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ON THE DYNAMIC COEFFICIENT OF LOAD GENERATING AN EXPANDING SPHERICAL STRESS WAVE IN ELASTIC MEDIUM

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In an unbounded, linearly-elastic, compressible and isotropic medium there is a spherical cavity. Its wall is loaded by the time-dependent pressure, which generates in the medium a spherical stress wave expanding from the cavity. The influence of the load character on the wave parameters was studied and the dynamic coefficient of load was regarded as the main compared parameter. Because of the spherical divergence of the wave, its parameters decrease in the inverse proportion to the square and the cube of the distance from the cavity center, so their maximum absolute values appear at the cavity wall and, therefore, the analysis was conducted there. For the pressure linearly increasing to the constant value two practical limiting values of increase time were found, which determinate three ranges of the load character. In the first, for short times, the load can be considered as surge for which the dynamic coefficient is the highest. In the third, for long times, the load can be considered as quasi-static, neglecting its dynamic effects. However, in the second range, the load has a transitional character and the parameters of the wave generated by it should be determined with the use of precise formulae presented in the paper. The maximum time of acting of the constant pressure pulse, for which the wave parameters do not exceed their static values yet, was also determined. However, a significant decrease of the cavity radius was observed as the effect of unloading.

Key words: expanding spherical stress wave, isotropic elastic medium, dynamic coefficient of load

1. Introduction

In the scientific-technical literature a great deal of attention was devoted to the problems of propagation of plastic-elastic disturbances generated by forces applied to the wall of a spherical cavity. An extensive review of this kind of studies relative to the ductile metal media was presented by Hopkins (1960). The problems that were investigated and presented in literature (Chadwick, 1962; Kolsky, 1953; Cristescu, 1967; Achenbach, 1975; Kaliski *et al.*, 1992b; Cole, 1948; Graff, 1975; Broberg, 1956; Baum *et al.*, 1975; Korobieiinikow, 1985) can be generally classified under two headings: problems of waves propagation, in which the material is subjected only to infinitesimal straining, and problems of one-dimensional explosions, in which the pressure generated inside the cavity is sufficiently strong in order to bring about large strains in the surrounding material.

Within the scope of the first problem, the solution to the problem of dynamic expansion of a spherical stress wave in the linearly-elastic isotropic medium was presented by Włodarczyk and Zielenkiewicz (2009a,b) in the closed analytical form. The wave was generated by the constant pressure suddenly created inside a spherical cavity. The extensive qualitative and quantitative analysis of variations of mechanical parameters of the medium surrounding the cavity was conducted in those papers. Among other things, the resonant influence of Poisson's ratio on the wave parameters was discovered.

The quantitative measure of the dynamic parameters of propagating disturbances is the dynamic coefficient of load generating the stress wave. As is known (Kaliski *et al.*, 1992a) its maximum value depends on the character of load variations in time applied to given construction. In the technical literature, this parameter is called the dynamic coefficient for short. It has the key significance in the design of constructions subjected to surge-loads. Taking this fact into account, an attempt of extensive qualitative and quantitative analysis of this parameter for the spherical stress wave expanding in the linearly-elastic isotropic medium was made in this paper.

2. Formulation of the problem

Let us consider the propagation of an elastic stress wave in an unbounded isotropic medium within the scope of linear elasticity theory (Nowacki, 1970). The wave is generated by the pressure $p(t)$ created in the spherical cavity of the initial radius a. Taking into account the spherical symmetry, the solution to the problem will depend only on two independent variables, the Lagrangian coordinate r and time t.

The states of stress and strain in the medium surrounding the cavity are represented by the following components: σ_r – radial stress, $\sigma_\varphi = \sigma_\theta$ – circumferential stresses, ε_r – radial strain and $\varepsilon_\varphi = \varepsilon_\theta$ – circumferential strains. The rest of the components of the stress and the strain tensors equal to zero in the considered coordinate system.

According to the linear elasticity theory and generalized Hooke's law (Nowacki, 1970), we have

$$\varepsilon_r = \frac{\partial u}{\partial r} \qquad\qquad \varepsilon_\varphi = \varepsilon_\theta = \frac{u}{r} \tag{2.1}$$

and

$$\begin{aligned} \sigma_r &= \frac{E}{(1+\nu)(1-2\nu)}\Big[(1-\nu)\frac{\partial u}{\partial r} + 2\nu\frac{u}{r}\Big] \\ \sigma_\varphi &= \frac{E}{(1+\nu)(1-2\nu)}\Big[\nu\frac{\partial u}{\partial r} + \frac{u}{r}\Big] \end{aligned} \tag{2.2}$$

where u is the radial displacement, E and ν denote Young's modulus and Poisson's ratio, respectively.

For an infinitesimal element of the linearly-elastic medium, the equation of motion can be written in the form

$$\frac{\partial \sigma_r}{\partial r} + 2\frac{\sigma_r - \sigma_\varphi}{r} = \rho_0\frac{\partial^2 u}{\partial t^2} \tag{2.3}$$

where ρ_0 is the initial density of the medium. Eliminating the stresses σ_r and σ_φ from Eq. (2.3) by means of expressions (2.2), we obtain

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} - \frac{2u}{r^2} = \frac{1}{c_e^2}\frac{\partial^2 u}{\partial t^2} \tag{2.4}$$

where

$$c_e^2 = n^2 c_0^2 \qquad\qquad n^2 = \frac{1-\nu}{(1+\nu)(1-2\nu)} \qquad\qquad c_0^2 = \frac{E}{\rho_0} \tag{2.5}$$

The quantity c_e denotes the velocity of spherical stress wave propagation in the linearly elastic medium.

The boundary conditions for Eq. (2.4) are

$$u(r,t) = 0 \qquad \text{for} \qquad r = a + c_e t \tag{2.6}$$

and

$$\begin{aligned} &\sigma_r(r,t) = -p(t) \qquad p(t) \geqslant 0 && \text{for} \qquad r = a \\ &\sigma_r(r,t) \equiv 0 && \text{for} \qquad r \to \infty \end{aligned} \tag{2.7}$$

3. Solution to the problem

3.1. General solution

The general solution to Eq. (2.4) fulfilling boundary conditions (2.6) and $(2.7)_2$ has the form (Achenbach, 1975; Włodarczyk and Zielenkiewicz, 2009a)

$$u(r,t) = \frac{\varphi'(r-a-c_e t)}{r} - \frac{\varphi(r-a-c_e t)}{r^2} \qquad \varphi'(0) = \varphi(0) = 0 \tag{3.1}$$

where

$$r - a = c_e t \tag{3.2}$$

is the trajectory of stress wave front propagating from the face of cavity into the medium (Fig. 1). The symbol φ' denotes the derivative of function φ with respect to its argument.

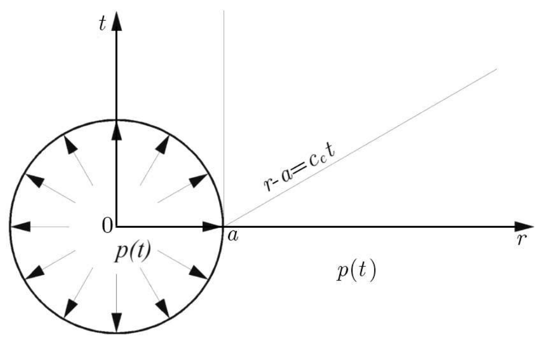

Fig. 1. Scheme of the studied initial-boundary problem

The variables r and t occuring in solution $(3.1)_1$ are contained within the intervals

$$a \leqslant r \leqslant \infty \qquad t \geqslant \frac{r-a}{c_e} \tag{3.3}$$

Substituting expression $(3.1)_1$ into boundary condition $(2.7)_1$, we obtain the following differential equation, which has to be fulfilled by the function $\varphi(x)$, namely

$$\varphi''(x_0) - 2h\varphi'(x_0) + \frac{2h}{a}\varphi(x_0) = -\frac{a}{n^2 E} p\Big(-\frac{x_0}{c_e}\Big) \tag{3.4}$$

where

$$h = \frac{1-2\nu}{1-\nu}\frac{1}{a} \geqslant 0 \qquad x_0 = -c_e t \tag{3.5}$$

The solution to this equation with homogeneous initial conditions $(3.1)_2$ is represented by the following expression

$$\varphi(x_0) = -\frac{a}{n^2\omega E}\int\limits_0^{x_0} p\Big(\frac{y-x_0}{c_e}\Big)\mathrm{e}^{hy}\sin\omega y\,dy \tag{3.6}$$

where

$$\omega = \frac{\sqrt{1-2\nu}}{(1-\nu)a}$$

The function $\varphi(x_0)$ and its derivatives uniquely determine all parameters of the expanding spherical stress wave, namely

$$\begin{aligned}
&u(r,t) = \frac{\varphi'(r-a-c_et)}{r} - \frac{\varphi(r-a-c_et)}{r^2}\\
&\varepsilon_r = \frac{\varphi''}{r} - 2\frac{\varphi'}{r^2} + 2\frac{\varphi}{r^3} \qquad\qquad \varepsilon_\varphi = \frac{\varphi'}{r^2} - \frac{\varphi}{r^3}\\
&\sigma_r = (2\mu+\lambda)\frac{\varphi''}{r} - 4\mu\frac{\varphi'}{r^2} + 4\mu\frac{\varphi}{r^3}\\
&\sigma_\varphi = \lambda\frac{\varphi''}{r} + 2\mu\frac{\varphi'}{r^2} - 2\mu\frac{\varphi}{r^3}\\
&\sigma_z = |\sigma_\varphi - \sigma_r| = \Big| -2\mu\frac{\varphi''}{r} + 6\mu\frac{\varphi'}{r^2} - 6\mu\frac{\varphi}{r^3}\Big|
\end{aligned} \tag{3.7}$$

where

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \qquad\qquad \mu = \frac{E}{2(1+\nu)}$$

are Lame's constants, and symbols φ' and φ'' denote respectively the first and second derivative of the function φ with respect to its argument. The quantity σ_z is the stress intensity. In the technical literature, it is also called the reduced stress.

3.2. Static solution

If the pressure p_0 inside the spherical cavity is created statically (increasing in theoretically infinite time), then the displacement of medium elements is only a function of the spatial coordinate r and Eq. (2.4) can be reduced to the form

$$\frac{d^2u_s}{dr^2} + 2\Big(\frac{1}{r}\frac{du_s}{dr} - \frac{u_s}{r^2}\Big) = 0 \tag{3.8}$$

with the boundary conditions

$$\sigma_r(a) = (2\mu + \lambda)\frac{du_s}{dr}\Big|_{r=a} + 2\lambda\frac{u_s}{a} = -p_0 \qquad p_0 > 0$$
$$\sigma_r(\infty) = 0 \tag{3.9}$$

The general integral of Eq. (3.8) is

$$u_s(r) = Cr + \frac{D}{r^2} \tag{3.10}$$

From conditions (3.9) and solution (3.10) it follows that

$$C = 0 \qquad\qquad D = \frac{1+\nu}{2}\frac{p_0}{E}a^3$$

Finally, the static parameters of the problem can be determined with the following formulae

$$\begin{aligned}
&u_s(r) = \frac{1+\nu}{2}\frac{p_0}{E}a\Big(\frac{a}{r}\Big)^2 \\
&\varepsilon_{rs}(r) = -(1+\nu)\frac{p_0}{E}\Big(\frac{a}{r}\Big)^3 \qquad && \varepsilon_{\varphi s}(r) = \frac{1+\nu}{2}\frac{p_0}{E}\Big(\frac{a}{r}\Big)^3 \\
&\sigma_{rs}(r) = -p_0\Big(\frac{a}{r}\Big)^3 && \sigma_{\varphi s}(r) = \frac{p_0}{2}\Big(\frac{a}{r}\Big)^3 \\
&\sigma_{zs}(r) = |\sigma_{\varphi s}(r) - \sigma_{rs}(r)| = \frac{3}{2}p_0\Big(\frac{a}{r}\Big)^3
\end{aligned} \tag{3.11}$$

3.3. Solution for constant pressure of limited duration

Consider the solution to the problem for constant pressure p_0 of limited duration t_g suddenly applied to the wall of cavity (Fig. 2), i.e.

$$p(t) \equiv \begin{cases} p_0 & \text{for} \quad 0 \leqslant t < t_g \\ 0 & \text{for} \quad t \geqslant t_g \end{cases} \tag{3.12}$$

In the first range of time course $(3.12)_1$, the solution overlaps the results obtained for the constant pressure p_0 suddenly applied to the wall of cavity and acting in infinite time. This case was thoroughly described and analysed in papers (Włodarczyk and Zielenkiewicz, 2009a,b). The following expressions for the function φ and its derivatives were obtained

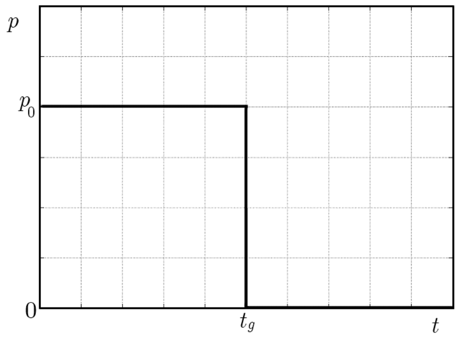

Fig. 2. Scheme of the finite pulse of constant pressure

$$
\begin{aligned}
\varphi_d(x) &= -\frac{a}{n^2\omega(h^2+\omega^2)}\frac{p_0}{E}[\omega + \mathrm{e}^{hx}(h\sin\omega x - \omega\cos\omega x)] \\
\varphi_d'(x) &= -\frac{a}{n^2\omega}\frac{p_0}{E}\mathrm{e}^{hx}\sin\omega x \\
\varphi_d''(x) &= -\frac{a}{n^2}\frac{p_0}{E}\mathrm{e}^{hx}\Big(\frac{h}{\omega}\sin\omega x + \cos\omega x\Big)
\end{aligned}
\tag{3.13}
$$

where

$$
\omega x = \frac{\sqrt{1-2\nu}}{1-\nu}\Big(\frac{r}{a} - 1 - n\frac{c_0 t}{a}\Big) \qquad\qquad hx = \frac{1-2\nu}{1-\nu}\Big(\frac{r}{a} - 1 - n\frac{c_0 t}{a}\Big) \tag{3.14}
$$

The parameters characterising this solution are identified by the subscript d, which indicates the dynamic (percussive) action of pressure on the cavity wall.

The dynamics of the studied medium is described by linear differential equation (2.4) with linear boundary conditions. Therefore, the solutions to the studied problem in the second range of time $(3.12)_2$ can be obtained by superposition of the results mentioned above with the solution obtained for identical pressure of opposite sign, applied suddenly to the cavity wall after time t_g. So to obtain the solution in the range $t \geqslant t_g$, it is enough to know results (3.13) and (3.14).

3.4. Solution for quasi-static pressure

The simplest mathematical model describing quasi-static pressure (reaching a constant value in finite time) is the function linearly increasing in the period t_g to the value p_0 (Fig. 3)

$$
\begin{aligned}
p(t) &= \frac{t}{t_g}p_0 && \text{for} \qquad 0 \leqslant t < t_g \\
p(t) &\equiv p_0 && \text{for} \qquad t \geqslant t_g
\end{aligned}
\tag{3.15}
$$

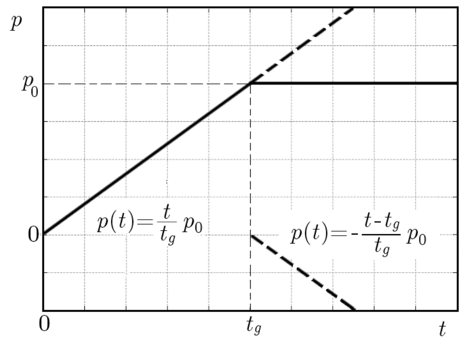

Fig. 3. Schematic variation of quasi-static pressure in time

In the first place, by means of expression (3.6), the forms of function φ and its derivatives φ' and φ'' for the pressure increasing linearly with time $(3.15)_1$ were determined, namely

$$\begin{aligned}
\varphi_l(x) &= \frac{1}{n^3\omega(h^2+\omega^2)^2}\frac{p_0}{E}\frac{a}{c_0t_g}\{2h\omega+\omega x(h^2+\omega^2)+\\
&\quad +e^{hx}[(h^2-\omega^2)\sin\omega x - 2h\omega\cos\omega x]\}\\
\varphi_l'(x) &= \frac{1}{n^3\omega(h^2+\omega^2)}\frac{p_0}{E}\frac{a}{c_0t_g}[\omega+e^{hx}(h\sin\omega x-\omega\cos\omega x)]\\
\varphi_l''(x) &= \frac{1}{n^3\omega}\frac{p_0}{E}\frac{a}{c_0t_g}e^{hx}\sin\omega x
\end{aligned} \tag{3.16}$$

The parameters obtained for such a load are distinguished by the subscript l.

Analogously to the case of constant pressure of limited duration, the solution for the range of time $(3.15)_2$ is obtained using superposition of results (3.16) with the solution obtained for identical pressure variation with the opposite sign and applied to the cavity wall after time t_g (Fig. 3).

3.5. Selected parameters of stress wave

In order to simplify the quantitative analysis of the stress wave parameters, the following dimensionless quantities were introduced

$$\begin{aligned}
\xi &= \frac{r}{a} & \eta &= \frac{c_0t}{a} & \eta_g &= \frac{c_0t_g}{a}\\
U &= \frac{u}{a} & U_s &= \frac{u_s}{a} & S_r &= \frac{\sigma_r}{p_0}
\end{aligned} \tag{3.17}$$

$$S_{rs} = \frac{\sigma_{rs}}{p_0} \qquad S_\varphi = \frac{\sigma_\varphi}{p_0} \qquad S_{\varphi s} = \frac{\sigma_{\varphi s}}{p_0}$$
$$S_z = \frac{\sigma_z}{p_0} \qquad S_{zs} = \frac{\sigma_{zs}}{p_0} \qquad P = \frac{p_0}{E}$$

According to (3.3) and (3.17), the dimensionless independent variables ξ and η are contained within the following intervals

$$1 \leqslant \xi \leqslant \infty \qquad \eta \geqslant \frac{\xi - 1}{n} \tag{3.18}$$

Using expressions (3.7) and (3.13), the parameters of the expanding stress wave, generated by the constant pressure of limited duration in range $(3.12)_1$, can be determined with the use of dimensionless quantities in the form

$$\begin{aligned}
U_{1p}(\xi,\eta) &= U_d(\xi,\eta) = \frac{1+\nu}{2}\frac{P}{\xi^2}\{1 - [A_1(\xi)\sin\omega x + \cos\omega x]\mathrm{e}^{hx}\}\\
S_{r1p}(\xi,\eta) &= S_{rd}(\xi,\eta) = -\frac{1}{\xi^3}\{1 + [A_2(\xi)\sin\omega x + A_3(\xi)\cos\omega x]\mathrm{e}^{hx}\}\\
S_{\varphi 1p}(\xi,\eta) &= S_{\varphi d}(\xi,\eta) = \frac{1}{2\xi^3}\{1 - [A_4(\xi)\sin\omega x + A_5(\xi)\cos\omega x]\mathrm{e}^{hx}\}\\
S_{z1p}(\xi,\eta) &= S_{zd}(\xi,\eta) = \Big|\frac{3}{2\xi^3}\{1 + [A_6(\xi)\sin\omega x + A_7(\xi)\cos\omega x]\mathrm{e}^{hx}\}\Big|
\end{aligned} \tag{3.19}$$

where

$$\omega x = \frac{\sqrt{1-2\nu}}{1-\nu}(\xi - 1) - \frac{1}{\sqrt{1-\nu^2}}\eta \qquad hx = \frac{1-2\nu}{1-\nu}(\xi - 1) - \sqrt{\frac{1-2\nu}{1-\nu^2}}\eta \tag{3.20}$$

and

$$\begin{aligned}
A_1(\xi) &= \sqrt{1-2\nu}(2\xi - 1) & A_2(\xi) &= \sqrt{1-2\nu}(\xi-1)^2\\
A_3(\xi) &= \xi^2 - 1 & &\\
A_4(\xi) &= \sqrt{1-2\nu}\Big(\frac{2\nu}{1-\nu}\xi^2 + 2\xi - 1\Big) & A_5(\xi) &= \frac{2\nu}{1-\nu}\xi^2 + 1\\
A_6(\xi) &= \sqrt{1-2\nu}\Big(\frac{2(1-2\nu)}{3(1-\nu)}\xi^2 - 2\xi + 1\Big) & A_7(\xi) &= \frac{2(1-2\nu)}{3(1-\nu)}\xi^2 - 1
\end{aligned} \tag{3.21}$$

They are marked by the subscript p. The dimensionless variables ξ and η are contained within the intervals

$$1 \leqslant \xi \leqslant \infty \qquad \frac{\xi - 1}{n} \leqslant \eta < \frac{\xi - 1}{n} + \eta_g \tag{3.22}$$

From the superposition of results discussed above it follows that the parameters of the stress wave generated by the finite pulse of constant pressure for $t \geqslant t_g$ can be expressed by the following functions

$$
\begin{aligned}
U_{2p}(\xi,\eta) &= U_d(\xi,\eta) - U_d(\xi,\eta-\eta_g) = \\
&= -\frac{1+\nu}{2}\frac{P}{\xi^2}\{[A_1(\xi) + (S_g - A_1(\xi)C_g)E_g]\sin\omega x + \\
&\quad +[1-(A_1(\xi)S_g + C_g)E_g]\cos\omega x\}\mathrm{e}^{hx} \\
S_{r2p}(\xi,\eta) &= S_{rd}(\xi,\eta) - S_{rd}(\xi,\eta-\eta_g) = \\
&= -\frac{1}{\xi^3}\{[A_2(\xi) + (A_3(\xi)S_g - A_2(\xi)C_g)E_g]\sin\omega x + \\
&\quad +[A_3(\xi) - (A_2(\xi)S_g + A_3(\xi)C_g)E_g]\cos\omega x\}\mathrm{e}^{hx} \\
S_{\varphi 2p}(\xi,\eta) &= S_{\varphi d}(\xi,\eta) - S_{\varphi d}(\xi,\eta-\eta_g) = \\
&= -\frac{1}{2\xi^3}\{[A_4(\xi) + (A_5(\xi)S_g - A_4(\xi)C_g)E_g]\sin\omega x + \\
&\quad +[A_5(\xi) - (A_4(\xi)S_g + A_5(\xi)C_g)E_g]\cos\omega x\}\mathrm{e}^{hx} \\
S_{z2p}(\xi,\eta) &= |S_{zd}(\xi,\eta) - S_{zd}(\xi,\eta-\eta_g)| = \\
&= \Big|\frac{3}{2\xi^3}\{[A_6(\xi) + (A_7(\xi)S_g - A_6(\xi)C_g)E_g]\sin\omega x + \\
&\quad +[A_7(\xi) - (A_6(\xi)S_g + A_7(\xi)C_g)E_g]\cos\omega x\}\mathrm{e}^{hx}\Big|
\end{aligned}
\tag{3.23}
$$

where

$$
E_g = \exp\Big(\sqrt{\frac{1-2\nu}{1-\nu^2}}\,\eta_g\Big) \qquad S_g = \sin\frac{\eta_g}{\sqrt{1-\nu^2}} \qquad C_g = \cos\frac{\eta_g}{\sqrt{1-\nu^2}}
\tag{3.24}
$$

and

$$
1 \leqslant \xi \leqslant \infty \qquad \eta \geqslant \frac{\xi-1}{n} + \eta_g
\tag{3.25}
$$

In an analogous way, the parameters of the expanding stress wave generated in the linearly-elastic medium by the quasi-static pressure were determined. For time $0 \leqslant t < t_g$ $(3.15)_1$, they can be expressed with the use of the following functions

$$
U_{1q}(\xi,\eta) = U_l(\xi,\eta) = \frac{1+\nu}{2}\frac{P}{\eta_g\xi^2}\Big\{\eta + \sqrt{\frac{1+\nu}{1-\nu}}[B_1(\xi)\sin\omega x - B_2(\xi)\cos\omega x]\mathrm{e}^{hx}\Big\}
$$

$$
\begin{aligned}
S_{r1q}(\xi,\eta) &= S_{rl}(\xi,\eta) = -\frac{1}{\eta_g\xi^3}\Big\{\eta - \sqrt{\frac{1+\nu}{1-\nu}}[B_3(\xi)\sin\omega x + B_2(\xi)\cos\omega x]\mathrm{e}^{hx}\Big\} \\
S_{\varphi 1q}(\xi,\eta) &= S_{\varphi l}(\xi,\eta) = \frac{1}{2\eta_g\xi^3}\Big\{\eta + \sqrt{\frac{1+\nu}{1-\nu}}[B_4(\xi)\sin\omega x - B_2(\xi)\cos\omega x]\mathrm{e}^{hx}\Big\} \\
S_{z1q}(\xi,\eta) &= S_{zl}(\xi,\eta) = \Big|\frac{3}{2\eta_g\xi^3}\Big\{\eta - \sqrt{\frac{1+\nu}{1-\nu}}[B_5(\xi)\sin\omega x + B_2(\xi)\cos\omega x]\mathrm{e}^{hx}\Big\}\Big|
\end{aligned}
\tag{3.26}
$$

where

$$
\begin{aligned}
B_1(\xi) &= (1-2\nu)\xi + \nu & B_2(\xi) &= \sqrt{1-2\nu}(\xi-1) \\
B_3(\xi) &= (\xi-1)[(1-\nu)\xi+\nu] & B_4(\xi) &= 2\nu\xi^2 + (1-2\nu)\xi + \nu \\
B_5(\xi) &= \frac{2}{3}(1-2\nu)\xi^2 - (1-2\nu)\xi - \nu
\end{aligned}
\tag{3.27}
$$

They are marked by the subscript q. In turn, for time $t \geqslant t_g$ $(3.15)_2$, we have

$$
\begin{aligned}
U_{2q}(\xi,\eta) &= U_l(\xi,\eta) - U_l(\xi,\eta-\eta_g) = \\
&= \frac{1+\nu}{2}\frac{P}{\xi^2}\Big\{1 + \frac{1}{\eta_g}\sqrt{\frac{1+\nu}{1-\nu}}[(B_1(\xi) - (B_2(\xi)S_g + B_1(\xi)C_g)E_g)\sin\omega x + \\
&\quad -(B_2(\xi) + (B_1(\xi)S_g - B_2(\xi)C_g)E_g)\cos\omega x]\mathrm{e}^{hx}\Big\} \\
S_{r2q}(\xi,\eta) &= S_{rl}(\xi,\eta) - S_{rl}(\xi,\eta-\eta_g) = \\
&= -\frac{1}{\xi^3}\Big\{1 - \frac{1}{\eta_g}\sqrt{\frac{1+\nu}{1-\nu}}[(B_3(\xi) + (B_2(\xi)S_g - B_3(\xi)C_g)E_g)\sin\omega x + \\
&\quad +(B_2(\xi) - (B_3(\xi)S_g + B_2(\xi)C_g)E_g)\cos\omega x]\mathrm{e}^{hx}\Big\} \\
S_{\varphi 2q}(\xi,\eta) &= S_{\varphi l}(\xi,\eta) - S_{\varphi l}(\xi,\eta-\eta_g) = \\
&= \frac{1}{2\xi^3}\Big\{1 + \frac{1}{\eta_g}\sqrt{\frac{1+\nu}{1-\nu}}[(B_4(\xi) - (B_2(\xi)S_g + B_4(\xi)C_g)E_g)\sin\omega x + \\
&\quad -(B_2(\xi) + (B_4(\xi)S_g - B_2(\xi)C_g)E_g)\cos\omega x]\mathrm{e}^{hx}\Big\} \\
S_{z2q}(\xi,\eta) &= |S_{zl}(\xi,\eta) - S_{zl}(\xi,\eta-\eta_g)| = \\
&= \Big|\frac{3}{2\xi^3}\Big\{1 - \frac{1}{\eta_g}\sqrt{\frac{1+\nu}{1-\nu}}[(B_5(\xi) + (B_2(\xi)S_g - B_5(\xi)C_g)E_g)\sin\omega x + \\
&\quad +(B_2(\xi) - (B_5(\xi)S_g + B_2(\xi)C_g)E_g)\cos\omega x]\mathrm{e}^{hx}\Big\}\Big|
\end{aligned}
\tag{3.28}
$$

4. Analysis of the parameters of expanding stress wave

As is known, due to spatial divergence of stress wave parameters, their absolute maximum values occur on the cavity wall and, therefore, we analyse them there. Moreover, in the analysis of phenomena, the relative (dimensionless) values of wave parameters are used. In order to shorten the descriptions, the word "relative" is omitted and the name of dimensional parameter is used.

4.1. Analysis of wave parameters for quasi-static load

The variation of cavity wall displacement $U(1,\eta)/P$ versus time η, caused by the pressure linearly increasing in the period of time $\eta_g = 10$ to the constant value p_0 for selected values of the parameter ν was presented in Fig. 4. For comparative purposes, graphs of the quantity $U(1,\eta)/P$ for the limiting

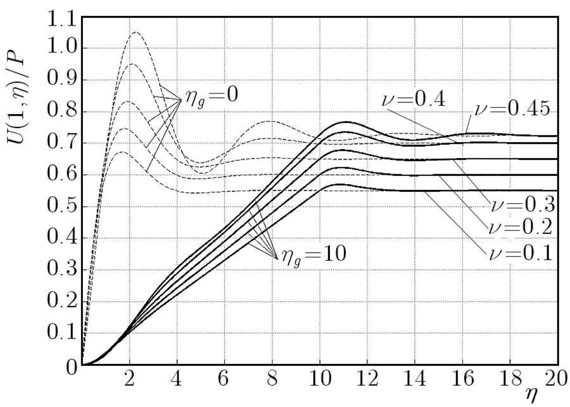

Fig. 4. Variation of the displacement (U/P) of the cavity wall $(\xi = 1)$ loaded by quasi-static pressure versus η for $\eta_g = 10$ and selected values of ν

case $\eta_g = 0$ (dashed lines), i.e. for the surge-load of cavity wall by the constant pressure p_0 were also shown. As can be seen, the presented courses of quantity $U(1,\eta)/P$ are similar for a wide range of Poisson's ratio ν variation. In the interval of linear increase of pressure, the displacement increases also approximately linearly. At the end of pressure increase, as a result of action of the medium inertial force, the quantity $U(1,\eta)/P$ continues to increase for a short time. It slightly exceeds the static value $(1+\nu)/2$ $(3.11)_1$, reaches the global maximum and next approaches the static value mentioned above with the damped oscillatory movement. The quantity U/P behaves similarly for the limiting case, i.e. for $\eta_g = 0$, but for the surge-load of cavity wall by the constant pressure p_0, the influence of medium inertia is much larger in comparison with the quasi-static load.

The graphs of variation of the reduced stress S_z were shown in Fig. 5 in the analogous way as for the displacement. The courses are similar, but the difference is that the static value is $S_{zs} = 1.5$ $(3.11)_6$ and does not depend on the medium compressibility (parameter ν). In order to keep the clarity of graphs, the number of values of the parameter ν was reduced to two extreme from the studied ones.

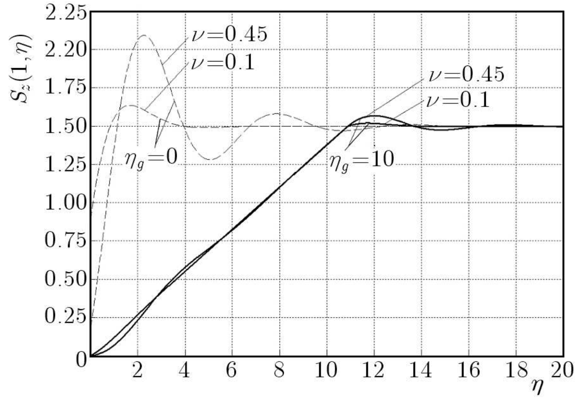

Fig. 5. Variation of the reduced stress S_z on the cavity wall $(\xi = 1)$ loaded by quasi-static pressure versus η for $\eta_g = 10$ and selected values of ν

The measure of influence of the medium inertia for the studied wave parameters is the dynamic coefficient of load Ψ, which is defined as the ratio of the maximum displacement to its static value, i.e.

$$\Psi = \frac{U_{2q}(1,\eta_e)}{U_s(1)} \tag{4.1}$$

where η_e is the root locus of the following equation

$$\begin{aligned} \frac{\partial U_{2q}(1,\eta)}{\partial \eta}\bigg|_{\eta=\eta_e} &= \frac{1+\nu}{2}\frac{P}{\eta_g}\exp\Big(-\sqrt{\frac{1-2\nu}{1-\nu^2}}\,\eta_e\Big)\cdot \\ &\cdot\Big\{[\sqrt{1-2\nu}+(S_g-\sqrt{1-2\nu}C_g)E_g]\sin\frac{\eta_e}{\sqrt{1-\nu^2}}+ \\ &-[1-(\sqrt{1-2\nu}S_g-C_g)E_g]\cos\frac{\eta_e}{\sqrt{1-\nu^2}}\Big\}=0 \end{aligned} \tag{4.2}$$

From Eq. (4.2), we obtain

$$\begin{aligned}\eta_e &= \sqrt{1-\nu^2}\Big(\arctan\frac{1-(\sqrt{1-2\nu}S_g - C_g)E_g}{\sqrt{1-2\nu}+(S_g-\sqrt{1-2\nu}C_g)E_g} + k\pi\Big)\\ \eta_g &< \eta_e < \eta_g + \frac{\pi}{\sqrt{1-\nu^2}}\end{aligned} \tag{4.3}$$

The static displacement of cavity wall, according to $(3.11)_1$, amounts

$$U_s(1) = \frac{1+\nu}{2}P \tag{4.4}$$

Formulae (4.1), (4.3), (4.4) and (3.28) allow for qualitative and quantitative description of the influence of the increase time of load η_g on the dynamic coefficient Ψ. The results are presented in Fig. 6 in the form of graphs plotted for selected values of Poisson's ratio ν. In order to facilitate the analysis of results for various orders of magnitude of the increase time, the logarithmic scale was used to describe the η_g axis.

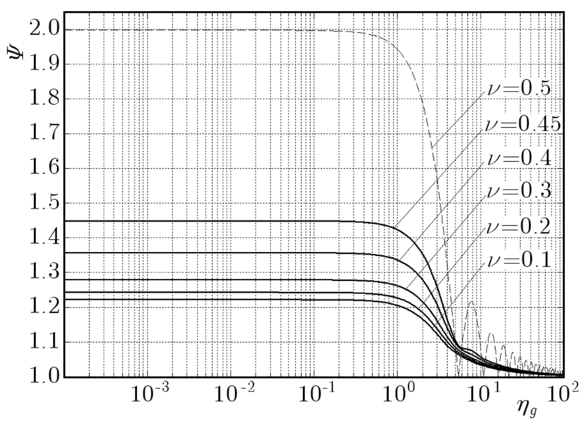

Fig. 6. Variation of the dynamic coefficient of load Ψ on the cavity wall $(\xi = 1)$ loaded by quasi-static pressure versus η_g for selected values of ν

As can be seen on the graph, with the rise of parameter ν, which means the fall of medium compressibility, the influence of inertia on the course of displacement increases. It is connected with the fact that for higher values of Poisson's ratio, despite the increase of the velocity of spherical stress wave front, the rate of transfer of the disturbance energy to further spherical layers of the medium is slower, which means that the energy is distributed in a wider zone after the wave front (Włodarczyk and Zielenkiewicz, 2009b). However, for the studied range of parameter ν, the differences do not exceed 20%.

It can be also noticed that regardless of the parameter ν value, for η_g less than about 1, variations of the dynamic coefficient in relation to the value for surge-load $(\eta_g \to 0)$ are insignificant, but in this range the differences between the results for the studied values of Poisson's ratio are the largest. Above the value of 1 both the dynamic coefficient and the differences begin to decrease intensively. In the close neighbourhood of $\eta_g = 10$, the coefficient falls below 1.05 and, next, monotonically approaches 1. The dynamic coefficient $\Psi = 1.05$ means that the maximum displacement exceeded by 5% its static value, which can be considered as insignificantly small. Therefore, the value of time $\eta_g = 10$ of the pressure increase can be assumed to be the conventional limit, above which the load can be called quasi-static.

For comparison, the calculations were performed also for the limiting case $\nu \to 0.5$ (dashed line). In such a medium, the wave character of its parameters propagation vanishes, because it becomes incompressible and behaves like a mechanical system of one degree of freedom (Włodarczyk and Zielenkiewicz, 2009a,b; Nowacki, 1970). After loading by the constant pressure suddenly applied to the cavity wall, all spherical sections of the medium oscillate with non-damped movement of common phase around the static values with amplitudes equal to these values. Therefore, as can be expected, $\Psi = 2$ for $\eta_g \to 0$ on the graph. The response of such a medium for the quasi-static load is also characteristic for a mechanical system of one degree of freedom. However, because the oscillations are not damped, the parameters do not approach asymptotically the static values. Only in the particular case, when the increase time of pressure is a multiple of the natural period of the medium, the parameters stabilize at the static level immediately at the moment of pressure stabilization, and this gives the value $\Psi = 1$ for these times and the characteristic shape of the graph with discontinuities of the derivative at these points. The slight outline of this tendency can be observed already on the plot of displacement for $\nu = 0.45$.

4.2. Analysis of solution for finite pulse of constant pressure

The graphs of displacement of the cavity wall suddenly loaded by the constant pressure of limited duration for two extreme analysed values of Poisson's ratio ν were presented in Fig. 7. The duration times of pulses were matched so as to not allow the maximum displacements to exceed the static values marked on the graph with horizontal dashed lines. The courses of displacement for the infinite time constant pressure pulse were also plotted with dashed lines.

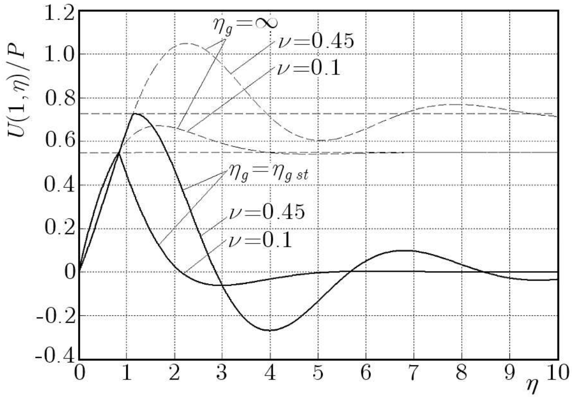

Fig. 7. Variation of the displacement (U/P) of the cavity wall $(\xi = 1)$ loaded by the finite time pressure pulse versus η for $\eta_g = \eta_{g\,st}$ and selected values of ν

The duration times of pulses can be determined using the function describing the course of cavity wall displacement for the infinite pulse

$$U(1,\eta)=\frac{1+\nu}{2}P\Big\{1+\Big[\sqrt{1-2\nu}\sin\frac{\eta}{\sqrt{1-\nu^2}}-\cos\frac{\eta}{\sqrt{1-2\nu}}\Big]\exp\Big(-\sqrt{\frac{1-2\nu}{1-\nu^2}}\eta\Big)\Big\} \tag{4.5}$$

and the value of static displacement $U_s(1)$ (4.4). Solving the equation

$$U(1,\eta_{g\,st}) = U_s(1) \tag{4.6}$$

we obtain the searched value of pulse duration time, namely

$$\eta_{g\,st} = \sqrt{1-\nu^2}\arctan\frac{1}{\sqrt{1-2\nu}} \tag{4.7}$$

As can be seen on the graph and from formulae $(3.19)_1$ and $(3.23)_1$, after applying suddenly the constant pressure of infinite acting time $(\eta_g \to \infty)$, the displacement asymptotically approaches the static value determined by this pressure. However, if after the finite time η_g there comes unloading with the pulse of opposite sign (pressure termination, $p_0 = 0$), the process will be stopped and the displacement will approach the static value determined by the new pressure, in this case being zero.

It can be also observed that for the analysed values of Poisson's ratio, the limiting times of pulse duration $\eta_{g\,st}$ are of the same order $\eta_g \approx 1$, but the differences between them cannot be neglected. The short time of displacement increase would cause the exceeding of static values even by 15%, which is already the significant value. It should also be noticed that the loading of this kind subsequently causes, after unloading, the occurrence of the negative displacement of cavity wall even to 30% of the static value, which means the decrease of cavity radius below the initial value. According to the analysis presented in the previous section, it is the result of significant part of inertia during surge-loading and the influence of inertia increasing with the decrease of compressibility. The graphs of reduced stress shown in Fig. 8 have the similar character, but at the initial instant and at the instant of unloading the discontinuities of describing functions occur. The differences of values between the functions in points of discontinuity decrease with the fall of medium compressibility.

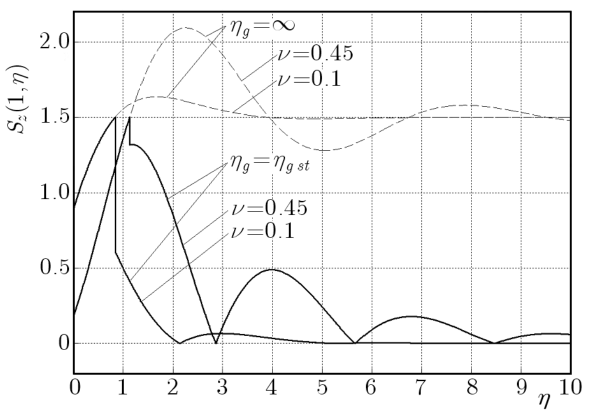

Fig. 8. Variation of the reduced stress S_z on the cavity wall $(\xi = 1)$ loaded by the finite time pressure pulse versus η for $\eta_g = \eta_{g\,st}$ and selected values of ν

5. Conclusions

From the analysis presented above, the following conclusions can be drawn:

- For the spherical cavity in an unbounded, linearly-elastic, compressible, isotropic medium loaded by the internal pressure linearly increasing to a constant value in a limited time, there exists a distinct limit of the time of increase of load, above which it can be considered as quasi-static, neglecting the dynamic factor. $\eta_g = 10$ can be assumed as the approximate limiting value. In turn, for the time of increase of pressure $\eta_g < 1$, the case can be considered by simplification as the surge-load. On the contrary, for times in the range $1 < \eta_g < 10$ it would be recommended to apply the exact formulae presented in the paper.

- For the steel medium of parameters $E = 210\,\mathrm{GPa}$, $\nu = 0.3$, $\rho = 7800\,\mathrm{kg/m^3}$ with the spherical cavity of radius a [m], the real limiting time of increase of pressure at $\eta_g = 10$ will be $t_g = 0.002a\,\mathrm{s}$, which means that it increases proportionally to the radius. For the cavity of radius 1 m, it will be $t_g = 2\,\mathrm{ms}$. This time is short enough to consider the load generated by the detonation of gaseous explosive mixture as quasi-static, but this assumption cannot be taken for high explosives.
- The loading of cavity by a constant pressure pulse of time of duration that is short enough indeed does not cause the displacement to exceed the static value, but after unloading its dynamic character generates negative displacements of the cavity wall reaching 30% of this value. This situation in some cases can be unacceptable.

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O współczynniku dynamiczności obciążenia generującego ekspandującą kulistą falę naprężenia w ośrodku sprężystym

Streszczenie

W nieograniczonym, liniowo-sprężystym i ściśliwym ośrodku izotropowym znajduje się kulista kawerna. Jej ścianka obciążona jest ciśnieniem zmiennym w czasie, które generuje w ośrodku ekspandującą z kawerny kulistą falę naprężenia. Zbadano wpływ charakteru obciążenia na charakterystyki parametrów fali, przy czym za główne kryterium porównawcze przyjęto współczynnik dynamiczności obciążenia. Ze względu na kulistą dywergencję fali, jej parametry maleją odwrotnie proporcjonalnie do drugiej i trzeciej potęgi odległości od centrum kawerny, tak więc ich maksymalne bezwzględne wartości występują na ściance kawerny i dlatego też analizę przeprowadzono w tym miejscu. Znaleziono dwie praktyczne graniczne wartości czasu liniowego narastania ciśnienie do stałej wartości, wyznaczające trzy obszary charakteru takiego obciążenia. W pierwszym z nich, dla krótkich czasów, może być ono traktowane jako skokowe, dla którego współczynnik dynamiczny jest największy. W trzecim, dla czasów długich, obciążenie to można traktować jako kwazistatyczne, pomijając jego skutki dynamiczne. Natomiast w obszarze drugim ma ono charakter przejściowy i parametry fali nim wywołanej należałoby opisywać wzorami dokładnymi zaprezentowanymi w artykule. Wyznaczono również maksymalną długość czasu działania impulsu stałego ciśnienia, dla której parametry fali nie przekraczają jeszcze wartości statycznych. Zaobserwowano jednak znaczne zmniejszanie promienia kawerny poniżej wartości początkowej na skutek odciążenia.

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