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VARIATIONAL PRINCIPLES AND NATURAL BOUNDARY CONDITIONS FOR MULTILAYERED ORTHOTROPIC GRAPHENE SHEETS UNDERGOING VIBRATIONS AND BASED ON NONLOCAL ELASTIC THEORY

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Variational principles are derived for multilayered orthotropic graphene sheets undergoing transverse vibrations based on the nonlocal elastic theory of orthotropic plates which provide a continuum model for graphene sheets. The variational formulation allows the derivation of natural boundary conditions which are expressed in the form of a set of coupled equations for multilayered sheets as opposed to uncoupled boundary conditions applicable to simply supported and clamped boundaries and also in the case of a formulation based on the local (classical) elasticity theory. For the free vibrations case, the Rayleigh quotient is derived. The methods for the variational formulation use techniques of calculus of variations and the semi-inverse method for deriving variational integrals. Variational formulations provide the basis for a number of approximate and numerical methods of solutions and improve the understanding of the physical phenomena.

Key words: variational formulation, multilayered graphene sheets, nonlocal theory, vibrations, semi-inverse method

1. Introduction

Graphene is a two-dimensional carbon nanostructure with many applications in several fields. The covalent bond of carbon atoms makes a graphene sheet one of the stiffest and strongest materials with a Young's modulus in the range of 1 TPa and higher as supported by the results given by Poot and van der Zant (2008), Sakhaee-Pour (2009), Gao and Hao (2009) and Shokrieh and Rafiee (2010) on the mechanical properties of graphene sheets. Its superior properties have already been put into use in a number of applications which include their use as sensing devices (Arsat *et al.*, 2009; Wu *et al.*, 2010), in lithium-ion batteries (Lian, 2010), for desalination of sea water (Mishra and Ramaprabhu, 2011), in electrochemical capacitors (Yuan *et al.*, 2011) for electrooxidation (Choi *et al.*, 2011) as well as for sensors for the detection of cancer cells (Yang *et al.*, 2010; Feng *et al.*, 2011). They are also used as reinforcements in composites, and a review of the graphene based polymer composites is given by Kuilla *et al.* (2010). Further applications and potential applications of graphene in the information technology and in other fields are discussed in the review articles by Soldano *et al.* (2010) and Terrones *et al.* (2010).

Experimental study of nano-scale structures has been a difficult field due to the size of the phenomenon. Similarly, molecular dynamics approach has its drawbacks in the form of extensive computer time and memory required to investigate even relatively small nano structures using nano time scales. This situation led to the development of continuum models for nano-sized components, e.g., carbon nanotubes, and in particular, graphene sheets to investigate their mechanical behaviour (He *et al.*, 2004; Kitipornchai *et al.*, 2005; Hemmasizadeh *et al.*, 2008), and these models were used extensively to investigate the mechanical behaviour of graphene sheets. However, the nano-scale thickness of the sheets leads to inaccurate results when the models are based on classical elastic constitutive relations. Classical elasticity is a scale free theory and as such neglects the size effects which become prominent at atomistic scale. Size effects have been observed in experimental and molecular dynamic simulations of carbon nanotubes due to the influences of interatomic and intermolecular interaction forces (Chang and Gao, 2003; Sun Zhang, 2003, Ni *et al.*, 2010).

The most often used continuum theory to analyze nano-scale structures is nonlocal elasticity developed in 70s to take small scale effects into account by formulating a constitutive relation with the stress at a point expressed as a function of the strains at all points of the domain instead of the strain at the same point as in the case with the classical elasticity theory (Edelen and Laws, 1971; Eringen, 1972, 1983). The recent book by Eringen (2002) provides a detailed account of the nonlocal theory. Continuum models were also implemented to study the mechanical behaviour of grahene sheets, and in particular, the buckling of single-layered graphene sheets by Pradhan and Murmu (2009), Sakhaee-Pour (2009) and Pradhan (2009) where nonlocal theories were employed. Vibrational behaviour of graphene sheets has been the subject of several studies due to its importance in many applications. Vibrations of single-layered graphene sheets using nonlocal models were studied by Murmu and Pradhan (2009), Shen *et al.* (2010) and Narendar and Gopalakrishnan

(2010). Vibrations of multilayered graphene sheets were investigated by He *et al.* (2005), Behfar and Naghdabadi (2005), Liew *et al.* (2006) and Jomehzadeh and Saidi (2011) based on the classical elasticity theory. More recently nonlocal continuum models were used in the study of vibrations of multilayered graphene sheets by Pradhan and Phadikar (2009), Pradhan and Kumar (2010), Ansari *et al.* (2010), and Pradhan and Kumar (2011).

The objective of the present study is to derive the variational principles and the applicable boundary conditions involving the transverse vibrations of multilayered orthotropic graphene sheets using the nonlocal theory of elasticity discussed above. Previous studies on variational principles involving nano-structures include multi-walled nanotubes under buckling loads (Adali, 2008), and undergoing linear and nonlinear vibrations (Adali, 2009a,b) which are based on nonlocal theory of Euler-Bernoulli beams. The corresponding results based on nonlocal Timoshenko theory are given in Adali (2011) for nanotubes under buckling loads and in Kucuk *et al.* (2010) for nanotubes undergoing vibrations. In the present study, these results are extended to the case of multilayered graphene sheets undergoing transverse vibrations, and natural boundary conditions are derived which are fairly involved due to coupling between the sheets and small size effects. Moreover Rayleigh quotient for freely vibrating graphene sheets is obtained. The governing equations of the vibrating multilayered graphene sheets constitute a system of partial differential equations and the variational formulation for this system is obtained by the semi-inverse method developed by He (1997, 2004). This method was applied to several problems of mathematical physics governed by a system of differential equations some examples of which can be found in He (2005, 2006, 2007), Liu (2005), Zhou (2006). The variational formulations given in Adali (2008, 2009a,b, 2011) and in Kucuk *et al.* (2010) were also obtained by the semi-inverse method.

2. Governing equations

A continuum model of multilayered graphene sheets is shown in Fig. 1a with the van der Waals interaction between the adjacent layers depicted as elastic springs. For an n-layered graphene, the top layer is numbered as $i = 1$ and the bottom layer as $i = n$. Top view of a graphene sheet is shown in Fig. 1b where a and b are the dimensions of the sheets in the x and y directions, respectively. Bending stiffnesses of the orthotropic sheets are given by D_{11}, D_{12}, D_{22} and D_{66} which are defined as (Pradhan and Phadikar, 2009)

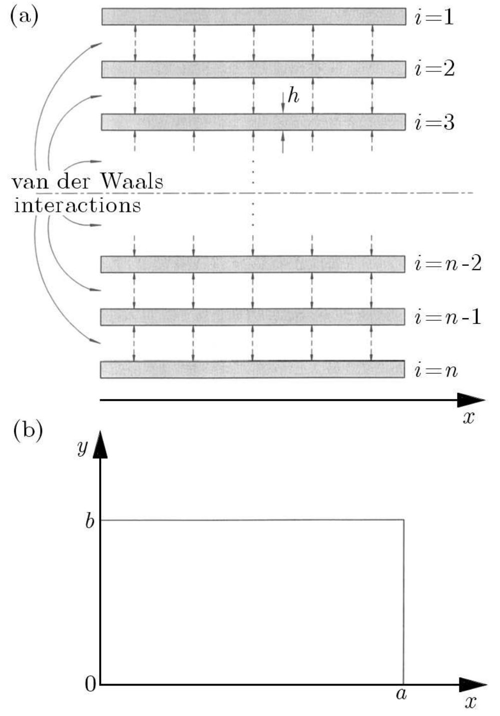

Fig. 1. Multileyered graphene sheets, (a) side view, (b) top view

$$\begin{aligned} D_{11} &= \frac{E_1 h^3}{12(1-\nu_{12}\nu_{21})} & D_{12} &= \frac{\nu_{12} E_2 h^3}{12(1-\nu_{12}\nu_{21})} \\ D_{22} &= \frac{E_2 h^3}{12(1-\nu_{12}\nu_{21})} & D_{66} &= \frac{G_{12} h^3}{12} \end{aligned} \tag{2.1}$$

where h is the thickness of the graphene sheet, E_1 and E_2 are Young's moduli in the x and y directions, respectively, G_{12} is the shear modulus, and ν_{12} and ν_{21} are Poisson's ratios. Let $w_i(x,y,t)$ indicate the transverse deflection of the i-th layer and η the small scale parameter of the nonlocal elastic theory as defined in Pradhan and Phadikar (2009), Pradhan and Kumar (2010, 2011). Then the differential equations governing the transverse vibrations of multilayered graphene sheets in the time interval $t_1 \leqslant t \leqslant t_2$ and based on the nonlocal theory of elasticity (Pradhan and Phadikar, 2009) are given as

$$\begin{aligned} D_1(w_1,w_2) &= L(w_1) - \eta^2 N(w_1) - c_{12}\varDelta w_1 + \eta^2 c_{12}\nabla^2(\varDelta w_1) \\ &= f(x,y,t) - \eta^2\nabla^2 f(x,y,t) \end{aligned}$$

$$
\begin{aligned}
D_i(w_{i-1},w_i,w_{i+1}) &= L(w_i)-\eta^2N(w_i)+c_{(i-1)i}\Delta w_{i-1}-c_{i(i+1)}\Delta w_i \\
&\quad -\eta^2c_{(i-1)i}\nabla^2(\Delta w_{i-1})+\eta^2c_{i(i+1)}\nabla^2(\Delta w_i)=0 \qquad \text{for} \quad i=2,3,\dots,n-1 \\
D_n(w_{n-1},w_n) &= L(w_n)-\eta^2N(w_n)+c_{(n-1)n}\Delta w_{n-1} \\
&\quad -\eta^2c_{(n-1)n}\nabla^2(\Delta w_{n-1})=0
\end{aligned}
\tag{2.2}
$$

where $f(x,y,t)$ is a transverse load acting on the topmost layer $(i=1)$ which can also be taken as acting at the bottommost layer $(i=n)$ due to the symmetry of the structure, the symbol Δw_i is defined as

$$
\Delta w_i \equiv w_{i+1}-w_i \tag{2.3}
$$

and $L(w_i)$ and $N(w_i)$ are differential operators given by

$$
\begin{aligned}
L(w_i) &= D_{11}\frac{\partial^4 w_i}{\partial x^4}+2(D_{12}+2D_{66})\frac{\partial^4 w_i}{\partial x^2\partial y^2}+D_{22}\frac{\partial^4 w_i}{\partial y^4} \\
&\quad +m_0\frac{\partial^2 w_i}{\partial t^2}-m_2\Big(\frac{\partial^4 w_i}{\partial x^2\partial t^2}+\frac{\partial^4 w_i}{\partial y^2\partial t^2}\Big) \\
N(w_i) &= \nabla^2\Big[m_0\frac{\partial^2 w_i}{\partial t^2}-m_2\Big(\frac{\partial^4 w_i}{\partial x^2\partial t^2}+\frac{\partial^4 w_i}{\partial y^2\partial t^2}\Big)\Big]
\end{aligned}
\tag{2.4}
$$

with $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$. In Eqs. (2.4), $m_0=\rho h$ and $m_2=\rho h^3/12$. The coefficient $c_{(i-1)i}$ is the interaction coefficient of van der Waals forces between the $(i-1)$-th and i-th layers with $i=2,\dots,n$. The constant $\eta=e_0\alpha$ is a material parameter defining the small scale effect in the nonlocal elastic theory where e_0 is an experimentally determined constant and has to be determined for each material independently (Eringen, 1983). α is an internal characteristic length such as lattice parameter, size of grain, granular distance, etc. (Narendar and Gopalakrishnan, 2010).

3. Variational functional

In the present section, the semi-inverse method (He, 1997, 2004) will be employed in order to derive the variational formulation of the problem. For this purpose, we first define a trial variational functional $V(w_1,w_2,\dots,w_n)$ given by

$$
\begin{aligned}
V(w_1,w_2,\dots,w_n) &= V_1(w_1,w_2)+V_2(w_1,w_2,w_3)+\dots \\
&\quad +V_{n-1}(w_{n-2},w_{n-1},w_n)+V_n(w_{n-1},w_n)
\end{aligned}
\tag{3.1}
$$

where

$$
\begin{aligned}
V_1(w_1,w_2) &= U(w_1) - T_a(w_1) - T_b(w_1) \\
&\quad + \int_{t_1}^{t_2}\int_0^b\int_0^a [(-f+\eta^2\nabla^2 f)w_1 + F_1(w_1,w_2)]\,dx\,dy\,dt \\
V_i(w_{i-1},w_i,w_{i+1}) &= U(w_i) - T_a(w_i) - T_b(w_i) \\
&\quad + \int_{t_1}^{t_2}\int_0^b\int_0^a F_i(w_{i-1},w_i,w_{i+1})\,dx\,dy\,dt \qquad \text{for}\quad i=2,3,\ldots,n-1
\end{aligned}
\tag{3.2}
$$

$$
V_n(w_{n-1},w_n) = U(w_n) - T_a(w_n) - T_b(w_n) + \int_{t_1}^{t_2}\int_0^b\int_0^a F_n(w_{n-1},w_n)\,dx\,dy\,dt
$$

with the functionals $U(w_i)$, $T_a(w_i)$ and $T_b(w_i)$ defined as

$$
U(w_i) = \frac{1}{2}\int_{t_1}^{t_2}\int_0^b\int_0^a \Big[D_{11}\Big(\frac{\partial^2 w_i}{\partial x^2}\Big)^2 + 2D_{12}\frac{\partial^2 w_i}{\partial x^2}\frac{\partial^2 w_i}{\partial y^2} + D_{22}\Big(\frac{\partial^2 w_i}{\partial y^2}\Big)^2
$$

$$
+4D_{66}\Big(\frac{\partial^2 w_i}{\partial x\partial y}\Big)^2\Big]\,dx\,dy\,dt
$$

$$
\begin{aligned}
T_a(w_i) &= \frac{1}{2}\int_{t_1}^{t_2}\int_0^b\int_0^a \Big\{m_0\Big(\frac{\partial w_i}{\partial t}\Big)^2 + m_2\Big[\Big(\frac{\partial^2 w_i}{\partial x\partial t}\Big)^2 + \Big(\frac{\partial^2 w_i}{\partial y\partial t}\Big)^2\Big]\Big\}\,dx\,dy\,dt \\
T_b(w_i) &= \frac{\eta^2}{2}\int_{t_1}^{t_2}\int_0^b\int_0^a \Big\{2m_0\Big(\frac{\partial^2 w_i}{\partial x^2}\frac{\partial^2 w_i}{\partial t^2} + \frac{\partial^2 w_i}{\partial y^2}\frac{\partial^2 w_i}{\partial t^2}\Big)
\end{aligned}
\tag{3.3}
$$

$$
+m_2\Big[\Big(\frac{\partial^3 w_i}{\partial x^2\partial t}\Big)^2 + 2\Big(\frac{\partial^3 w_i}{\partial x\partial y\partial t}\Big)^2 + \Big(\frac{\partial^3 w_i}{\partial y^2\partial t}\Big)^2\Big]\Big\}\,dx\,dy\,dt
$$

where $i = 1,2,\ldots,n$. It is observed that $U(w_i)$ represents the strain energy and $T_a(w_i)$ the kinetic energy of the i-th layer of the multilayered graphene sheet. The functional $T_b(w_i)$ arises due to small scale effects, i.e. the nonlocal theory used in modeling of the graphene sheet. Similarly, the expression $\int_{t_1}^{t_2}\int_0^b\int_0^a F_i(w_{i-1},w_i,w_{i+1})dx\,dy\,dt$ in equation $(3.2)_1$ represents the potential energy due to van der Waals forces between the layers. Similarly the term $\int_{t_1}^{t_2}\int_0^b\int_0^a[(-f+\eta^2\nabla^2 f)w_1]dx\,dy\,dt$ represents the work done by external forces where the second term arises due to small scale effects. In equations (3.2), $F_i(w_{i-1},w_i,w_{i+1})$ denotes the unknown functions of w_{i-1}, w_i and w_{i+1}, and their derivatives and should be determined such that the Euler-Lagrange equations of variational functional $(3.2)_1$ correspond to differential equations (2.2). These equations are given by

$$
\begin{aligned}
& L(w_1)-\eta^2 N(w_1)+\sum_{j=1}^{2}\frac{\delta F_j}{\delta w_1}=L(w_1)-\eta^2 N(w_1)\\
& \qquad +\sum_{j=1}^{2}\frac{\partial F_j}{\partial w_1}-\sum_{j=1}^{2}\frac{\partial}{\partial x}\Big(\frac{\partial F_j}{\partial w_{1x}}\Big)-\sum_{j=1}^{2}\frac{\partial}{\partial y}\Big(\frac{\partial F_j}{\partial w_{1y}}\Big)=0\\
& L(w_i)-\eta^2 N(w_i)+\sum_{j=i-1}^{i+1}\frac{\delta F_j}{\delta w_i}=L(w_i)-\eta^2 N(w_i)\\
& \qquad +\sum_{j=i-1}^{i+1}\frac{\partial F_j}{\partial w_i}-\sum_{j=i-1}^{i+1}\frac{\partial}{\partial x}\Big(\frac{\partial F_j}{\partial w_{ix}}\Big)-\sum_{j=i-1}^{i+1}\frac{\partial}{\partial y}\Big(\frac{\partial F_j}{\partial w_{iy}}\Big)=0\\
& L(w_n)-\eta^2 N(w_n)+\sum_{j=n-1}^{n}\frac{\delta F_j}{\delta w_n}=L(w_n)-\eta^2 N(w_n)\\
& \qquad +\sum_{j=n-1}^{n}\frac{\partial F_j}{\partial w_n}-\sum_{j=n-1}^{n}\frac{\partial}{\partial x}\Big(\frac{\partial F_j}{\partial w_{nx}}\Big)-\sum_{j=n-1}^{n}\frac{\partial}{\partial y}\Big(\frac{\partial F_j}{\partial w_{ny}}\Big)=0
\end{aligned}
\tag{3.4}
$$

where $i=2,3,\ldots,n-1$ and the subscripts x, y and t denote differentiations with respect to that variable, and $\delta F_i/\delta w_i$ is the variational derivative defined as

$$
\frac{\delta F_i}{\delta w_i}=\frac{\partial F_i}{\partial w_i}-\sum_{k=1}^{k=3}\frac{\partial}{\partial \xi_k}\Big(\frac{\partial F_i}{\partial w_{i\xi_k}}\Big)+\sum_{k=1}^{k=3}\sum_{j=k}^{j=3}\frac{\partial^2}{\partial \xi_k\partial \xi_j}\Big(\frac{\partial F_i}{\partial w_{i\xi_k\xi_j}}\Big)+\ldots \tag{3.5}
$$

where $\xi_1=x$, $\xi_2=y$ and $\xi_3=t$. It is noted that the variational derivative $\delta F_i/\delta w_i$ of $F_i(w_{i-1},w_i,w_{i+1})$ follows from the Euler-Lagrange equations of the functional $\int_{t_1}^{t_2}\int_0^b\int_0^a F_i(w_{i-1},w_i,w_{i+1})dx\,dy\,dt$. Comparing equations (3.4) with equations (2.2), we observe that the following equations have to be satisfied for Euler-Lagrange equations (3.4) to represent the governing equations (2.2)

$$
\sum_{j=1}^{2}\frac{\delta F_j}{\delta w_1}=-c_{12}\Delta w_1+\eta^2 c_{12}\Big(\frac{\partial^2\Delta w_1}{\partial x^2}+\frac{\partial^2\Delta w_1}{\partial y^2}\Big)
$$

$$
\begin{aligned}
\sum_{j=i-1}^{i+1} \frac{\delta F_j}{\delta w_i} &= c_{(i-1)i}\Delta w_{i-1} - c_{i(i+1)}\Delta w_i - \eta^2 c_{(i-1)i}\Big(\frac{\partial^2 \Delta w_{i-1}}{\partial x^2} + \frac{\partial^2 \Delta w_{i-1}}{\partial y^2}\Big) \\
&\quad + \eta^2 c_{i(i+1)}\Big(\frac{\partial^2 \Delta w_i}{\partial x^2} + \frac{\partial^2 \Delta w_i}{\partial y^2}\Big) \qquad (3.6) \\
\sum_{j=n-1}^{n} \frac{\delta F_j}{\delta w_n} &= c_{(n-1)n}\Delta w_{n-1} - \eta^2 c_{(n-1)n}\Big(\frac{\partial^2 \Delta w_{n-1}}{\partial x^2} + \frac{\partial^2 \Delta w_{n-1}}{\partial y^2}\Big)
\end{aligned}
$$

From equations (3.6), it follows that

$$
\begin{aligned}
F_1(w_1, w_2) &= \frac{c_{12}}{4}(\Delta w_1)^2 + \frac{c_{12}}{4}\eta^2\Big[\Big(\frac{\partial \Delta w_1}{\partial x}\Big)^2 + \Big(\frac{\partial \Delta w_1}{\partial y}\Big)^2\Big] \\
F_i(w_{i-1}, w_i, w_{i+1}) &= \frac{c_{(i-1)i}}{4}(\Delta w_{i-1})^2 + \frac{c_{i(i+1)}}{4}(\Delta w_i)^2 \\
&\quad + \frac{\eta^2 c_{(i-1)i}}{4}\Big[\Big(\frac{\partial \Delta w_{i-1}}{\partial x}\Big)^2 + \Big(\frac{\partial \Delta w_{i-1}}{\partial y}\Big)^2\Big] \qquad (3.7) \\
&\quad + \frac{\eta^2 c_{i(i+1)}}{4}\Big[\Big(\frac{\partial \Delta w_i}{\partial x}\Big)^2 + \Big(\frac{\partial \Delta w_i}{\partial y}\Big)^2\Big] \qquad \text{for} \quad i = 2, 3, \ldots, n-1 \\
F_n(w_{n-1}, w_n) &= \frac{c_{(n-1)n}}{4}(\Delta w_{n-1})^2 + \frac{\eta^2 c_{(n-1)n}}{4}\Big[\Big(\frac{\partial \Delta w_{n-1}}{\partial x}\Big)^2 + \Big(\frac{\partial \Delta w_{n-1}}{\partial y}\Big)^2\Big]
\end{aligned}
$$

With F_i given by equations (3.7), we observe that equations (3.4) are equivalent to equations (2.2), viz.

$$
\begin{aligned}
D_1(w_1, w_2) &= L(w_1) - \eta^2 N(w_1) + \sum_{j=1}^{2} \frac{\delta F_j}{\delta w_1} = f - \eta^2 \nabla^2 f \\
D_i(w_{i-1}, w_i, w_{i+1}) &= L(w_i) - \eta^2 N(w_i) + \sum_{j=i-1}^{i+1} \frac{\delta F_j}{\delta w_i} = 0 \qquad (3.8) \\
D_n(w_{n-1}, w_n) &= L(w_n) - \eta^2 N(w_n) + \sum_{j=n-1}^{n} \frac{\delta F_j}{\delta w_n} = 0
\end{aligned}
$$

4. Free vibrations

In the present Section, the variational principle and the Rayleigh quotient are given for the case of freely vibrating graphene sheets. Let the harmonic motion of the i-th layer be expressed as

$$
w_i(x, y, t) = W_i(x, y)\mathrm{e}^{\sqrt{-1}\omega t} \qquad (4.1)
$$

where ω is the vibration frequency and $W_i(x,y)$ is the deflection amplitude. The equations governing the free vibrations are obtained by substituting equation (4.1) into equations (2.2) with $f(x,y,t)=0$ and replacing the deflection $w_i(x,y,t)$ by $W_i(x,y)$. The operators $L(w_i)$ and $N(w_i)$ now become $L_{FV}(W_i)$ and $N(W_i)$ given by

$$\begin{aligned}
L_{FV}(W_i) &= D_{11}\frac{\partial^4 W_i}{\partial x^4} + 2(D_{12}+2D_{66})\frac{\partial^4 W_i}{\partial x^2\partial y^2} + D_{22}\frac{\partial^4 W_i}{\partial y^4} - m_0\omega^2 W_i \\
&\quad + m_2\omega^2\Big(\frac{\partial^2 W_i}{\partial x^2} + \frac{\partial^2 W_i}{\partial y^2}\Big) \\
N(W_i) &= \nabla^2\Big[-m_0\omega^2 W_i + m_2\omega^2\Big(\frac{\partial^2 W_i}{\partial x^2} + \frac{\partial^2 W_i}{\partial y^2}\Big)\Big]
\end{aligned} \tag{4.2}$$

The variational principle for the case of free vibrations is the same as the one given by equations (3.1) and (3.2) with the deflection $w_i(x,y,t)$ replaced by $W_i(x,y)$, the triple integrals replaced by the double integrals with respect to x and y, i.e., $\int_0^b\int_0^a F_i(W_{i-1},W_i,W_{i+1})\,dx\,dy$, and $U(w_i)$, $T_a(w_i)$ and $T_b(w_i)$ replaced by $U_{FV}(W_i)$, $T_{FVa}(W_i)$ and $T_{FVb}(W_i)$ given by

$$\begin{aligned}
U_{FV}(W_i) &= \frac{1}{2}\int_0^b\int_0^a\Big[D_{11}\Big(\frac{\partial^2 W_i}{\partial x^2}\Big)^2 + 2D_{12}\frac{\partial^2 W_i}{\partial x^2}\frac{\partial^2 W_i}{\partial y^2} + D_{22}\Big(\frac{\partial^2 W_i}{\partial y^2}\Big)^2 \\
&\quad + 4D_{66}\Big(\frac{\partial^2 W_i}{\partial x\partial y}\Big)^2\Big]\,dx\,dy \\
T_{FVa}(W_i) &= \frac{1}{2}\int_0^b\int_0^a\Big\{m_0 W_i^2 + m_2\Big[\Big(\frac{\partial W_i}{\partial x}\Big)^2 + \Big(\frac{\partial W_i}{\partial y}\Big)^2\Big]\Big\}dx\,dy \\
T_{FVb}(W_i) &= -\frac{\eta^2}{2}\int_0^b\int_0^a\Big\{m_0\Big[\Big(\frac{\partial W_i}{\partial x}\Big)^2 + \Big(\frac{\partial W_i}{\partial y}\Big)^2\Big] \\
&\quad + m_2\Big[\Big(\frac{\partial^2 W_i}{\partial x^2}\Big)^2 + 2\Big(\frac{\partial^2 W_i}{\partial x\partial y}\Big)^2 + \Big(\frac{\partial^2 W_i}{\partial y^2}\Big)^2\Big]\Big\}dx\,dy
\end{aligned} \tag{4.3}$$

The functions $F_i(W_{i-1},W_i,W_{i+1})$ are of the same form as given by equations (3.7) since the functions $F_i(W_{i-1},W_i,W_{i+1})$ are independent of time. Next the Rayleigh quotient is obtained for the vibration frequency ω from equations (3.1), (3.2) and (4.3) as

$$\omega^2 = \min_{W_i}\frac{\sum_{i=1}^n U_{FVi}(W_i) + \sum_{i=1}^n \int_0^b\int_0^a F_i\,dx\,dy}{\sum_{i=1}^n [T_{FVa}(W_i) + T_{FVb}(W_i)]} \tag{4.4}$$

where F_i $(i = 1, 2, \ldots, n)$ are given by equations (3.7) with $w_i(x, y, t)$ replaced by $W_i(x, y)$.

5. Boundary conditions

After substituting equations (3.2) into functional (3.1), we take its first variation with respect to w_i in order to derive the natural and geometric boundary conditions. The first variations of $V(w_1, w_2, \ldots, w_n)$ with respect to w_i, denoted by $\delta_{w_i} V$, are given by

$$\begin{aligned}
&\delta_{w_1} V(w_1, w_2, \ldots, w_n) = \delta_{w_1} V_1(w_1, w_2) + \delta_{w_1} V_2(w_1, w_2, w_3) \\
&\delta_{w_i} V(w_1, w_2, \ldots, w_n) = \delta_{w_i} V_{i-1}(w_{i-2}, w_{i-1}, w_i) + \delta_{w_i} V_i(w_{i-1}, w_i, w_{i+1}) \\
&\qquad + \delta_{w_i} V_{i+1}(w_i, w_{i+1}, w_{i+2}) \qquad \text{for} \quad i = 2, 3, \ldots, n-1 \\
&\delta_{w_n} V(w_1, w_2, \ldots, w_n) = \delta_{w_n} V_{n-1}(w_{n-2}, w_{n-1}, w_n) + \delta_{w_n} V_n(w_{n-1}, w_n)
\end{aligned} \tag{5.1}$$

The first variation of $V_i(w_{i-1}, w_i, w_{i+1})$ with respect to w_i is given by

$$\begin{aligned}
&\delta_{w_i} V_i(w_{i-1}, w_i, w_{i+1}) = \delta_{w_i} U(w_i) - \delta_{w_i} T_a(w_i) - \delta_{w_i} T_b(w_i) \\
&\qquad + \delta_{w_i} \int\limits_{t_1}^{t_2} \int\limits_0^b \int\limits_0^a \Big[F_i(w_{i-1}, w_i, w_{i+1})\Big] \, dx \, dy \, dt
\end{aligned} \tag{5.2}$$

for $i = 1, 2, \ldots, n-1, n$. Let δw_i denote the variation of w_i satisfying the boundary conditions

$$\begin{aligned}
\delta w_{ix}(x, 0, t) &= 0 \qquad & \delta w_{ix}(x, b, t) &= 0 \\
\delta w_{iy}(0, y, t) &= 0 \qquad & \delta w_{iy}(a, y, t) &= 0
\end{aligned} \tag{5.3}$$

where the following notation was used $\delta(\partial w_i/\partial x) = \delta w_{ix}$, $\delta(\partial w_i/\partial y) = \delta w_{iy}$. Moreover, the deflections $w_i(x, y, t)$ and their space derivatives vanish at the end points $t = t_1$ and $t = t_2$, i.e., $\delta w_i(x, y, t_1) = 0$, $\delta w_i(x, y, t_2) = 0$, $\delta w_{i_x}(x, y, t_1) = 0$, $\delta w_{i_x}(x, y, t_2) = 0$, etc.

Next using the subscript notation for differentiation, i.e., $w_{ix} = \partial w_i/\partial x$, $w_{iy} = \partial w_i/\partial y$ etc., we derive the first variations $\delta_{w_i} U(w_i)$, $\delta_{w_i} T_a(w_i)$, $\delta_{w_i} T_b(w_i)$ and $\delta_{w_i} \int_{t_1}^{t_2} \int_0^b \int_0^a [F_i(w_{i-1}, w_i, w_{i+1})] dx \, dy \, dt$ by integration by parts to obtain

$$
\begin{aligned}
\delta_{w_i} U(w_i) &= \int\limits_{t_1}^{t_2}\int\limits_0^b\int\limits_0^a \Big(D_{11} w_{ixx}\delta w_{ixx} + D_{12} w_{ixx}\delta w_{iyy} + D_{12} w_{iyy}\delta w_{ixx} \\
&\quad + D_{22} w_{iyy}\delta w_{iyy} + 4D_{66} w_{ixy}\delta w_{ixy}\Big)\,dx\,dy\,dt \\
&= \int\limits_{t_1}^{t_2}\int\limits_0^b\int\limits_0^a [D_{11} w_{ixxxx} + 2(D_{12}+2D_{66}) w_{ixxyy} + D_{22} w_{iyyyy}]\delta w_i\,dx\,dy\,dt \\
&\quad + B_1(w_i,\delta w_i) \\
\delta_{w_i} T_a(w_i) &= \int\limits_{t_1}^{t_2}\int\limits_0^b\int\limits_0^a [m_0 w_{it}\delta w_{it} + m_2(w_{ixt}\delta w_{ixt} + w_{iyt}\delta w_{iyt})]\,dx\,dy\,dt \\
&= -\int\limits_{t_1}^{t_2}\int\limits_0^b\int\limits_0^a [m_0 w_{itt} - m_2(w_{ixxtt} + w_{iyytt})]\delta w_i dx\,dy\,dt + B_2(w_i,\delta w_i) \\
\delta_{w_i} T_b(w_i) &= \eta^2 \int\limits_{t_1}^{t_2}\int\limits_0^b\int\limits_0^a m_0(w_{itt}\delta w_{ixx} + w_{ixx}\delta w_{itt} + w_{itt}\delta w_{iyy} \\
&\quad + w_{iyy}\delta w_{itt})\,dx\,dy\,dt \ldots \\
&\quad + \eta^2 \int\limits_{t_1}^{t_2}\int\limits_0^b\int\limits_0^a m_2(w_{ixxt}\delta w_{ixxt} + 2w_{ixyt}\delta w_{ixyt} + w_{iyyt}\delta w_{iyyt})\,dx\,dy\,dt \\
&= \eta^2 \int\limits_{t_1}^{t_2}\int\limits_0^b\int\limits_0^a \nabla^2[m_0 w_{itt} - m_2(w_{ixxtt} + w_{iyytt})]\delta w_i dx\,dy\,dt + B_3(w_i,\delta w_i) \\
\int\limits_{t_1}^{t_2}\int\limits_0^b\int\limits_0^a & \delta_{w_i}[F_i(w_{i-1}, w_i, w_{i+1})]\,dx\,dy\,dt \\
&= \frac{1}{2}\int\limits_{t_1}^{t_2}\int\limits_0^b\int\limits_0^a (c_{(i-1)i}\Delta w_{i-1} - c_{i(i+1)}\Delta w_i)\delta w_i\,dx\,dy\,dt \ldots \\
&\quad + \frac{\eta^2}{2}\int\limits_{t_1}^{t_2}\int\limits_0^b\int\limits_0^a [-c_{(i-1)i}\nabla^2(\Delta w_{i-1}) + c_{i(i+1)}\nabla^2(\Delta w_i)]\delta w_i\,dx\,dy\,dt \\
&\quad + B_4(w_i,\delta w_i)
\end{aligned}
\tag{5.4}
$$

where $B_k(w_i,\delta w_i)$, $k=1,\ldots,4$ are the boundary terms. $B_1(w_i,\delta w_i)$ is given by

$$B_1(w_i,\delta w_i) = \sum_{k=1}^{k=3} B_{1k}(w_i,\delta w_i) \tag{5.5}$$

where

$$\begin{aligned}
B_{11}(w_i,\delta w_i) &= \int_{t_1}^{t_2}\int_0^b [D_{11}(w_{ixx}\delta w_{ix} - w_{ixxx}\delta w_i) \\
&\quad + D_{12}(w_{iyy}\delta w_{ix} - w_{ixyy}\delta w_i)]\Big|_{x=0}^{x=0} dy\,dt \\
B_{12}(w_i,\delta w_i) &= \int_{t_1}^{t_2}\int_0^a [D_{12}(w_{ixx}\delta w_{iy} - w_{ixxy}\delta w_i) \\
&\quad + D_{22}(w_{iyy}\delta w_{iy} - w_{iyyy}\delta w_i)]\Big|_{y=0}^{y=b} dx\,dt \\
B_{13}(w_i,\delta w_i) &= -2D_{66}\int_{t_1}^{t_2}\int_0^b w_{ixyy}\delta w_i\Big|_{x=0}^{x=0} dy\,dt - 2D_{66}\int_{t_1}^{t_2}\int_0^a w_{ixxy}\delta w_i\Big|_{y=0}^{y=b} dx\,dt
\end{aligned} \tag{5.6}$$

Similarly, $B_2(w_i,\delta w_i)$ and $B_3(w_i,\delta w_i)$ are given by

$$\begin{aligned}
B_2(w_i,\delta w_i) &= -m_2\int_{t_1}^{t_2}\int_0^b w_{ixtt}\delta w_i\Big|_{x=0}^{x=0} dy\,dt - m_2\int_{t_1}^{t_2}\int_0^a w_{iytt}\delta w_i\Big|_{y=0}^{y=b} dx\,dt \\
B_3(w_i,\delta w_i) &= \sum_{k=1}^{k=3} B_{3k}(w_i,\delta w_i)
\end{aligned} \tag{5.7}$$

where

$$\begin{aligned}
B_{31}(w_i,\delta w_i) &= \eta^2 m_0\int_{t_1}^{t_2}\int_0^b (w_{itt}\delta w_{ix} - w_{ixtt}\delta w_i)\Big|_{x=0}^{x=0} dy\,dt \\
&\quad + \eta^2 m_0\int_{t_1}^{t_2}\int_0^a (w_{itt}\delta w_{iy} - w_{iytt}\delta w_i)\Big|_{y=0}^{y=b} dx\,dt \\
B_{32}(w_i,\delta w_i) &= \eta^2 m_2\int_{t_1}^{t_2}\int_0^b [-w_{ixxtt}\delta w_{ix} + (w_{ixxxtt} + w_{ixxytt})\delta w_i]\Big|_{x=0}^{x=0} dy\,dt \\
B_{33}(w_i,\delta w_i) &= \eta^2 m_2\int_{t_1}^{t_2}\int_0^a [-w_{iyytt}\delta w_{iy} + (w_{iyyytt} + w_{ixyytt})\delta w_i]\Big|_{y=0}^{y=b} dx\,dt
\end{aligned} \tag{5.8}$$

Finally, we have

$$
\begin{aligned}
B_4(w_1,\delta w_1) &= \frac{\eta^2}{2}\Big(\int\limits_{t_1}^{t_2}\int\limits_0^b c_{12}\varDelta w_{1x}\delta w_1\Big|_{x=0}^{x=0} dy\,dt + \int\limits_{t_1}^{t_2}\int\limits_0^a c_{12}\varDelta w_{1y}\delta w_1\Big|_{y=0}^{y=b} dx\,dt\Big)\\
B_4(w_i,\delta w_i) &= \frac{\eta^2}{2}\int\limits_{t_1}^{t_2}\int\limits_0^b (c_{(i-1)i}\varDelta w_{(i-1)x} + c_{i(i+1)}\varDelta w_{ix})\delta w_i\Big|_{x=0}^{x=0} dy\,dt\\
&\quad +\frac{\eta^2}{2}\int\limits_{t_1}^{t_2}\int\limits_0^a (c_{(i-1)i}\varDelta w_{(i-1)y} + c_{i(i+1)}\varDelta w_{iy})\delta w_i\Big|_{y=0}^{y=b} dx\,dt \quad \text{for } i=2,\dots,n-1\\
B_4(w_n,\delta w_n) &= \frac{\eta^2}{2}\Big(\int\limits_{t_1}^{t_2}\int\limits_0^b c_{(n-1)n}\varDelta w_{(n-1)x}\delta w_n\Big|_{x=0}^{x=0} dy\,dt\\
&\quad +\int\limits_{t_1}^{t_2}\int\limits_0^a c_{(n-1)n}\varDelta w_{(n-1)y}\delta w_n\Big|_{y=0}^{y=b} dx\,dt\Big)
\end{aligned}
\tag{5.9}
$$

Using the fundamental lemma of calculus of variations, the boundary conditions at $x = 0, a$ and $y = 0, b$ are obtained from equations (5.5)-(5.9) for $i = 2,\dots,n-1$. The boundary conditions at $x = 0, a$ are given by

$$
\begin{aligned}
& D_{11}w_{ixx} + D_{12}w_{iyy} + \eta^2(-m_0w_{itt} + m_2w_{ixxtt}) = 0 \qquad \text{or} \qquad w_{ix} = 0\\
& -D_{11}w_{1xxx} - D_{12}w_{1xyy} - 2D_{66}w_{1xyy} + m_2w_{1xtt}\\
& \qquad +\eta^2[m_0w_{1xtt} - m_2(w_{1xxxtt} + w_{1xxytt} + c_{12}\varDelta w_{1x})] = 0 \qquad \text{or} \qquad w_1 = 0\\
& -D_{11}w_{ixxx} - D_{12}w_{ixyy} - 2D_{66}w_{ixyy} + m_2w_{ixtt}\\
& \qquad +\eta^2[m_0w_{ixtt} - m_2(w_{ixxxtt} + w_{ixxytt})]\\
& \qquad +\eta^2(c_{(i-1)i}\varDelta w_{(i-1)x} + c_{i(i+1)}\varDelta w_{ix}) = 0 \quad \text{o} \quad w_i = 0 \ \text{ for } \ i=2,\dots,n-1\\
& -D_{11}w_{nxxx} - D_{12}w_{nxyy} - 2D_{66}w_{nxyy} + m_2w_{nxtt}\\
& \qquad +\eta^2[m_0w_{nxtt} - m_2(w_{nxxxtt} + w_{nxxytt} + c_{(n-1)n}\varDelta w_{(n-1)nx})] = 0\\
& \qquad \text{or} \qquad w_n = 0
\end{aligned}
\tag{5.10}
$$

and at $y = 0, b$ by

$$
\begin{aligned}
& D_{12}w_{ixx} + D_{22}w_{iyy} + \eta^2(-m_0w_{itt} + m_2w_{iyytt}) = 0 \qquad \text{or} \qquad w_{iy} = 0\\
& -D_{12}w_{1xxy} - D_{22}w_{1yyy} - 2D_{66}w_{1xxy} + m_2w_{1ytt}\\
& \qquad +\eta^2[m_0w_{1ytt} - m_2(w_{1yyytt} + w_{1xyytt})] + \eta^2c_{12}\varDelta w_{1y} = 0 \qquad \text{or} \qquad w_1 = 0
\end{aligned}
$$

$$
\begin{aligned}
&-D_{12}w_{ixxy}-D_{22}w_{iyyy}-2D_{66}w_{ixxy}+m_2w_{iytt}\\
&\quad+\eta^2[m_0w_{iytt}-m_2(w_{iyyytt}+w_{ixyytt})]\\
&\quad+\eta^2(c_{(i-1)i}\varDelta w_{(i-1)y}+c_{i(i+1)}\varDelta w_{iy})=0 \quad \text{or} \quad w_i=0 \text{ for } i=2,\dots,n-1\\
&-D_{12}w_{nxxy}-D_{22}w_{nyyy}-2D_{66}w_{nxxy}+m_2w_{nytt}\\
&\quad+\eta^2[m_0w_{nytt}-m_2(w_{nyyytt}+w_{nxyytt}+c_{(n-1)n}\varDelta w_{(n-1)ny})]=0\\
&\quad \text{or} \quad w_n=0
\end{aligned}
\tag{5.11}
$$

It is observed that when $\eta \neq 0$, the natural boundary conditions are coupled, that is, the nonlocal formulation of the problem leads to natural boundary conditions which contain derivatives of w_{i-1} and w_{i+1} in the expression for w_i, e.g. see the first equations of $(5.10)_3$ and $(5.11)_4$.

6. Conclusions

The variational formulations for the free and forced vibrations of multilayered graphene sheets were derived using a continuum formulation based on the nonlocal orthotropic plate theory. The nonlocal theory used in the formulation allows the inclusion of small size effects and as such improves the accuracy of the model. A semi-inverse approach was employed in the derivation of the variational principles and the Rayleigh quotient for free vibrations was obtained. The formulation was used to obtain the natural boundary conditions. The variational principles presented here may form the basis of approximate and numerical methods of solution such as the Rayleigh-Ritz and finite element methods based on the energy functional of the problem and may facilitate the implementation of complicated boundary conditions. It was observed that the nonlocal theory leads to coupled boundary conditions as opposed to uncoupled natural boundary conditions in the case of local theory of graphene sheets.

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Zasady wariacyjne i naturalne warunki brzegowe dla wielowarstwowych ortotropowych paneli grafenowych poddanych drganiom, sformułowane w ramach nielokalnej teorii sprężystości

Streszczenie

W pracy zajęto się problemem drgań poprzecznych ortotropowych paneli grafenowych, dla których sformułowano zasady wariacyjne na podstawie nielokalnej teorii sprężystości, co pozwoliło na budowę ciągłego modelu takich struktur. Formuła

wariacyjna umożliwiła konstrukcję naturalnych warunków brzegowych wyrażonych zbiorem sprzężonych równań opisujących grafenowe panele wielowarstwowe w odróżnieniu od rozprzężonych warunków brzegowych stosowanych jedynie do zamocowań typu swobodne podparcie lub zamurowanie, jednocześnie przy zastosowaniu lokalnej (klasycznej) teorii sprężystości. Dla przypadku drgań swobodnych wyznaczono iloraz Rayleigha układu z grafenu. W prezentowanym sformułowaniu użyto odpowiednich technik obliczania funkcjonałów i półodwrotnej metody wyznaczania całek. Wykazano, że postać wariacyjna stanowi podstawę dla numerycznych metod poszukiwania przybliżonych rozwiązań i pogłębia zrozumienie zachodzących zjawisk fizycznych w takich układach.

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