depends on λ as seen in Eq. (3.4).

Actually, it is Eq. (3.1) that ensures $\sigma_e/\sigma_y \equiv 1$ everywhere within the plastic zone and $\sigma_e/\sigma_y = k_j^2/x^2$ within the elastic zone irrespective of k_j and k. Substituting Eq. (3.1) into Eq. (5.2), results in

$$\frac{\sigma_e^p}{\sigma_y} = \frac{k^2 - k_j^2 + k^2 \ln k_j^2}{(k^2-1)x^2} \tag{5.8}$$

Substituting Eqs. (5.8) and (2.2) into Eq. (5.5), one just obtains $\sigma_e/\sigma_y \equiv 1$; substituting Eqs. (5.8) and (2.4) into Eq. (5.5), one just obtains $\sigma_e/\sigma_y = k_j^2/x^2$. Therefore, as long as $p/\sigma_y = (k^2 - k_j^2 + k^2 \ln k_j^2)/(\sqrt{3}k^2)$ (i.e. Eq. (3.1)), the results for $\sigma_e/\sigma_y \equiv 1$ within the plastic zone and for $\sigma_e/\sigma_y = k_j^2/x^2$ within the elastic zone have nothing to do with the magnitude of k_j and k. In other words, providing that $p/\sigma_y = (k^2 - k_j^2 + k^2 \ln k_j^2)/(\sqrt{3}k^2)$, for any k_j and k, which are not needed to be related by Eq. (3.4), $\sigma_e/\sigma_y \equiv 1$ within the plastic zone and $\sigma_e/\sigma_y = k_j^2/x^2$ $(0 < k_j^2/x^2 < 1$, for $k_j \leqslant x \leqslant k)$ within the elastic zone are inevitable. However, k_j and k affect the residual stresses. Inadequate k_j for a certain k may cause compressive yield when a cylinder is being treated with the autofrettage, and it is necessary for k_j to be less than the value determined by Eq. (3.4), otherwise the compressive yield occurs.

Generally, from Eqs (2.2), (2.4), (5.2) and (5.5), the equivalent total stresses are:

— within the plastic zone

$$\frac{\sigma_e}{\sigma_y} = 1 - \frac{k^2 - k_j^2 + k^2 \ln k_j^2}{(k^2-1)(r/r_i)^2} + \frac{\sqrt{3}k^2}{x^2(k^2-1)}\frac{p}{\sigma_y} \tag{5.9}$$

— within the elastic zone

$$\frac{\sigma_e}{\sigma_y} = \frac{k^2(k_j^2 - 1 - \ln k_j^2)}{(k^2-1)(r/r_i)^2} + \frac{\sqrt{3}k^2}{x^2(k^2-1)}\frac{p}{\sigma_y} \tag{5.10}$$

At the elastic-plastic juncture $(x = k_j)$, Eqs. (5.9) and (5.10) both become

$$\frac{\sigma_e}{\sigma_y} = \frac{k^2(k_j^2 - 1 - \ln k_j^2)}{(k^2-1)k_j^2} + \frac{\sqrt{2}k^2 p/\sigma_y}{(k^2-1)k_j^2} \tag{5.11}$$

Letting $d(\sigma_e/\sigma_y)/dk_j = 0$ in Eq. (5.11), one obtains

$$\frac{p}{\sigma_y} = \frac{1}{\sqrt{3}} \ln k_j^2 \qquad \text{or} \qquad k_j = \exp\Big(\frac{\sqrt{3}p}{2\sigma_y}\Big) \tag{5.12}$$

This is the relation between p/σ_y and k_j when σ_{ej}/σ_y is the minimum at the elastic-plastic juncture. Combining Eq. (5.11) with Eq. (3.1) results in the entire yield loading $p_y/\sigma_y = \ln k^2/\sqrt{3}$. This means that if $\sigma_{ej} = \sigma_y$ and concurrently it is the minimum, then $p = p_y = \sigma_y \ln k^2/\sqrt{3}$, or the cylinder is entirely yielded. Nevertheless, only when $k \leqslant k_{c\lambda}$, this can be realistic. Besides, letting $k_j = k$ in Eq. (3.1), one also obtains the entire yield loading $p_y/\sigma_y = \ln k^2/\sqrt{3}$. In addition, letting $k_j = 1$ in Eq. (3.1), one obtains p_e/σ_y.

6. The effect of λ on $k_{j\lambda}$ and ε

The effect of λ on $k_{j\lambda}$ and ε is shown by Eq. (3.4) and (3.6), which is graphed in Fig. 9.

From Eqs. (3.4), (3.6) and Fig. 9, it can be concluded that:

(1) When $k \leqslant k_{c\lambda}$ $(k_{c\lambda\,max} = k_{c2\,max} = k_c)$, $k_{j\lambda\,max} = k = k_{c\lambda}$, $\varepsilon_{\lambda\,max} = 1$ and $\lambda_{max} = \eta$, when $k \geqslant k_c$, $k_{j\lambda\,max} < k$ $(k_{c\lambda})$, $\varepsilon_{\lambda\,max} \leqslant 1$, $k_{j\lambda\,max}$ is determined by $k^2 \ln k_{j\lambda}^2 - k^2 - k_{j\lambda}^2 + 2 = 0$, $\lambda = \lambda_{max} = 2$, $\varepsilon_{\lambda\,max} = (k_{j\lambda\,max} - 1)/(k-1)$.

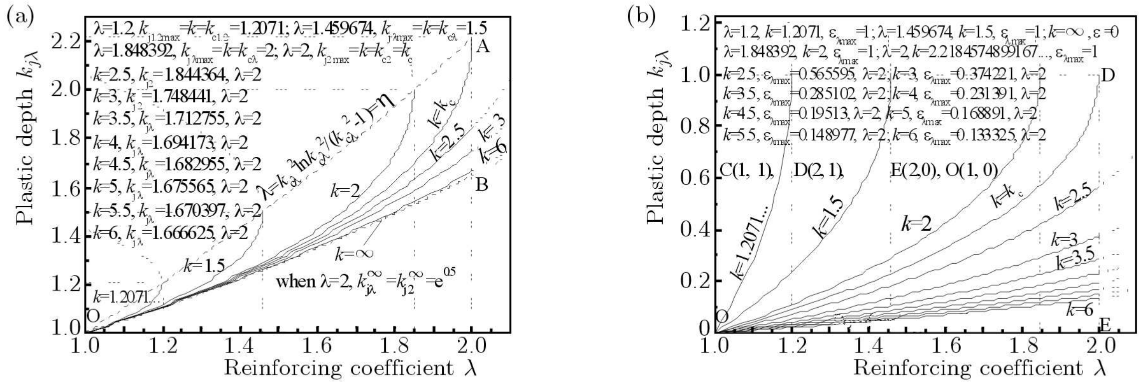

Fig. 9. Effect of λ on $k_{j\lambda}$ (a) and ε (b)

(2) For a certain k, the greater the λ, the greater the $k_{j\lambda}$ and ε_λ. So, for a pressure vessel to contain a higher pressure, the plastic depth should be deeper.

(3) The extended dotted (not dash) curves of the corresponding solid curves are nothing but mathematical results, which are meaningless in practice.

(4) With k getting greater and greater, the curves get closer and closer. For $k = \infty$ (the dash curve), if $\lambda = 2$, $k_{j2}^{\infty} = \sqrt{\mathrm{e}}$. The curve showing $k = 6$ and the curve showing $k = \infty$ almost coincide.

(5) For various λ and k, the meaningful and possible optimum plastic depth k_j is within the curved triangle OAB, for which the side OA is a linked curve of $k_{c\lambda}$, the equation of the side OA is just Eq. (3.7): $k_{j\lambda}^2 \ln k_{j\lambda}^2 - \lambda(k_{j\lambda}^2 - 1) = 0$, resulting from letting $k_{j\lambda} = k$ in Eq. (3.4), the equation of the side OB is $k_{j\lambda} = \mathrm{e}^{(\lambda-1)/2} = k_{j\lambda}^{\infty}$ and the equation of the side AB is $\lambda = 2$. The coordinate of point O is $(1, 1)$, the coordinate of point B is $(2, \sqrt{\mathrm{e}})$, the coordinate of point A is $(2, k_c)$. Correspondingly, the meaningful and possible optimum overstrain is within the rectangle CDEO. The coordinate of four vertexes of this rectangle are shown in Fig. 9b.

(6) $k = k_c = 2.218\,457\,489\,916\,7\ldots$ is the solution to the equation $k^2 \ln k/(k^2 - 1) = 1$.

(7) Point O can be regarded as a curve for $k = 1$.

(8) Factually, the dash curve OA in Fig. 9a is the solid curve OA in Fig. 2.

7. Discussion on load-bearing capacity

When $k > k_c$, the load-bearing capacity is $p/\sigma_y = \lambda p_e/\sigma_y = \lambda(k^2 - 1)/(\sqrt{3}k^2)$ for a cylinder with k and $k_{j\lambda}$; when $k < k_c$, $p/\sigma_y = p_y/\sigma_y = \ln k^2/\sqrt{3}$, which is plotted in Fig. 10.

In view of some data given in Fig. 10, the load-bearing capacity is explained as follows to show the application of the figure.

If $\lambda = 1.2$, when $k \leqslant 1.2071\ldots$ $(k_{c\lambda})$, $|\sigma'_{ei}\sigma_y| > 1$ never occurs irrespective of k_j even if $k_j = k$; when $k_j = k$, $p/\sigma_y = p_y/\sigma_y = \ln k^2/\sqrt{3}$ $(< \lambda(k^2 - 1)/(\sqrt{3}k^2))$. When $k \geqslant 1.2071\ldots$, if $k_j \leqslant k_{j\lambda}$, $|\sigma'_{ei}\sigma_y| > 1$ never occurs; if $k_j = k_{j\lambda}$, $p/\sigma_y = \lambda p_e/\sigma_y = 1.2p_e/\sigma_y$ $(< \ln k^2/\sqrt{3})$.

If $\lambda = 1.8$, when $k \leqslant 1.93322\ldots$ $(k_{c\lambda})$, $|\sigma'_{ei}\sigma_y| > 1$ never occurs irrespective of k_j even if $k_j = k$; when $k_j = k$, $p/\sigma_y = p_y/\sigma_y = \ln k^2/\sqrt{3}$ $(< \lambda(k^2 - 1)/(\sqrt{3}k^2))$. When $k \geqslant 1.93322\ldots$, if $k_j \leqslant k_{j\lambda}$, $|\sigma'_{ei}\sigma_y| > 1$ never occurs; if $k_j = k_{j\lambda}$, $p/\sigma_y = \lambda p_e/\sigma_y = 1.8p_e/\sigma_y$ $(< \ln k^2/\sqrt{3})$.

If $\lambda = 2$, when $k \leqslant k_c$, $|\sigma'_{ei}\sigma_y| > 1$ never occurs irrespective of k_j even if $k_j = k$; when $k_j = k$, $p/\sigma_y = p_y/\sigma_y = \ln k^2/\sqrt{3}$ $(< \lambda(k^2 - 1)/(\sqrt{3}k^2))$. When $k \geqslant k_c$, if $k_j \leqslant k_{j\lambda}$, $|\sigma'_{ei}\sigma_y| > 1$

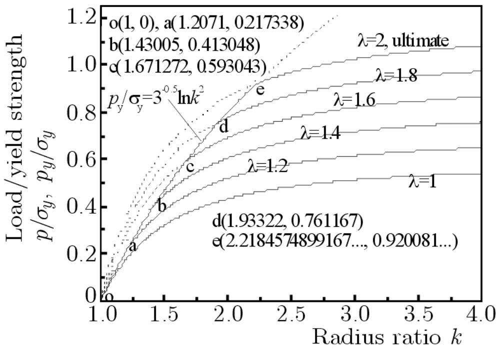

Fig. 10. Load-bearing capacity of a cylinder

never occurs, if $k_j = k_{j\lambda}$, $p/\sigma_y = 2p_e/\sigma_y$ $(< \ln k^2/\sqrt{3})$. When $\lambda = 2$, $k_{j\lambda}$ is marked as k_{j*} by Zhu (2008).

Substituting Eq. (3.7) into $p/\sigma_y = \lambda(k^2 - 1)/(\sqrt{3}k^2)$, one obtains $p/\sigma_y = \ln k_{c\lambda}^2/\sqrt{3}$. It is easy to prove that when $k \leqslant k_{c\lambda}$, $\ln k \leqslant (k^2 - 1)/k^2$.

8. Conclusions

- The optimum operation conditions are: for any k, the plastic depth is determined by $k^2 \ln k_{j\lambda}^2 - (\lambda - 1)k^2 - k_{j\lambda}^2 + \lambda = 0$, and the load-bearing capacity is determined by $p = \lambda p_e$, where $\lambda = \eta = k^2 \ln k^2/(k^2 - 1)$ when $k \leqslant k_c$ ($\lambda = \eta = 1 \sim 2$ calculated by $k^2 \ln k^2/(k^2 - 1)$ for $k = 1 \sim k_c$) and $\lambda = 1 \sim 2$ for choosing when $k \geqslant k_c$. $\lambda \leqslant 2$ is required for $|\sigma'_{ei}| \leqslant \sigma_y$.
- When $k \leqslant k_{c\lambda}$, $|\sigma'_{ei}\sigma_y| > 1$ never occurs irrespective of k_j even if $k_j = k$, if $k_j = k$, the ultimate load-bearing capacity $p/\sigma_y = p_y/\sigma_y = \ln k^2/\sqrt{3}$ $(< \lambda(k^2 - 1)/k^2)$. When $k \geqslant k_{c\lambda}$, if $k_j \leqslant k_{j\lambda}$, $|\sigma'_{ei}\sigma_y| > 1$ never occurs, if $k_j = k_{j\lambda}$, the load-bearing capacity $p/\sigma_y = \lambda p_e/\sigma_y$ $(< \ln k^2/\sqrt{3})$. When $\lambda = 2$, $k_{j\lambda}$ and p reach the maxima: $k^2 \ln k_j^2 - k^2 - k_{j2}^2 + 2 = 0$ and $p/\sigma_y = 2p_e/\sigma_y$.
- The possible and optimum plastic depth k_j is situated in the quasi-infinite area constructed of the horizontal axis, the straight line $k_j = k$ and the curve $k^2 \ln k_{j\lambda}^2 - k^2 - k_{j\lambda}^2 + 2 = 0$.
- If $k \leqslant k_c$ (or $k_{c\lambda}$ for $\lambda = 2$), $k_{j\lambda\,max} = k = k_{c\lambda}$; if $k \geqslant k_c$, $k_{j\lambda\,max} < k$ $(k_{c\lambda})$, $k_{j\lambda\,max}$ is determined by $k^2 \ln k_{j\lambda}^2 - k^2 - k_{j\lambda}^2 + 2 = 0$ $(\lambda = 2)$. The greater the λ, the greater the $k_{j\lambda}$.
- As long as $p/\sigma_y = (k^2 - k_j^2 + k^2 \ln k_j^2)/(\sqrt{3}k^2)$, irrespective of k_j, $\sigma_e \equiv \sigma_y$ within the whole plastic zone, or σ_e is even, and the equivalent stress of total stress within the elastic zone is always lower than σ_y. However, if k_j is outside the quasi-infinite area of the possible and optimum plastic depth, the compressive yield occurs.
- Due to the equation $k^2 \ln k_{j\lambda}^2 - (\lambda - 1)k^2 - k_{j\lambda}^2 + \lambda = 0$, the relations between various parameters and their varying tendency become concise and clearer, and the equations concerned with the autofrettage are simplified greatly.

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