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BUCKLING OF CYLINDRICAL SHELLS UNDER EXTERNAL PRESSURE IN A HAMILTONIAN SYSTEM

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In this article, the elastic buckling behavior of cylindrical shells under external pressure is studied by using a symplectic method. Based on Donnell's shell theory, the governing equations which are expressed in stress function and radial displacement are re-arranged into the Hamiltonian canonical equations. The critical loads and buckling modes are reduced to solving for symplectic eigenvalues and eigenvectors. The buckling solutions are mainly grouped into four categories according to the natures of the buckling modes. The effects of geometrical parameters and boundary conditions on the buckling loads and modes are examined in detail.

Keywords: buckling, cylindrical shells, external pressure, Hamiltonian system

1. Introduction

With the development of the ocean science and underwater project, the safety criterion for predicting the collapse of the pressurized vessels and pipelines is necessary in the engineering application. In the past, this kind of problem was investigated theoretically and experimentally by many researchers (Mises, 1914; Flügge, 1973; Batdorf, 1947; Nash, 1954; Galletly and Bart, 1956, Armenakas and Herrmann, 1963; Soong, 1967). In these studies, only a simple one-term mode function was adopted, and the problem was solved under some special boundary conditions which need to be satisfied by an assumed expression. By employing the similar method, Yamaki (1969) and Vodenitcharova and Ansourian (1996) carried out more extensive research and presented some thought-provoking solutions for this problem under various boundary conditions. Meanwhile, for the sake of resolving more practical problems, numerous approximate and numerical methods have sprung up and exhibited an excellent performance in handling complicated situations. With the aid of the Ritz method, Tian *et al.* (1999) investigated elastic buckling of cylindrical shells with ring-stiffeners under pressure. It was appropriate for any combination of end conditions by using polynomial functions multiplied by boundary equations raised to appropriate powers as the Ritz functions. Pinna and Ronalds (2000)] examined eigenvalue buckling of cylindrical shells subjected to hydrostatic load under various boundary conditions through an energy method. The effect of ends conditions, including radial elastic restraint at the open end, was discussed in detail. Xue and Fatt (2002) obtained analytical solutions for elastic buckling of a non-uniform, long cylindrical shell subjected to external hydrostatic pressure. The finite element method was applied to examine the validity of analytical method and the results were found to be in close agreement with the numerical method (Goncalves *et al.*, 2008). A set of experimental tests were also conducted by Hübner *et al.* (2007) to improve the assessment procedure for cylindrical shells. For the post-buckling analysis of cylindrical shells under external pressure, the numerical results obtained by means of the nonlinear finite element method were compared with the results of the experimental study (Aghajari *et al.*, 2006). In order to trace the nonlinear equilibrium paths, the "Arc-Length-Type Method" was used in the study. Shen (2008) also developed a boundary layer theory for the similar problem and applied the perturbation technique to determine the buckling pressure and post-buckling equilibrium paths. More comprehensive results and discussions can also be found in the monographs by Teng and Rotter (2004), Ventsel and Krauthammer (2001).

However, most of the traditional analytic methods mentioned above belong to the Lagrange solving system. It involves only one set of variables and can be resolved by the force method or the displacement method. In this system, the fundamental equations exist in form of high-order partial differential equations which are difficult to be analytically or numerically worked out. Recently, Zhong (2004) developed a symplectic analytical method for some fundamental problems in solid mechanics. Through the Legendre transformation, Lagrange formulations can be transformed into Hamiltonian dual equations by introducing dual variables. By employing separation of the variables, the fundamental problem can be boiled down to solving for symplectic eigenvalues and eigenvectors. According to the completeness theorem of the symplectic system, all solutions can be sought out for the current problem. The symplectic solving approach is rigorous and rational in solving the problem, and boundary conditions are satisfied in a natural manner. Xu *et al.* (2006) investigated the local buckling and the propagation (and reflection) of axial stress waves by introducing a Hamiltonian system or a symplectic system into dynamic buckling of cylindrical shells.

In this study, a new Hamiltonian system is established to investigate buckling of cylindrical shells under external pressure. Hamiltonian canonical equations are derived from the Hamiltonian principle of mixed energy. According to rational deduction, buckling characteristic parameters should be determined by solving for eigenequations in a symplectic space. A uniform solving process is developed for this problem under symmetric and non-symmetric boundary conditions. The factors which influence buckling results are also discussed in detail.

2. Basic equations

Consider a thin-walled cylindrical shell with radius R, length l, thickness h, Young's modulus E and Poisson's ratio ν (Fig. 1), compressed by uniform lateral pressure P. A circular cylindrical coordinate with an x-axis along the central axis is adopted. And the corresponding displacements can be denoted that x-direction is u, θ-direction is v, z-direction is w, respectively. The membrane internal forces are given by

$$
\begin{array}{lll}
N_x = K(\varepsilon_x + \nu\varepsilon_\theta) & N_\theta = K(\varepsilon_\theta + \nu\varepsilon_x) & N_{x\theta} = \dfrac{1}{2}K\varepsilon_{x\theta}(1-\nu) \\
M_x = D(\kappa_x + \nu\kappa_\theta) & M_\theta = D(\kappa_\theta + \nu\kappa_x) & M_{x\theta} = D(1-\nu)\kappa_{x\theta}
\end{array}
\tag{2.1}
$$

where $D = Eh^3/[12(1-\nu^2)]$ and $K = Eh/(1-\nu^2)$. $\{N_x, N_\theta, N_{x\theta}\}$ and $\{M_x, M_\theta, M_{x\theta}\}$ are the resultant membrane forces and bending moments. The strain components $\{\varepsilon_x, \varepsilon_\theta, \varepsilon_{x\theta}\}$ and curvature components $\{\kappa_x, \kappa_\theta, \kappa_{x\theta}\}$ are expressed by

$$\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2}\Big(\frac{\partial w}{\partial x}\Big)^2 & \kappa_x &= -\frac{\partial^2 w}{\partial x^2} \\ \varepsilon_\theta &= \frac{1}{R}\Big(\frac{\partial v}{\partial \theta} - w\Big) + \frac{1}{2R^2}\Big(\frac{\partial w}{\partial \theta}\Big)^2 & \kappa_\theta &= -\frac{1}{R^2}\frac{\partial^2 w}{\partial \theta^2} \\ \varepsilon_{x\theta} &= \frac{1}{R}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial x} + \frac{1}{R}\frac{\partial w}{\partial x}\frac{\partial w}{\partial \theta} & \kappa_{x\theta} &= -\frac{1}{R}\frac{\partial^2 w}{\partial x \partial \theta} \end{aligned} \tag{2.2}$$

By introducing a stress function F, the internal forces can be written into

$$N_x = \frac{1}{R^2}\frac{\partial^2 F}{\partial \theta^2} \qquad N_\theta = \frac{\partial^2 F}{\partial x^2} \qquad N_{x\theta} = -\frac{1}{R}\frac{\partial^2 F}{\partial x \partial \theta} \tag{2.3}$$

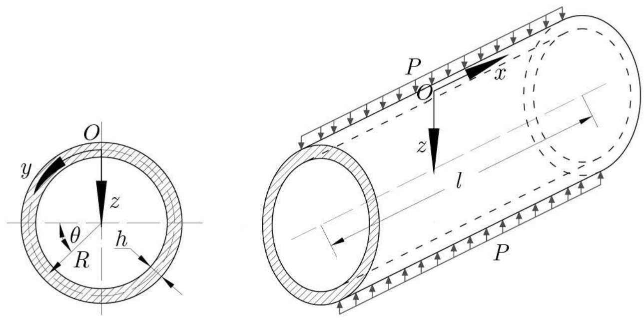

Fig. 1. Geometric parameters of a cylindrical shell under pressure

Assuming the membrane pre-buckling state, the pre-buckling internal forces are $\widetilde{N}_x = 0$, $\widetilde{N}_\theta = -PR$, $\widetilde{N}_{x\theta} = 0$. Total potential energy, caused by the incremental buckling displacements (u, v, w) and the stress function F, consists of the extension potential energy, bending potential energy and the potential of external forces. Neglecting higher-order nonlinear terms, it can be obtained as

$$\begin{aligned} \varPi &= \varPi_\varepsilon + \varPi_k - \varPi_w \\ &= \iint\limits_S \Big[\frac{1}{R^2}\frac{\partial^2 F}{\partial \theta^2}\frac{\partial u}{\partial x} + \frac{\partial^2 F}{\partial x^2}\frac{1}{R}\Big(\frac{\partial v}{\partial \theta} - w\Big) - \frac{1}{R}\frac{\partial^2 F}{\partial x \partial \theta}\Big(\frac{1}{R}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial x}\Big)\Big] \\ &\quad - \frac{1}{2Eh}\Big\{\Big(\frac{\partial^2 F}{\partial x^2} + \frac{1}{R^2}\frac{\partial^2 F}{\partial \theta^2}\Big)^2 - 2(1+\nu)\Big[\frac{\partial^2 F}{\partial x^2}\frac{1}{R^2}\frac{\partial^2 F}{\partial \theta^2} - \Big(\frac{1}{R}\frac{\partial^2 F}{\partial x \partial \theta}\Big)^2\Big]\Big\} \\ &\quad + \frac{1}{2}D\Big\{\Big(\frac{\partial^2 w}{\partial x^2} + \frac{1}{R^2}\frac{\partial^2 w}{\partial \theta^2}\Big)^2 - 2(1-\nu)\Big[\frac{\partial^2 w}{\partial x^2}\frac{1}{R^2}\frac{\partial^2 w}{\partial \theta^2} - \Big(\frac{1}{R}\frac{\partial^2 w}{\partial x \partial \theta}\Big)^2\Big]\Big\} \\ &\quad - \frac{\widetilde{N}_\theta}{2R^2}\Big(\frac{\partial w}{\partial \theta}\Big)^2 R\,dx\,d\theta \end{aligned} \tag{2.4}$$

A Lagrange density function can be derived from Eq. (2.4) as

$$\begin{aligned} L &= \frac{1}{R^2}\frac{\partial^2 F}{\partial \theta^2}\frac{\partial u}{\partial x} + \frac{\partial^2 F}{\partial x^2}\frac{1}{R}\Big(\frac{\partial v}{\partial \theta} - w\Big) - \frac{1}{R}\frac{\partial^2 F}{\partial x \partial \theta}\Big(\frac{1}{R}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial x}\Big) \\ &\quad - \frac{1}{2Eh}\Big(\frac{\partial^2 F}{\partial x^2} + \frac{1}{R^2}\frac{\partial^2 F}{\partial \theta^2}\Big)^2 + \frac{1}{2}D\Big(\frac{\partial^2 w}{\partial x^2} + \frac{1}{R^2}\frac{\partial^2 w}{\partial \theta^2}\Big)^2 - \frac{\widetilde{N}_\theta}{2R^2}\Big(\frac{\partial w}{\partial \theta}\Big)^2 \end{aligned} \tag{2.5}$$

Based on the Hamiltonian principle, the variational equation is expressed as

$$\delta \varPi = \delta \int\limits_0^{2\pi} R\,d\theta \int\limits_{-l/2}^{l/2} L(F, w)\,dx = 0 \tag{2.6}$$

Then Eq. (2.5) is substituted into Eq. (2.6) and the variation with respect to F and w, is respectively taken. The compatibility condition and the equilibrium equation are obtained as

$$\frac{\delta\Pi}{\delta F}=\frac{1}{Eh}\nabla^4 F+\frac{1}{R}\frac{\partial^2 w}{\partial x^2}=0 \qquad \frac{\delta\Pi}{\delta w}=D\nabla^4 w-\frac{1}{R}\frac{\partial^2 F}{\partial x^2}+\frac{\widetilde{N}_\theta}{R^2}\frac{\partial^2 w}{\partial\theta^2}=0 \tag{2.7}$$

where $\nabla^2=\partial^2/\partial x^2+\partial^2/(R^2\partial\theta^2)$ is the Laplacian operator.

3. Symplectic system

The dimensionless terms $X=x/R$, W/R, $\phi=F/(Eh^3)$, $L=l/R$, $H=h/R$, $\beta=\alpha H^2$ and $N_{cr}=\widetilde{N}_\theta/D$ are adopted. The dimensionless critical pressure N_{cr} relates physical parameters with geometric parameters. An over-dot represents differentiation with respect to θ, namely $\dot{W}=\partial W/\partial\theta$ in which the θ-coordinate is chosen as a time-equivalent coordinate and, $\partial_X W=\partial W/\partial X$ in which X-coordinate is taken as a transverse coordinate.

Define two new variables $\xi=-\dot{W}$ and $\varphi=-\dot{\phi}$. The dimensionless Lagrange density function is expressed as

$$L=-\alpha W\partial_X^2\phi-\frac{1}{2}\beta(\partial_X^2\phi+\ddot{\phi})^2+\frac{1}{2}(\partial_X^2 W+\ddot{W})^2-\frac{1}{2}N_{cr}(\dot{W})^2 \tag{3.1}$$

Define a vector $\mathbf{q}=\{W,\xi,\phi,\varphi\}^{\mathrm{T}}=\{q_1,q_2,q_3,q_4\}^{\mathrm{T}}$, the dual vector $\mathbf{p}$ can be deducted as

$$\begin{aligned} p_1&=\frac{\delta L}{\delta\dot{q}_1}=-(\dddot{W}+\partial_X^2\dot{W})-N_{cr}\dot{W} \qquad & p_2&=\frac{\delta L}{\delta\dot{q}_2}=-(\ddot{W}+\partial_X^2 W)\\ p_3&=\frac{\delta L}{\delta\dot{q}_3}=\beta(\phi+\partial_X^2\dot{\phi}) & p_4&=\frac{\delta L}{\delta\dot{q}_4}=\beta(\ddot{\phi}+\partial_X^2\phi) \end{aligned} \tag{3.2}$$

The dual variables denote the equivalent transverse shear stress, bending moment, in-plane shear stress and normal stress, respectively. Substituting the dual variables into Eq. (3.1), the Hamiltonian density function can be obtained as

$$\begin{aligned} H(\mathbf{q},\mathbf{p})&=\mathbf{p}^{\mathrm{T}}\dot{\mathbf{q}}(\mathbf{q},\mathbf{p})-L(\mathbf{q},\mathbf{p})\\ &=-p_1q_2+\frac{1}{2}p_2^2+p_2\partial_X^2 q_1-p_3q_4-\frac{1}{2\beta}p_4^2+p_4\partial_X^2 q_3+\alpha q_1\partial_X^2 q_3+\frac{1}{2}N_{cr}(q_2)^2 \end{aligned} \tag{3.3}$$

Substituting Eq. (3.3) into Eq. (2.6), we have

$$\delta\int[\mathbf{p}^{\mathrm{T}}\dot{\mathbf{q}}(\mathbf{q},\mathbf{p})-H(\mathbf{q},\mathbf{p})]\,ds=0 \tag{3.4}$$

Then the Hamiltonian canonical equations are obtained by integration by parts as

$$\dot{\mathbf{q}}=\frac{\delta H}{\delta\mathbf{p}} \qquad \dot{\mathbf{p}}=-\frac{\delta H}{\delta\mathbf{q}} \tag{3.5}$$

Equations (3.5) can be expressed in the matrix form as

$$\begin{Bmatrix}\dot{\mathbf{q}}\\ \dot{\mathbf{p}}\end{Bmatrix}=\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & -\mathbf{A}^{\mathrm{T}}\end{bmatrix}\begin{Bmatrix}\mathbf{q}\\ \mathbf{p}\end{Bmatrix} \tag{3.6}$$

where

$$\mathbf{A}=\begin{bmatrix}0 & -1 & 0 & 0\\ \partial_X^2 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & \partial_X^2 & 0\end{bmatrix} \qquad \mathbf{B}=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -\frac{1}{\beta}\end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 0 & 0 & -\alpha\partial_X^2 & 0 \\ 0 & -N_{cr} & 0 & 0 \\ -\alpha\partial_X^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Define a state vector $\boldsymbol{\psi} = \{\mathbf{q}^{\mathrm{T}}, \mathbf{p}^{\mathrm{T}}\}^{\mathrm{T}}$, then Eq. (3.6) can be simplified as

$$\dot{\boldsymbol{\psi}} = \mathbf{H}\boldsymbol{\psi} \tag{3.7}$$

To discuss the property of the matrix $\mathbf{H}$, an inner product needs to be defined as

$$\langle \boldsymbol{\psi}_1, \boldsymbol{\psi}_2 \rangle = \int\limits_{-L/2}^{L/2} (\mathbf{q}_1^{\mathrm{T}}\mathbf{p}_2 - \mathbf{q}_2^{\mathrm{T}}\mathbf{p}_1)\, dX \tag{3.8}$$

It can be proved that $\mathbf{H}$ is a Hamiltonian operator matrix (Zhong, 2004).

4. Symplectic eigensolutions and orthogonality relation

In a symplectic system, by separation of variables, the solution of Eq. (3.7) is expressed as

$$\boldsymbol{\psi}(X,\theta) = \boldsymbol{\eta}(X)\mathrm{e}^{\mu\theta} \tag{4.1}$$

where $\boldsymbol{\eta} = \{\overline{q}_1, \overline{q}_2, \overline{q}_3, \overline{q}_4, \overline{p}_1, \overline{p}_2, \overline{p}_3, \overline{p}_4\}$ and μ are the eigenvector and the eigenvalue. Substitute Eq. (4.1) into Eq. (3.7), the eigenvalue equation can be obtained as

$$\mathbf{H}\boldsymbol{\eta}(X) = \mu\boldsymbol{\eta}(X) \tag{4.2}$$

According to the property of revolutionary shell, Eq. (4.1) needs to satisfy the closed condition

$$\boldsymbol{\psi}(X,0) = \boldsymbol{\psi}(X,2\pi) \tag{4.3}$$

So it is proved that $\mu_n = n\mathrm{i}$ $(n = 0, \pm 1, \pm 2, \ldots)$. Substituting the corresponding eigenvalues $\mu_n = n\mathrm{i}$ into Eq. (4.2), the characteristic polynomial can be written as

$$\lambda^8 + a\lambda^6 + b\lambda^4 + c\lambda^2 + d = 0 \tag{4.4}$$

where $a = -4n^2$, $b = 6n^4 - n^2N_{cr} + \alpha^2/\beta$, $c = 2n^4N_{cr} - 4n^6$ and $d = n^8 - n^6N_{cr}$. If $n \neq 0$, the eigensolutions can be classified into four sorts. They can be given by:

<u>Sort 1:</u> If λ_1, λ_2, λ_3 and λ_4 are complex roots, besides, $\lambda_1 = \overline{\lambda_3}$, $\lambda_2 = \overline{\lambda_4}$, $|\lambda_1| \neq |\lambda_2|$, α_i and β_i are absolute values of the real and imaginary part of the characteristic root λ_i, respectively, $\lambda_i = -\lambda_{i+4}$ $(i = 1, 2, 3, 4)$. It is expressed by

$$\begin{aligned} \boldsymbol{\eta}_n &= \mathbf{c}_1\mathrm{e}^{\alpha_1 X}\cos(\beta_1 X) + \mathbf{c}_2\mathrm{e}^{\alpha_1 X}\sin(\beta_1 X) + \mathbf{c}_3\mathrm{e}^{-\alpha_1 X}\cos(\beta_1 X) + \mathbf{c}_4\mathrm{e}^{-\alpha_1 X}\sin(\beta_1 X) \\ &\quad + \mathbf{c}_5\mathrm{e}^{\alpha_2 X}\cos(\beta_2 X) + \mathbf{c}_6\mathrm{e}^{\alpha_2 X}\sin(\beta_2 X) + \mathbf{c}_7\mathrm{e}^{-\alpha_2 X}\cos(\beta_2 X) + \mathbf{c}_8\mathrm{e}^{-\alpha_2 X}\sin(\beta_2 X) \end{aligned} \tag{4.5}$$

<u>Sort 2:</u> If λ_1, λ_2, λ_3, λ_4 are different real roots, λ_5, λ_6 are complex roots, besides, $\lambda_5 = \overline{\lambda_6}$, α_1 and β_1 are absolute values of the real and imaginary part of the characteristic root λ_5, respectively, $\lambda_i = -\lambda_{i+2}$ $(i = 5, 6)$. It is given by

$$\begin{aligned} \boldsymbol{\eta}_n &= \mathbf{c}_1\mathrm{e}^{\lambda_1 X} + \mathbf{c}_2\mathrm{e}^{\lambda_2 X} + \mathbf{c}_3\mathrm{e}^{\lambda_3 X} + \mathbf{c}_2\mathrm{e}^{\lambda_4 X} + \mathbf{c}_5\mathrm{e}^{\alpha_1 X}\cos(\beta_1 X) + \mathbf{c}_6\mathrm{e}^{\alpha_1 X}\sin(\beta_1 X) \\ &\quad + \mathbf{c}_7\mathrm{e}^{-\alpha_1 X}\cos(\beta_1 X) + \mathbf{c}_8\mathrm{e}^{-\alpha_1 X}\sin(\beta_1 X) \end{aligned} \tag{4.6}$$

Sort 3: If λ_1, λ_2 are conjugate pure imaginary roots, λ_3, λ_4 are different real roots, λ_5, λ_6 are complex roots, besides, $\lambda_5 = \overline{\lambda}_6$, β_1 are absolute value of λ_1, α_2 and β_2 are absolute values of the real and imaginary part of the characteristic root λ_5, respectively, $\lambda_i = -\lambda_{i+2}$ $(i = 5, 6)$. It is that

$$\begin{aligned}\boldsymbol{\eta}_n &= \mathbf{c}_1 \cos(\beta_1 X) + \mathbf{c}_2 \sin(\beta_1 X) + \mathbf{c}_3 \mathrm{e}^{\lambda_3 X} + \mathbf{c}_4 \mathrm{e}^{\lambda_4 X} + \mathbf{c}_5 \mathrm{e}^{\alpha_2 X} \cos(\beta_2 X)\\ &\quad + \mathbf{c}_6 \mathrm{e}^{\alpha_2 X} \sin(\beta_2 X) + \mathbf{c}_7 \mathrm{e}^{-\alpha_2 X} \cos(\beta_2 X) + \mathbf{c}_8 \mathrm{e}^{-\alpha_2 X} \sin(\beta_2 X)\end{aligned} \tag{4.7}$$

Sort 4: If λ_1, λ_2 are conjugate pure imaginary roots, $\lambda_3, \lambda_4, \ldots, \lambda_8$ are different real roots, β_1 are absolute values of λ_1. It is expressed by

$$\boldsymbol{\eta}_n = \mathbf{c}_1 \cos(\beta_1 X) + \mathbf{c}_2 \sin(\beta_1 X) + \mathbf{c}_3 \mathrm{e}^{\lambda_3 X} + \mathbf{c}_4 \mathrm{e}^{\lambda_4 X} + \mathbf{c}_5 \mathrm{e}^{\lambda_5 X} + \mathbf{c}_6 \mathrm{e}^{\lambda_6 X} + \mathbf{c}_7 \mathrm{e}^{\lambda_7 X} + \mathbf{c}_8 \mathrm{e}^{\lambda_8 X} \tag{4.8}$$

where $\mathbf{c}_k = \{c_k^1, c_k^2, \ldots, c_k^8\}^{\mathrm{T}}$ $(k = 1, 2, \ldots, 8)$ are eight constant vectors which can be determined from boundary conditions.

For a special case $n = 0$, $\lambda = 0$ is a quadruple root of Eq. (4.4). So the eigenvector can be written into

$$\boldsymbol{\eta}_0 = \mathbf{c}_1 \mathrm{e}^{\lambda_1 X} + \mathbf{c}_2 \mathrm{e}^{\lambda_2 X} + \mathbf{c}_3 \mathrm{e}^{\lambda_3 X} + \mathbf{c}_4 \mathrm{e}^{\lambda_4 X} + \mathbf{c}_5 X^3 + \mathbf{c}_6 X^2 + \mathbf{c}_7 X + \mathbf{c}_8 \tag{4.9}$$

By considering the specified boundary condition, it can be proved that this equation have only a trivial solution for incremental components. So, there are no axisymmetric buckling modes for buckling of cylindrical shells under pressure. Any solutions of the buckling problem can be expanded as

$$\boldsymbol{\psi}(X, \theta) = \sum a_n(\theta) \boldsymbol{\eta}_n(X) \tag{4.10}$$

where $a_n(\theta)$ is an undetermined function which can be found by considering boundary conditions.

5. Boundary conditions and the buckling bifurcation condition

All boundary conditions described at the two ends $(X = \pm L/2)$ can be derived from the variational principle, Eq. (2.6). It is well known that transverse boundary conditions are generally defined by the displacement or the internal force. In a symplectic system, they need to be expressed in terms of Hamiltonian dual variables as:

— the clamped boundary condition

$$W = q_1\Big|_{X=\pm L/2} = 0 \qquad \partial_X W = \partial_X q_1\Big|_{X=\pm L/2} = 0 \tag{5.1}$$

— the simply supported boundary condition

$$W = q_1\Big|_{X=\pm L/2} = 0 \qquad \partial_X^2 W = \partial_X^2 q_1\Big|_{X=\pm L/2} = 0 \tag{5.2}$$

— the free boundary condition

$$Q_X = (1-\nu)\partial_X^3 q_1 + (2-\nu)\partial_X p_2\Big|_{X=\pm L/2} = 0 \qquad M_X = \nu p_2 - (1-\nu)\partial_X^2 q_1\Big|_{X=\pm L/2} = 0 \tag{5.3}$$

Meanwhile, the internal displacement and the force should also be satisfy the following four in-plane conditions. For the displacement conditions $U = 0$ and $V = 0$, it needs to be replaced by $\partial_\theta^2 U = 0$ and $\partial_\theta V = 0$, which can be deducted by considering Eq. (4.1). They are expressed as:

— Condition 1

$$\begin{aligned} &\partial_\theta^2 U = -(1+\nu)\partial_X^3 q_3 + \frac{2+\nu}{\beta}\partial_X p_4 + \frac{1}{H^2}\partial_X q_1\Big|_{X=\pm L/2} = 0 \\ &\partial_\theta V = (1+\nu)\partial_X^2 q_3 - \frac{\nu}{\beta} p_4 + \frac{1}{H^2} q_1\Big|_{X=\pm L/2} = 0 \end{aligned} \tag{5.4}$$

— Condition 2

$$\begin{aligned} &\partial_\theta^2 U = (1+\nu)\partial_X^3 q_3 - \frac{\nu}{\beta}\partial_X p_4 + \frac{1}{H^2}\partial_X q_1\Big|_{X=\pm L/2} = 0 \\ &N_{X\theta} = \partial_X q_4\Big|_{X=\pm L/2} = 0 \end{aligned} \tag{5.5}$$

— Condition 3

$$N_X = \frac{p_4}{\beta} - \partial_X^2 q_3\Big|_{X=\pm L/2} = 0 \qquad \partial_\theta V = \partial_X^2 q_3 + \frac{1}{H^2} q_1\Big|_{X=\pm L/2} = 0 \tag{5.6}$$

— Condition 4

$$N_X = \frac{p_4}{\beta} - \partial_X^2 q_3\Big|_{X=\pm L/2} = 0 \qquad N_{X\theta} = \partial_X q_4\Big|_{X=\pm L/2} = 0 \tag{5.7}$$

By making eigenvectors Eqs. (4.5)-(4.8) satisfy the specific boundary conditions, a set of eight homogeneous linear equations can be obtained as

$$\mathbf{D}\mathbf{c}^1 = 0 \tag{5.8}$$

where $\mathbf{c}^1 = \{c_1^1, c_2^1, \ldots, c_8^1\}$ represents the unknown coefficients in the original variable $\overline{q}_1$. In order that they have non-trivial solutions, the determinant of Eq. (5.8) should vanish. Then the bifurcation condition can be given by

$$\begin{vmatrix} D_{11}^{(i)}(N_{cr},n,-L/2) & D_{12}^{(i)}(N_{cr},n,-L/2) & \cdots & D_{18}^{(i)}(N_{cr},n,-L/2) \\ D_{21}^{(i)}(N_{cr},n,-L/2) & D_{22}^{(i)}(N_{cr},n,-L/2) & \cdots & D_{28}^{(i)}(N_{cr},n,-L/2) \\ D_{31}^{(j)}(N_{cr},n,-L/2) & D_{32}^{(j)}(N_{cr},n,-L/2) & \cdots & D_{38}^{(j)}(N_{cr},n,-L/2) \\ D_{41}^{(j)}(N_{cr},n,-L/2) & D_{42}^{(j)}(N_{cr},n,-L/2) & \cdots & D_{48}^{(j)}(N_{cr},n,-L/2) \\ D_{51}^{(i)}(N_{cr},n,L/2) & D_{52}^{(i)}(N_{cr},n,L/2) & \cdots & D_{58}^{(i)}(N_{cr},n,L/2) \\ D_{61}^{(i)}(N_{cr},n,L/2) & D_{62}^{(i)}(N_{cr},n,L/2) & \cdots & D_{68}^{(i)}(N_{cr},n,L/2) \\ D_{71}^{(j)}(N_{cr},n,L/2) & D_{72}^{(j)}(N_{cr},n,L/2) & \cdots & D_{78}^{(j)}(N_{cr},n,L/2) \\ D_{81}^{(j)}(N_{cr},n,L/2) & D_{82}^{(j)}(N_{cr},n,L/2) & \cdots & D_{88}^{(j)}(N_{cr},n,L/2) \end{vmatrix}_{8\times 8} = 0 \tag{5.9}$$

where $i = 1,2,3$; $j = 1,2,\ldots,4$ indicate three transverse boundaries and four in-plane boundaries, respectively. The critical load and the corresponding buckling mode can be determined from Eq. (5.9) and Eqs. (4.5)-(4.8).

6. Buckling results and discussion

6.1. Results of cylindrical shells under symmetric boundary conditions

A dimensionless curvature parameter was introduced by Batdorf (1947) as $Z = \sqrt{1-\nu^2}L^2/H$. In the following analysis, thickness $H = h/R = 1/405$, Poisson's ratio $\nu = 0.3$ is selected. Eight sets of symmetric boundary conditions are described as: $C1$: clamped edges and Condition 1, $C2$: clamped edges and Condition 2, $C3$: clamped edges and Condition 3,

$C4$: clamped edges and Condition 4, $S1$: simply supported edges and Condition 1, $S2$: simply supported edges and Condition 2, $S3$: simply supported edges and Condition 3, $S4$: Simply supported edges and Condition 4.

For some fixed geometrical parameters, the minimum critical pressure can always be determined for different eigenvalues $\mu_n = n\mathrm{i}$ $(n = \pm 1, \pm 2, \ldots)$. According to the uniform buckling deflection, Eq. (4.1), it is obvious that the integer n indicates the number of circumferential buckling waves. So the corresponding buckling modes can also be referred to as the n-th order buckling modes (Xu *et al.*, 2006). Variations of the minimum critical pressures N_{cr} determined by bifurcation condition Eq. (5.9) with Z are displayed in Fig. 2. It is seen there that the minimum buckling loads decrease rapidly with the increase of Z. For an extremely short cylindrical shell with $Z < 5$, the minimum critical loads of cases $C1$-$C4$, calculated for Sort 4, is smaller than that of cases $S1$-$S4$, and the in-plane boundary conditions have tiny influence on the minimum critical pressure. For Z greater than 5, the effect of the in-plane boundary conditions becomes more significant than that of transverse boundary conditions, and the results of $C1$, $C2$, $S1$, $S2$, belonging to Sort 3, are greater than those for $C3$, $C3$, $S3$, $S3$. The longer the cylindrical shell is, the more distinctly this discrepancy becomes. Meanwhile, variations of the corresponding circumferential waves with the geometrical parameter Z are presented in Fig. 3. In order to distinguish between different cases, a dimensionless wave factor $N = Ln/\pi$ is introduced. The tendency of variations of the wave factors N with Z is similar to that of variations of N_{cr} with Z.

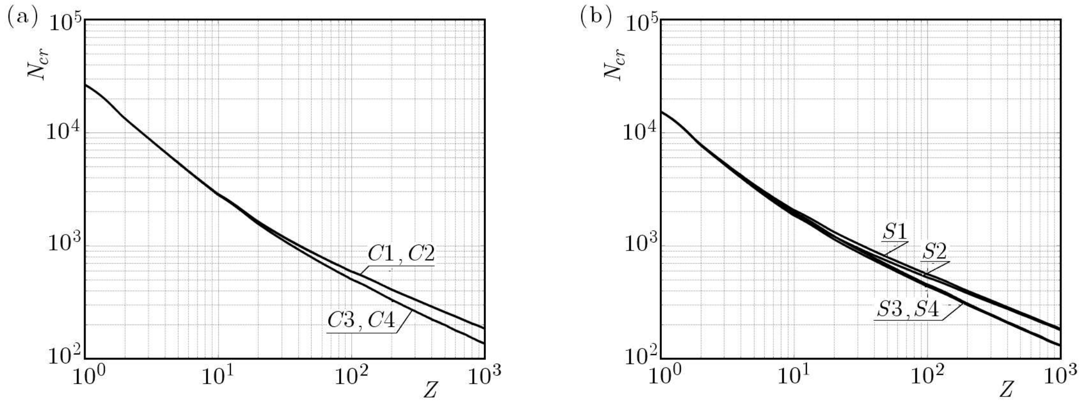

Fig. 2. N_{cr} vs. Z under symmetric boundary conditions: (a) $C1$, $C3$, $S1$ and $S3$; (b) $C2$, $C4$, $S2$ and $S4$

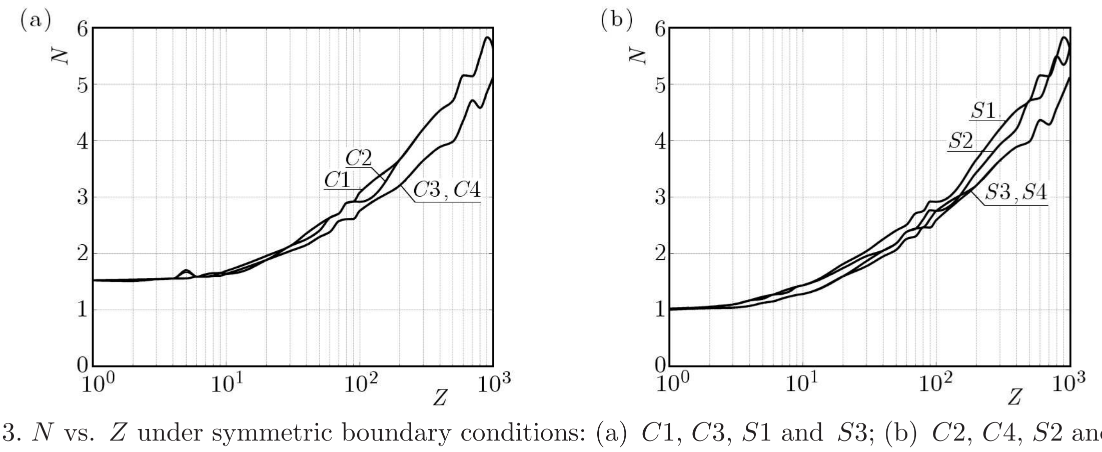

Fig. 3. N vs. Z under symmetric boundary conditions: (a) $C1$, $C3$, $S1$ and $S3$; (b) $C2$, $C4$, $S2$ and $S4$

Corresponding to Fig. 2 and Fig. 3, the buckling modes of cylindrical shells with $Z = 10^3$ under each boundary condition are illustrated in Fig. 4. It is shown in Fig. 4 that the in-plane and transverse boundary conditions have no distinct influence on the symmetric buckling deflections, and there is only one half-wave in the axial direction. In addition, the buckling modes of cylindrical shells subjected to typical boundary conditions $C1$ are displayed in Fig. 5 when Z is equal to 20, 50, 100, 200, 500, respectively. It is found that regardless how long is the cylindrical shell, the waveforms in the axial direction do not exhibit any change. But the number of circumferential waves should decrease dramatically with the increasing Z.

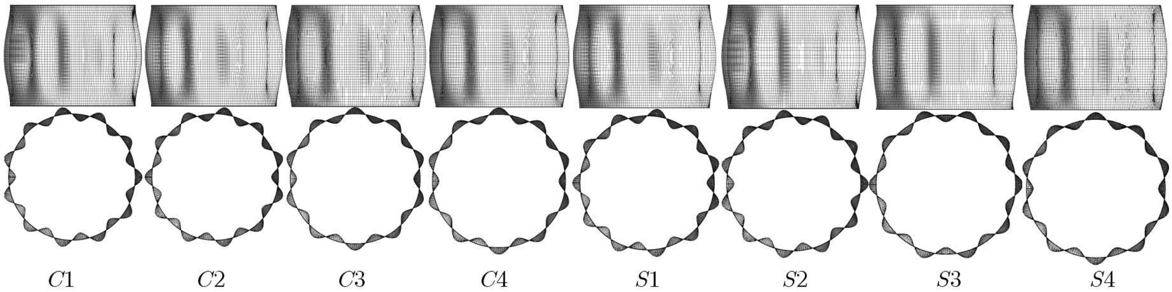

Fig. 4. Buckling modes for $Z = 10^3$ under symmetric boundary conditions

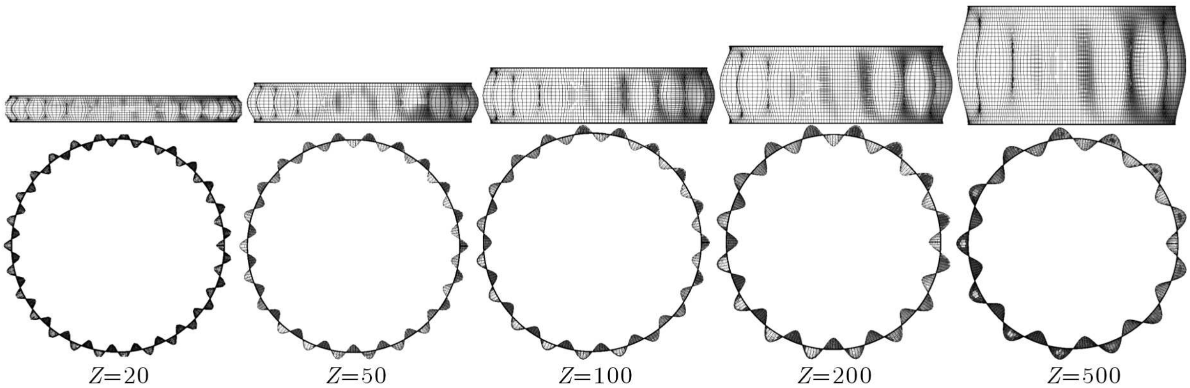

Fig. 5. Buckling modes vs. Z under boundary condition $C1$

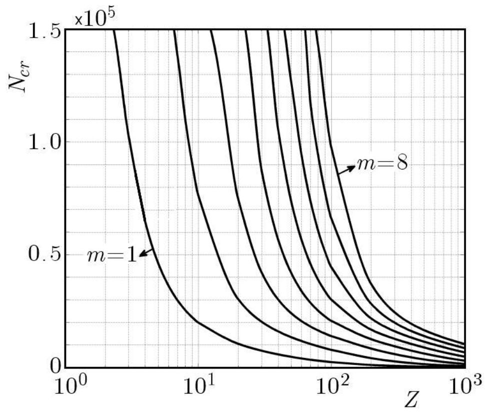

Fig. 6. The first eight branches vs. Z for the order $n = 10$

For fixed eigenvalues $\mu_n = n\mathrm{i}$ $(n = \pm 1, \pm 2, \ldots)$, a series of critical pressures N_{cr} should be determined by bifurcation condition Eq. (5.9) and can be marked as different branches, such as the first branch for $m = 1$, etc. Figure 6 represents variation of the first eight branches versus Z for the number of waves $n = 10$ under boundary conditions $C1$. From these curves, it is shown that each branch decreases to some distinct value rapidly with the increase of Z. The higher the branch number is, the greater the critical pressure becomes.

Figure 7 presents the variations of the first branches for the orders $n = 5$, 10, 15, 20, 25, 30 versus Z under typical boundary condition $C1$. With the increase of Z, the first branches, belonging to different orders, would intersect each other. And intersections between these branches indicate that cylindrical shells subjected to the same pressure should buckle into two different modes.

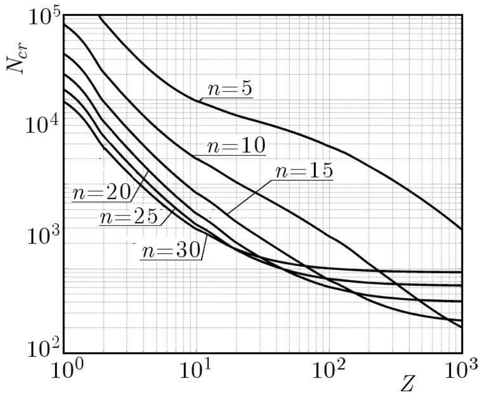

Fig. 7. The first branches for the orders $n = 5$, 10, 15, 20, 25, 30 vs. Z under $C1$

The relation between the minimum critical pressures $\overline{N}_\theta$ and the dimensionless thickness H of cylindrical shells with different lengths $L = 0.25$, 0.5, 1,2 is presented in Fig. 8a. Here, the dimensionless parameter $\overline{N}_\theta = H^2 N_{cr}$ is introduced to repersent the actual pressure. The corresponding circumferential waves are also shown in Fig. 8b. With the increase of thickness, the critical prssures become more higher, and the circumferential waves are lower. And this tendency is more clear for a shorter shell.

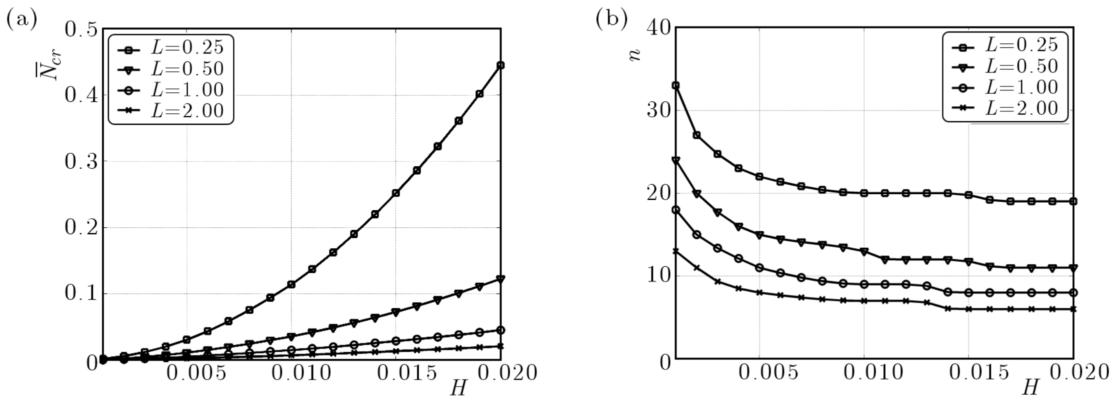

Fig. 8. Buckling versus thickness for different lengths: (a) the minimum buckling loads, (b) circumferential waves

6.2. Results of cylindrical shells under non-symmetric boundary conditions

Assume now that $H = 1/200$ and $\nu = 0.3$. The results are discussed for the following boundary conditions:

- C-$S1$: (i) for $X = L/2$, clamped edge and Condition 1; (ii) for $X = -L/2$, simply supported edge and Condition 1.
- C-$F1$: for $X = L/2$, clamped edge and Condition 1; for $X = -L/2$, free edge and Condition 4.

In order to compare with the results mentioned above, Fig. 9 represents the variations of the minimum critical pressures and the corresponding circumferential wave factors versus Z under typical symmetric and non-symmetric boundary conditions, respectively. It is seen that the curves have the same tendency with the variation of Z. The results of case C-$F1$ are far less than the others due to the relaxation of the edge $X = -L/2$. The corresponding buckling modes of the cylindrical shell for $Z = 100$ are represented in Fig. 10 under four typical boundary conditions. It is shown in Fig. 10 that the buckling deformation of case C-$F1$ is totally different from the others. And the buckling deformations present a "bell mouth" shape on the free edge.

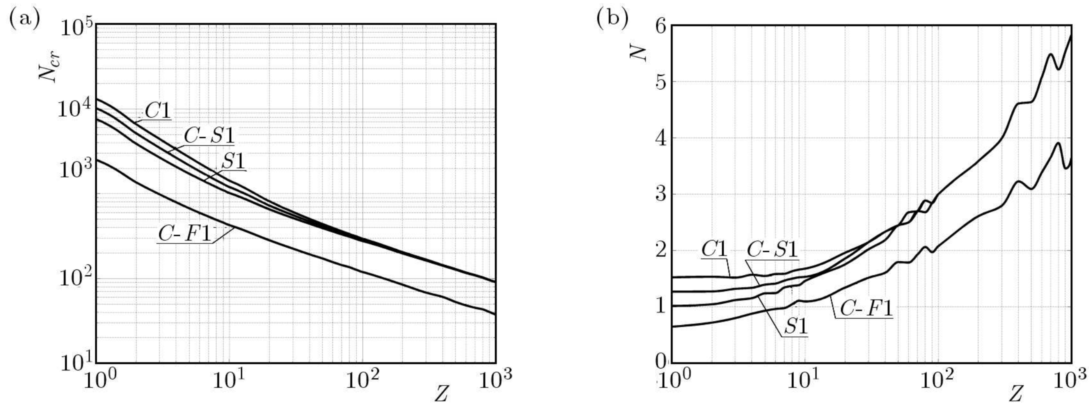

Fig. 9. Buckling results for different non-symmetry boundary conditions: (a) the minimum buckling loads; (b) circumferential waves

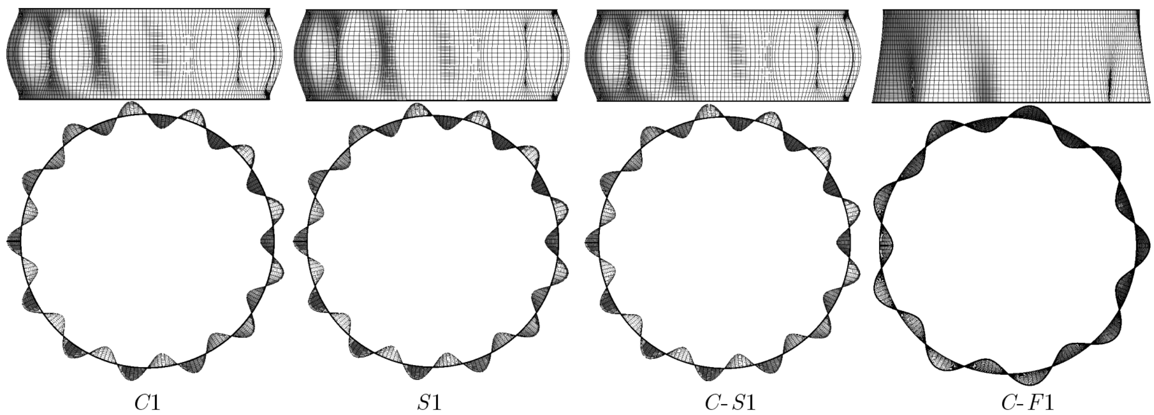

Fig. 10. Buckling modes of cylindrical shell for $Z = 100$ under non-symmetric boundary conditions

7. Conclusion

For buckling analysis of cylindrical shells under symmetric boundary conditions, the minimum critical pressures and circumferential waves of shorter cylindrical shells are mainly affected by transverse boundary conditions. With the increase of length, the effect caused by in-plane boundary conditions becomes more significant than that caused by transverse boundary conditions. Regardless how long are the cylindrical shells, the waveforms of buckling modes always appear one half-wave in the axial direction and are not influenced by any symmetric boundary conditions. With regard to the results of cylindrical shells subjected to non-symmetric boundary conditions, there is no obvious discrepancy with those mentioned above except for case C-$F1$.

The minimum critical pressures of cylindrical shells under free boundary conditions are far less than those for other cases. And the buckling deflections disclose a "bell mouth" shape on the free edge. All analytical results of the cylindrical shells under external pressure belong to Sort 3 and 4. For the influence of thickness, it is found that shorter shells should be more significantly affected by it. Furthermore, some other interesting buckling results are also discussed in detail. The Hamiltonian system and the solution methodology developed here is effective and can be extended to other engineering fields.

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