JOURNAL OF THEORETICAL AND APPLIED MECHANICS
55, 1, pp. 213-226, Warsaw 2017
DOI: 10.15632/jtam-pl.55.1.213

MODELING AND ANALYSIS OF COUPLED FLEXURAL-TORSIONAL SPINNING BEAMS WITH UNSYMMETRICAL CROSS SECTIONS

Jie Wang, Dongxu Li, Jianping Jiang
College of Aerospace Science and Engineering, National University of Defense Technology, Changsha, China
e-mail: wangjie@nudt.edu.cn

The structural modeling and dynamic properties of a spinning beam with an unsymmetrical cross section are studied. Due to the eccentricity and spinning, transverse deflections along the two principal directions and the torsional motion about the longitudinal axis are coupled. The structural model of the beam is established based on the Hamilton principle and by incorporating the torsional inertia. Moreover, because of its significant influence on characteristics for the non-circular cross-sectional beam, the warping effect is considered in the formulation. The proposed model is effectively validated in two cases: the spinning beam with a symmetric cross section and the cantilevered beam with an unsymmetrical cross section. Then the effects of the spinning speed on natural frequencies and mode shapes are investigated. Numerical results reveal that the critical speed is altered with respect to noncoincidence of the centroid and the shear center. For the beams with strong warping rigidities, the warping effect cannot be neglected due to significant influence on natural frequencies.

Keywords: spinning beam, critical speed, warping, coupled flexural-torsional vibration

1. Introduction

Spinning beams are important components of turbine blades, propellers, elastic linkages, satellite booms and are widespread in various branches of structural engineering. Dynamic characteristics such as natural frequencies and mode shapes of these systems are meaningful for analysis of position accuracy, throughput, fatigue and safety. As a consequence, it is essential to accurately establish the dynamic model of spinning beams and predict its vibration characteristics.

For the last decades, there has been a growing interest in the investigation of structural modeling, and excellent work has been done on the dynamic analysis of spinning beams. The fully flexural-torsional coupling model for spinning beams has been successfully established based on the analytical method by Bishop (1959), Dimentberg (1961), Kane (1961), Newland (1972) and Zu and Han (1992). Bishop (1959) utilized Newton's method to derive the characteristic equation of a bent shaft in the Euler-Bernoulli beam model and investigated the stability of the system. Lagrangian approach (Shiau *et al.*, 2006) and Hamilton's principle (Yoon and Kim, 2002) were also utilized to derive the governing equations for the system. Besides, different methods were proposed by researchers in order to solve the governing equations, i.e. assumed-modes method, finite element method, and dynamic stiffness method. Shiau *et al.* (2006) studied the dynamic behavior of a spinning Timoshenko beam with general boundary conditions based on the global assumed mode method. Yoon and Kim (2002) utilized the finite element method to analyze the dynamic stability of an unconstrained spinning beam subjected to a pulsating follower force. Banerjee and Su (2004) developed the dynamic stiffness method and the Wittrick-Williams algorithm was applied to compute natural frequencies and mode shapes. This method was also used in the free vibration analysis of a spinning composite beam (Banerjee and Su, 2006).

Based on the proposed methods, many researchers dealt with problems of spinning beams subjected to different kinds of loads (Ho and Chen, 2006; Lee, 1995; Zu and Han, 1994) under various boundary conditions (Choi *et al.*, 2000; Zu and Melanson, 1998). Sheu and Yang (2005) studied the dynamic response of a spinning Rayleigh beam with rotary inertia and gyroscopic effects in general boundary conditions. The relationship between the critical speed and the hollowness ratio and length-to-radius ratio was investigated by Sheu (2007). Ouyang and Wang (2007) presented a dynamic model for vibration of a rotating Timoshenko beam subjected to a three-directional load moving in the axial direction. Popplewell and Chang (1997) investigated free vibrations of a simply supported but stepped spinning Timoshenko beam with the Galerkin method. Ho and Chen (2006) discussed the vibration problems of a spinning axially loaded pre-twisted Timoshenko beam. Na *et al.* (2006) established the model of a tapered thin-walled composite spinning beam subjected to an axial compressive force. Moreover, dynamic stability of spinning structures around the longitudinal axis such as a shaft or an unconstrained beam has been widely investigated (Lee, 1996; Tylikowski, 2008). Experimental investigations on a cantilevered spinning shaft have been reported. Qian *et al.* (2010) conducted a non-contact dynamic testing of a highly flexible spinning vertical shaft.

In all of these studies, spinning beams had symmetric cross-sections and the shear center and centroid were assumed to superpose each other. In practical applications, the cross-section of the spinning beam can be eccentric due to errors during processing. Moreover, in some circumstances the cross-section is intended to be eccentric to meet the requirement of the engineering. For a beam with an arbitrary uniform cross section, the coupling of bending and torsion may occur when the beam experiences rotating motion. Yoo and Shin (1998) studied the eigenvalue loci veerings and mode shape variations for a rotating cantilever beam with the coupling effect considered. Latalski *et al.* (2014) investigated a rotating composite beam with piezoelectric active elements. An analysis of a rotor with several flexible blades was conducted with the spin softening effects and the centrifugal stiffening effects considered through a pre-stressed potential (Lesaffre *et al.*, 2007). Sinha and Turner (2011) further researched the characteristics of a rotating pre-twisted blade. In these studies, the direction of rotation is vertical to the longitudinal direction. Literature focused on the analysis of a beam rotating about its longitudinal direction is few. Filipich *et al.* (1987) studied the free vibration coupling of bending and torsion of a uniform spinning beam having one axis of symmetry. Then they extended the approach to a beam having no symmetric axis and developed a dynamic model of coupled torsional an bending deformations (Filipich and Rosales, 1990). The model accounted for the dynamic coupling terms due to the rotation and the eccentricity.

This paper further discusses the bending-torsion coupling effects with the warping effect considered. Natural characteristics including natural frequencies and mode shapes with respect to the spinning velocity and the eccentricity are investigated. And the critical spinning speed variation is observed in the presence of coupling effects. The paper is organized as follows. In Section 2, differential equations of the beam are formulated based on the Hamilton principle. The formulations are built on Euler-Bernoulli beam theory with the warping effect and torsional rigidity while neglecting the effect of shear rigidities. In Section 3, we calculate mode shape functions and natural frequencies of the system by applying the assumed mode method. In Section 4, the present model is validated by comparing with literature and numerically simulated with examples. The effects of spinning speed and warping on natural frequencies, mode shapes and critical speed are examined.

2. Governing differential equations

This Section deals with the formulation of differential equations for a spinning beam with an arbitrary cross section based on Hamilton's principle.

2.1. System description

A homogeneous slender beam with a uniform arbitrary cross-section is depicted in Fig. 1a. The left end of the beam is fixed to a base which rotates about the longitudinal axis at a constant angular velocity designated as Ω while the right end is free. When in undeformed configuration, the longitudinal axis of the beam goes through the shear center of the cross section.

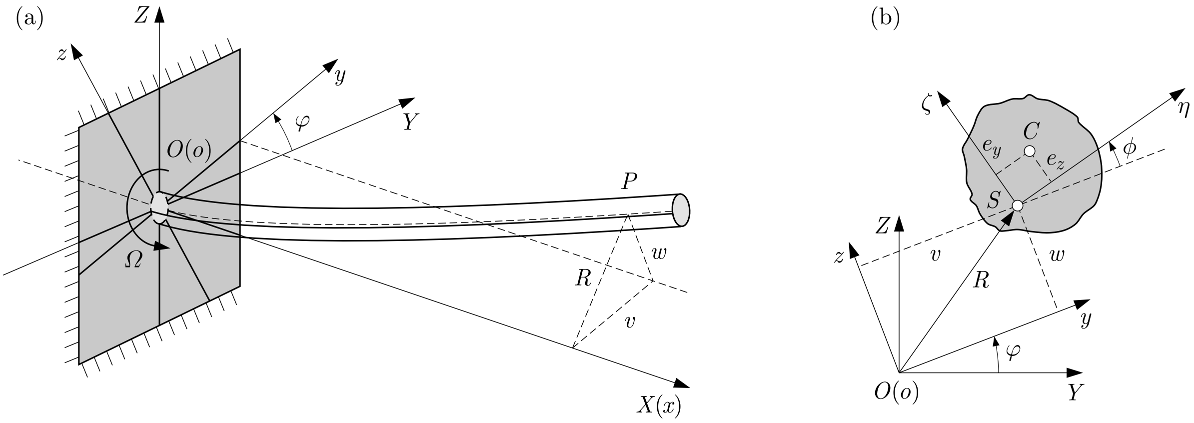

Fig. 1. Deformed configuration of a spinning beam with arbitrary cross section

Three sets of orthogonal right-handed coordinate frames are defined in order to describe the position vector $\mathbf{R}$ of a differential element dM at a generic point P. The rectangular coordinate system XYZ is fixed with the inertial frame and the origin O is placed at the shear center of the cross-section on the clamped end. The frame xyz is a rotating frame whose origin o remains coincident with the point O and the x axis remains parallel to the X axis. When the beam spins, directions of the y and z axes are time-varying. The angle between the y axis and the Y axis is represented by the symbol φ. The third reference frame $\xi\eta\zeta$ is the element coordinate of the differential beam element which is attached to the shear center S of the beam section. C represents the center of mass of the beam section and (e_y, e_z) denote the coordinates of C in the frame $S\eta\zeta$.

The orientations of these frames at a time during free vibration are shown in Fig. 1b. The deformation of a differential beam element located at a distance x from the left end is defined by spatial displacement $v(x,t)$, $w(x,t)$ and rotation $\phi(x,t)$ about x-axis. The $v(x,t)$ and $w(x,t)$ represent lateral displacements in the y and z directions, respectively.

2.2. Equations of motion

The governing equation of the flexible beam is formulated based on the following assumptions: (1) for the elementary case of beam flexure and torsion using the Euler-Bernoulli beam theory with torsional inertia but not shear deformation or axial-force effects, (2) the warping effect is considered due to the fact that the torsion induced warping occurs when the beam section is not circular, (3) the axial displacement of the beam is neglected. Moreover, the deformation of the beam is small and yields to the linear conditions. Physical properties of the material are elastic and constant.

We consider the spinning beam undergoing transverse displacements and torsional motion. Also, it is assumed that the shear center S and the center of mass C of the cross section are not coincident. For such a beam, the position vector of a representative point after beam deformation can be defined as

$$\mathbf{R}(x) = v\mathbf{j} + w\mathbf{k} + e_y\mathbf{j}_1 + e_z\mathbf{k}_1 \tag{2.1}$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors in the x, y and z directions, respectively. And $\mathbf{i}_1$, $\mathbf{j}_1$ and $\mathbf{k}_1$ are unit vectors in the ξ, η and ζ directions, respectively.

The velocity of the point can be obtained as follows

$$\boldsymbol{v}(x) = \dot{v}\mathbf{j} + \dot{w}\mathbf{k} + \Omega\mathbf{i} \times (v\mathbf{j} + w\mathbf{k}) + (\Omega + \dot{\phi})\mathbf{i} \times (e_y\mathbf{j}_1 + e_z\mathbf{k}_1) \tag{2.2}$$

The overhead dot denotes partial derivatives with respect to time t.

The kinetic energy can be simplified as

$$\begin{aligned} T &= \frac{1}{2}\int_0^L \rho A(\dot{v}^2 + \dot{w}^2)\,dx + \frac{1}{2}\rho J_p \int_0^L \dot{\phi}^2\,dx + \frac{1}{2}\int_0^L \rho A[\Omega^2(v^2 + w^2) - 2\Omega\dot{v}w + 2\Omega v\dot{w}]\,dx \\ &+ \frac{1}{2}\int_0^L \rho A[(e_y^2 + e_z^2)\dot{\phi}^2 - 2e_z\dot{v}\dot{\phi} + 2e_y\dot{w}\dot{\phi} + 2e_z\Omega w\dot{\phi} + 2e_y\Omega v\dot{\phi}]\,dx \\ &+ \int_0^L \rho A[(e_y^2 + e_z^2)\Omega\dot{\phi} - e_z\Omega\dot{v} + e_z\Omega^2 w + e_y\Omega\dot{w} + e_y\Omega^2 v]\,dx \\ &+ \int_0^L \rho A(-e_y\Omega\dot{v}\phi + e_y\Omega^2 w\phi - e_z\Omega\dot{w}\phi - e_z\Omega^2 v\phi)\,dx + \frac{1}{2}\int_0^L \rho A(e_y^2 + e_z^2)\Omega^2\,dx \end{aligned} \tag{2.3}$$

The symbols ρ, E and A denote density, Young's modulus and cross sectional area. J_p is the polar moment of inertia and is given by

$$J_p = \iint_A r_p^2\,d\eta\,d\zeta \tag{2.4}$$

where r_p represents the distance between a certain point in the section and the center.

The potential strain energy of the beam including the warping effect is considered as below

$$U = \frac{1}{2}\int_0^L E(I_z v''^2 + I_y w''^2)\,dx + \frac{1}{2}\int_0^L GJ_p\phi'^2\,dx + \frac{1}{2}\int_0^L E\Gamma\phi''^2\,dx \tag{2.5}$$

where G denotes the shear modulus. I_y and I_z show the second moments of area about the z-axis and y-axis, $E\Gamma$ is warping rigidity. Primes denote partial derivatives with respect to x. For uniform beams, A, I_y, I_z, J_p and $E\Gamma$ are constant throughout the span.

Then the Lagrangian function of the beam system can be expressed as

$$\begin{aligned} L = T - U &= \frac{1}{2}\int_0^L \rho A(\dot{v}^2 + \dot{w}^2)\,dx + \frac{1}{2}\int_0^L \rho A[\Omega^2(v^2 + w^2) - 2\Omega\dot{v}w + 2\Omega v\dot{w}]\,dx \\ &+ \frac{1}{2}\rho J_p\int_0^L \dot{\phi}^2\,dx + \frac{1}{2}\int_0^L \rho A(e^2\dot{\phi}^2 - 2e_z\dot{v}\dot{\phi} + 2e_y\dot{w}\dot{\phi} + 2e_z\Omega w\dot{\phi} + 2e_y\Omega v\dot{\phi})\,dx \\ &+ \int_0^L \rho A(e^2\Omega\dot{\phi} - e_z\Omega\dot{v} + e_z\Omega^2 w + e_y\Omega\dot{w} + e_y\Omega^2 v)\,dx \\ &+ \int_0^L \rho A(-e_y\Omega\dot{v}\phi + e_y\Omega^2 w\phi - e_z\Omega\dot{w}\phi - e_z\Omega^2 v\phi)\,dx + \frac{1}{2}\int_0^L \rho A e^2\Omega^2\,dx \\ &- \frac{1}{2}\int_0^L E(I_z v''^2 + I_y w''^2)\,dx - \frac{1}{2}\int_0^L GJ_p\phi'^2\,dx - \frac{1}{2}\int_0^L E\Gamma\phi''^2\,dx \end{aligned} \tag{2.6}$$

Using Hamilton's principle, the dynamic model of the system can be obtained

$$
\begin{aligned}
&EI_z\frac{\partial^4 v}{\partial x^4}+\rho A(\ddot{v}-\Omega^2 v-2\Omega\dot{w}-e_z\ddot{\phi}-2\Omega e_y\dot{\phi}+e_z\Omega^2\phi)=\rho Ae_x\Omega^2\\
&EI_y\frac{\partial^4 w}{\partial x^4}+\rho A(\ddot{w}-\Omega^2 w+2\Omega\dot{v}+e_y\ddot{\phi}-2\Omega e_z\dot{\phi}-e_y\Omega^2\phi)=\rho Ae_z\Omega^2\\
&E\Gamma\frac{\partial^4\phi}{\partial x^4}-GJ_p\frac{\partial^2\phi}{\partial x^2}+(\rho Ae^2+\rho J_p)\ddot{\phi}+\rho Ae_y(2\Omega\dot{v}+\ddot{w}-\Omega^2 w)\\
&\qquad+\rho Ae_z(-\ddot{v}+\Omega^2 v+2\Omega\dot{w})=0
\end{aligned}
\tag{2.7}
$$

When skipping the eccentricity of the cross section, Eq. (2.7) has the following form

$$
\begin{aligned}
&EI_z\frac{\partial^4 v}{\partial x^4}+\rho A(\ddot{v}-\Omega^2 v-2\Omega\dot{w})=0 \qquad\qquad EI_y\frac{\partial^4 w}{\partial x^4}+\rho A(\ddot{w}-\Omega^2 w+2\Omega\dot{v})=0\\
&E\Gamma\frac{\partial^4\phi}{\partial x^4}-GJ_p\frac{\partial^2\phi}{\partial x^2}+\rho J_p\ddot{\phi}=0
\end{aligned}
\tag{2.8}
$$

The first two equations in Eq. (2.8) are fully consistent with the results by Banerjee and Su (2004). Also, it can be concluded that the eccentricity induces the coupling between transverse deformations and torsional motion.

When skipping the spinning, Eq. (2.7) has the following form, which is consistent with the results by Tanaka and Bercin (1999)

$$
\begin{aligned}
&EI_z\frac{\partial^4 v}{\partial x^4}+\rho A(\ddot{v}-e_z\ddot{\phi})=0 \qquad\qquad EI_y\frac{\partial^4 w}{\partial x^4}+\rho A(\ddot{w}+e_y\ddot{\phi})=0\\
&E\Gamma\frac{\partial^4\phi}{\partial x^4}-GJ_p\frac{\partial^2\phi}{\partial x^2}+(\rho Ae^2+\rho J_p)\ddot{\phi}+\rho A(-e_z\ddot{v}+e_y\ddot{w})=0
\end{aligned}
\tag{2.9}
$$

It is obvious that the coupling between v and w takes place due to spinning.

3. Mode shape and frequency equation

For a free homogeneous vibration problem, a sinusoidal oscillation is assumed

$$
v(x,t)=V(x)\mathrm{e}^{\mathrm{j}\omega t} \qquad w(x,t)=W(x)\mathrm{e}^{\mathrm{j}\omega t} \qquad \phi(x,t)=\Phi(x)\mathrm{e}^{\mathrm{j}\omega t} \qquad \mathrm{j}=\sqrt{-1}
\tag{3.1}
$$

where ω is the circular frequency of oscillation, V, W and Φ are amplitudes of v, w and ϕ, respectively. Substituting Eq. (3.1) into differential equation (2.7) leads to

$$
\begin{aligned}
&\frac{EI_z}{\rho A}V^{(4)}-(\omega^2+\Omega^2)V-2\mathrm{j}\omega\Omega W+e_z(\omega^2+\Omega^2)\Phi-2\mathrm{j}\omega\Omega e_y\Phi=0\\
&\frac{EI_y}{\rho A}W^{(4)}-(\omega^2+\Omega^2)W+2\mathrm{j}\omega\Omega V-e_y(\omega^2+\Omega^2)\Phi-2\mathrm{j}\omega\Omega e_z\Phi=0\\
&\frac{E\Gamma}{\rho A}\Phi^{(4)}-\frac{GJ_p}{\rho A}\Phi''-\Big(e^2+\frac{J_p}{A}\Big)\omega^2\Phi+(\omega^2+\Omega^2)(e_zV-e_yW)+2\mathrm{j}\omega\Omega(e_yV+e_zW)=0
\end{aligned}
\tag{3.2}
$$

For convenience, we consider a beam with a monosymmetric cross-section with the symmetry axis y. The centroid C is on the axis y and the scalar e_z is equal to zero. Then Eq. (3.2) can be simplified as

$$
\begin{aligned}
&\frac{EI_z}{\rho A}V^{(4)} - (\omega^2 + \Omega^2)V - 2\mathrm{j}\omega\Omega W - 2\mathrm{j}\omega\Omega e_y\Phi = 0\\
&\frac{EI_y}{\rho A}W^{(4)} - (\omega^2 + \Omega^2)W + 2\mathrm{j}\omega\Omega V - e_y(\omega^2 + \Omega^2)\Phi = 0\\
&\frac{E\Gamma}{\rho A}\Phi^{(4)} - \frac{GJ_p}{\rho A}\Phi'' - \Big(e_y^2 + \frac{J_p}{A}\Big)\omega^2\Phi + 2\mathrm{j}\omega\Omega e_y V - e_y(\omega^2 + \Omega^2)W = 0
\end{aligned}
\tag{3.3}
$$

Then introducing the differential operator D and subsequent variables as follows

$$
\begin{aligned}
&D = \frac{d}{dx} && L_{11} = \frac{EI_z}{\rho A}D^4 - (\omega^2 + \Omega^2)\\
&L_{12} = -2\mathrm{j}\omega\Omega && L_{13} = -2\mathrm{j}\omega\Omega e_y\\
&L_{21} = 2\mathrm{j}\omega\Omega && L_{22} = \frac{EI_y}{\rho A}D^4 - (\omega^2 + \Omega^2)\\
&L_{23} = -e_y(\omega^2 + \Omega^2) && L_{31} = 2\mathrm{j}\omega\Omega e_y\\
&L_{32} = -e_y(\omega^2 + \Omega^2) && L_{33} = \frac{E\Gamma}{\rho A}D^4 - \frac{GJ_p}{\rho A}D^2 - \Big(e_y^2 + \frac{J_p}{A}\Big)\omega^2
\end{aligned}
\tag{3.4}
$$

It can be seen that Y, Z and Ψ satisfy the equation

$$
\mathbf{\Delta}\begin{bmatrix} V\\ W\\ \Phi\end{bmatrix} = \mathbf{0}
\tag{3.5}
$$

where

$$
\mathbf{\Delta} = \begin{bmatrix} L_{11} & L_{12} & L_{13}\\ L_{21} & L_{22} & L_{23}\\ L_{31} & L_{32} & L_{33}\end{bmatrix}
\tag{3.6}
$$

Introducing

$$
\begin{aligned}
&\kappa_1 = \frac{\rho A}{EI_z} && \kappa_2 = \frac{\rho A}{EI_y} && \kappa_3 = \frac{\rho A}{E\Gamma}\\
&\kappa_4 = \frac{GJ_p}{E\Gamma} && \kappa_5 = \Big(e_y^2 + \frac{J_p}{A}\Big) && \kappa_6 = \omega^2 + \Omega^2
\end{aligned}
\tag{3.7}
$$

and setting the determinant of differential operator matrix (3.6) equal to zero leads to the following twelvth order differential equation:

$$
\begin{aligned}
&(D^4 - \kappa_6\kappa_1)[(D^4 - \kappa_6\kappa_2)(D^4 - \kappa_4 D^2 - \kappa_3\kappa_5\omega^2) - e_y^2\kappa_2\kappa_3\kappa_6^2]\\
&\qquad - 4\omega^2\Omega^2\kappa_1\kappa_2[(D^4 - \kappa_4 D^2 - \kappa_3\kappa_5\omega^2) + e_y^2\kappa_3\kappa_6] + 4\omega^2\Omega^2 e_y^2\kappa_1\kappa_3 D^4 = 0
\end{aligned}
\tag{3.8}
$$

The solution to the above equation can be expressed in an exponential form

$$
R(x) = \mathrm{e}^{rx}
\tag{3.9}
$$

Specifying $s = r^2$, then substituting Eq. (3.9) into (3.8), the following characteristic equation can be obtained

$$
\begin{aligned}
&s^6 - \kappa_4 s^5 - (\kappa_3\kappa_5\omega^2 + \kappa_2\kappa_6 + \kappa_1\kappa_6)s^4 + (\kappa_2\kappa_4\kappa_6 + \kappa_1\kappa_4\kappa_6)s^3\\
&\qquad + (\kappa_1\kappa_2\kappa_6^2 - 4\kappa_1\kappa_2\omega^2\Omega^2 - \kappa_2\kappa_3\kappa_6^2 e_y^2 - 4\kappa_1\kappa_3 e_y^2\omega^2\Omega^2 + \kappa_1\kappa_3\kappa_5\kappa_6\omega^2\\
&\qquad + \kappa_2\kappa_3\kappa_5\kappa_6\omega^2)s^2 + \kappa_1\kappa_2\kappa_4(4\omega^2\Omega^2 - \kappa_6^2)s - \kappa_1\kappa_2\kappa_3(\kappa_5\kappa_6^2\omega^2 - \kappa_6^3 e_y^2\\
&\qquad + 4\kappa_6 e_y^2\omega^2\Omega^2 - 4\kappa_5\omega^4\Omega^2) = 0
\end{aligned}
\tag{3.10}
$$

s_1-s_6 are solutions to Eq. (3.10). The twelve roots of Eq. (3.8) can be written as

$$\pm r_i \qquad r_i = \mathrm{j}\sqrt{s_i} \qquad i = 1,2,\dots,6 \tag{3.11}$$

Then the general solutions of V, W and Φ are expressed as

$$\begin{aligned}
V(x) &= A_1\cosh r_1x + A_2\sinh r_1x + A_3\cosh r_2x + A_4\sinh r_2x + A_5\cosh r_3x + A_6\sinh r_3x \\
&\quad + A_7\cos r_4x + A_8\sin r_4x + A_9\cos r_5x + A_{10}\sin r_5x + A_{11}\cos r_6x + A_{12}\sin r_6x \\
W(x) &= B_1\cosh r_1x + B_2\sinh r_1x + B_3\cosh r_2x + B_4\sinh r_2x + B_5\cosh r_3x + B_6\sinh r_3x \\
&\quad + B_7\cos r_4x + B_8\sin r_4x + B_9\cos r_5x + B_{10}\sin r_5x + B_{11}\cos r_6x + B_{12}\sin r_6x \\
\Phi(x) &= C_1\cosh r_1x + C_2\sinh r_1x + C_3\cosh r_2x + C_4\sinh r_2x + C_5\cosh r_3x + C_6\sinh r_3x \\
&\quad + C_7\cos r_4x + C_8\sin r_4x + C_9\cos r_5x + C_{10}\sin r_5x + C_{11}\cos r_6x + C_{12}\sin r_6x
\end{aligned} \tag{3.12}$$

where A_i, B_i and C_i (i = 1-12) are three different sets of constants.

Substituting Eq. (3.12) into Eq. (3.2), relations between A_i, B_i and C_i can be derived

$$\begin{array}{llll}
B_1 = p_1A_1 & B_2 = p_1A_2 & B_3 = p_2A_3 & B_4 = p_2A_4 \\
B_5 = p_3A_5 & B_6 = p_3A_6 & B_7 = p_4A_7 & B_8 = p_4A_8 \\
B_9 = p_5A_9 & B_{10} = p_5A_{10} & B_{11} = p_6A_{11} & B_{12} = p_6A_{12} \\
C_1 = q_1A_1 & C_2 = q_1A_2 & C_3 = q_2A_3 & C_4 = q_2A_4 \\
C_5 = q_3A_5 & C_6 = q_3A_6 & C_7 = q_4A_7 & C_8 = q_4A_8 \\
C_9 = q_5A_9 & C_{10} = q_5A_{10} & C_{11} = q_6A_{11} & C_{12} = q_6A_{12}
\end{array} \tag{3.13}$$

where

$$\begin{aligned}
p_i &= \frac{\kappa_2\kappa_6}{2\mathrm{j}\omega r_i^4\Omega}\Big(\frac{1}{\kappa_1}r_i^4 - \kappa_6 + \frac{4\omega^2\Omega^2}{\kappa_6}\Big) \qquad i = 1,2,\dots,6 \\
q_i &= \begin{cases}
\dfrac{\kappa_2\kappa_6^2e(\frac{1}{\kappa_1}r_i^4 - \kappa_6 + \frac{4\omega^2\Omega^2}{\kappa_6}\Big) + 4e_y\omega^2\Omega^2r_i^4}{2\mathrm{j}\omega r_i^4\Omega\Big(\frac{1}{\kappa_3}r_i^4 - \frac{\kappa_4}{\kappa_3}r_i^2 - \kappa_5\omega^2\Big)} & i = 1,2,3 \\
\dfrac{\kappa_2\kappa_6^2e\Big(\frac{1}{\kappa_1}r_i^4 - \kappa_6 + \frac{4\omega^2\Omega^2}{\kappa_6}\Big) + 4e_y\omega^2\Omega^2r_i^4}{2\mathrm{j}\omega r_i^4\Omega\Big(\frac{1}{\kappa_3}r_i^4 + \frac{\kappa_4}{\kappa_3}r_i^2 - \kappa_5\omega^2\Big)} & i = 4,5,6
\end{cases}
\end{aligned} \tag{3.14}$$

The constants A_1-A_{12} can be determined from the boundary conditions. For a clamped-free beam, the boundaries are as follows

$$\begin{array}{lll}
\text{clamped end} & (x = 0): & V = 0, V' = 0, W = 0, W' = 0, \Phi = 0, \Phi' = 0 \\
\text{free end} & (x = L): & V'' = 0, V''' = 0, W'' = 0, W''' = 0, \kappa_4\Phi' - \Phi''' = 0, \Phi'' = 0
\end{array} \tag{3.15}$$

Using boundary condition (3.15), a set of twelve homogeneous equations in terms of the constants A_1-A_{12} will be generated. The natural frequencies ω can be numerically solved by setting the determinant of the coefficient matrix of A_1-A_{12} to be equal to zero.

4. Numerical applications and results

In this Section, firstly some limiting cases are examined to validate the model presented here. Secondly, the dynamic characteristics of the beam with unsymmetrical cross sections are investigated using the proposed method.

4.1. Validation

The example for validating is taken from literature (Banerjee and Su, 2004). The beam has a rectangular cross section and possesses equal flexural rigidities in the two principal directions of the cross section. The properties are given by: $EI_{yy} = 582.996\,\mathrm{Nm}^2$, $EI_{zz} = 582.996\,\mathrm{Nm}^2$, $\rho A = 2.87\,\mathrm{kg/m}$, $L = 1.29\,\mathrm{m}$.

The non-dimensional natural frequency and the spinning speed parameter are defined as in literature (Banerjee and Su, 2004)

$$\omega_i^* = \frac{\omega_i}{\omega_0} \qquad \Omega^* = \frac{\Omega}{\omega_0} \tag{4.1}$$

where

$$\omega_0 = \sqrt{\frac{\sqrt{EI_{yy}EI_{zz}}}{\rho A L^4}} \tag{4.2}$$

Comparison of the first three natural frequencies in the current study with those given in published literature is listed in Table 1. Both examples apply to cantilever end conditions, and the effect of warping stiffness is excluded in the analysis. It is concluded that the resulting frequencies are in good agreement with the one given in the previous work. Because EI_{yy} equals EI_{zz} in this example, the natural frequency parameters of the first two modes are equal when the spinning speed parameter is zero.

Table 1. Natural frequencies of the spinning beam: (1) Banerjee and Su (2004), (2) present method

Spinning speed parameter (Ω^*)	Natural frequency parameters (ω_i^*)					
	ω_1^*		ω_2^*		ω_3^*	
	(1)	(2)	(1)	(2)	(1)	(2)
0	3.516	3.516	3.516	3.516	22.034	22.034
2	1.516	1.516	5.516	5.516	24.034	24.034
3.5	0	0	7.016	7.016	25.534	25.534
4	–	–	7.516	7.516	26.034	26.034

Then, to investigate characteristics of the beam with unsymmetrical cross section, two uniform beams with a semi-circular open cross section and with a channel cross section showed in Fig. 2, are considered. Physical properties of the beams for validation are derived from (Bercin and Tanaka, 1997), as shown in Table 2.

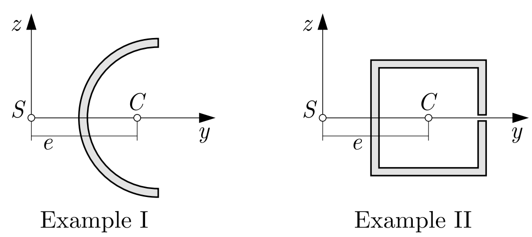

Fig. 2. The cross sections of the two beams studied

The first seven natural frequencies for the beams given in Fig. 2 are obtained by including and excluding the effect of warping stiffness when the spinning speed is zero, and compared with the results by Bercin and Tanaka (1997), as shown in Table 3. It is observed that when the effect of warping is neglected, the errors associated with it become increasingly large as the modal index increases.

Table 2. Physical properties of the beams studied

Parameters	Example I	Example II
EI_y [Nm^2]	6380	$1.436 \cdot 10^5$
EI_z [Nm^2]	2702	$2.367 \cdot 10^5$
GJ [N]	43.46	346.71
$E\Gamma$ [Nm^4]	0.10473	536.51
ρ [kg/m^3]	2712	2712
A [m^2]	$3.08 \cdot 10^{-4}$	$1.57 \cdot 10^{-3}$
L [m]	0.82	2.7
e [m]	0.0155	0.0735

Table 3. Natural frequencies [Hz] of the beam: (1) Bercin and Tanaka (1997); (2) present approach including warping; (3) present approach excluding warping

Modal index	Example I			Example II		
	(1)	(2)	(3)	(1)	(2)	(3)
1	63.79	63.79	62.65	11.03	11.02	8.332
2	137.7	137.7	130.4	–	18.10	18.10
3	–	149.7	149.7	39.02	39.02	23.92
4	278.4	278.4	261.5	58.19	58.20	36.74
5	484.8	484.8	422.5	–	113.4	47.42
6	663.8	663.8	613.3	152.4	152.4	67.41
7	–	768.4	656.3	209.4	209.4	86.64

4.2. Spinning speed

To examine the effect of the spinning speed on natural frequencies of the beam with an unsymmetrical cross section, various values with the interval $[0, 4]$ for the spinning speed parameter are considered for Example I and Example II, and the corresponding frequencies are presented in Tables 4 and 5.

Table 4. Natural frequencies of Example I versus the spinning speed parameter

Spinning speed parameter (Ω^*)	Natural frequency parameters (ω_i^*)					
	ω_1^*	ω_2^*	ω_3^*	ω_4^*	ω_5^*	ω_6^*
0	2.149	4.639	5.044	9.378	16.333	22.365
1	1.783	4.607	5.453	9.355	16.319	22.340
2	0.760	4.729	6.235	9.287	16.278	22.268
2.25	0	4.783	6.453	9.262	16.264	22.244
3	–	4.988	7.132	9.174	16.212	22.163
4	–	5.327	8.069	9.027	16.122	22.043

It is found that the spinning speed alters the natural frequencies, especially at the lower vibration modes. With an increase of the spinning speed, the coupling between y-axial and z-axial deformations becomes larger, which is demonstrated in Eq. (3.3). Therefore, mode shapes of the system change due to larger coupling and natural frequencies vary correspondingly. Mostly, as the modal index rises, the effect of spinning speed on natural frequencies weakens since the motion amplitudes become smaller with an increasing frequency, which corresponds to an insignificant change in the reference kinetic energy.

Table 5. Natural frequencies of Example II versus the spinning speed parameter

Spinning speed parameter (Ω^*)	Natural frequency parameters (ω_i^*)					
	ω_1^*	ω_2^*	ω_3^*	ω_4^*	ω_5^*	ω_6^*
0	2.426	3.984	8.587	12.809	24.966	33.541
1	1.943	4.461	8.737	12.741	25.038	33.517
2	1.024	5.335	9.166	12.547	25.249	33.448
2.63	0	5.937	9.553	12.372	25.449	33.380
3	–	6.300	9.811	12.255	25.588	33.332
4	–	7.303	10.532	11.971	26.038	33.171

Figures 3a and 3b show variations of the first four non-dimensional natural frequencies with respect to the spinning speed parameter. Because of the large difference between the bending rigidities in the two principal planes, the natural frequencies start off with different values. The fundamental frequencies of both examples decrease with the increasing spinning speed while the others decrease or increase. At a certain spinning speed, which is defined as the critical speed, the first natural frequency becomes negative, resulting in instability. For the spinning beam with circular or rectangular cross-section, the natural frequencies are obtained by subtracting or adding the natural frequencies when $\Omega^* = 0$ to the spinning speed parameter (Banerjee and Su, 2004). So the value of the critical spinning speed when the beam becomes unstable equals to the first frequency of the beam with $\Omega^* = 0$. For the spinning beam with an unsymmetrical cross-section, the noncoincidence of mass center and shear center induces coupled flexural-torsional modes and alters the critical speed. Both values of the critical speed are larger than the first frequencies for the examples studied.

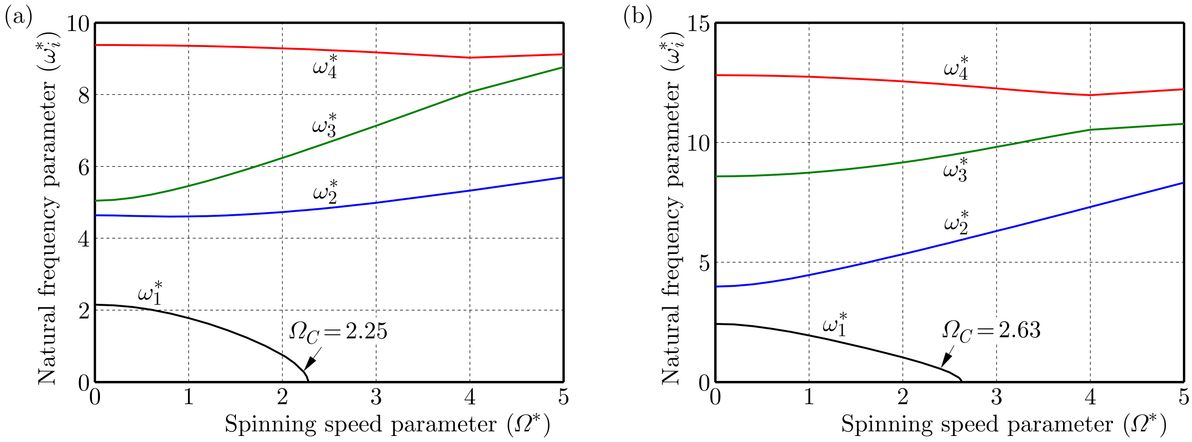

Fig. 3. Natural frequencies versus the spinning speed for (a) Example I, (b) Example II

4.3. Warping effect

The relative errors of natural frequencies due to the warping effect are discussed in this Section. Figures 4a and 4b show changes of natural frequencies with respect to the spinning speed with inclusion and exclusion of the warping for Example I and II, respectively.

It is evident that the inclusion of the warping effect increases the natural frequencies. And when the warping effect is neglected, the errors associated with it become increasingly larger as the modal index increases. Additionally, errors in Example II are more severe than in Example I. This is because the proportion of warping rigidity to bending rigidity in Example II is larger than that in Example I. It is also observed that the exclusion of warping makes the critical speed decrease.

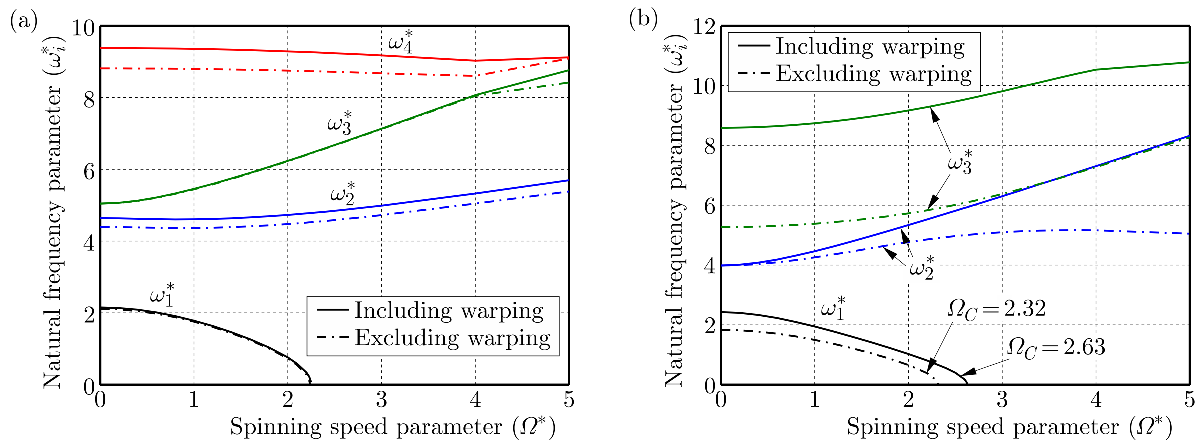

Fig. 4. Natural frequencies versus the spinning speed for (a) Example I, (b) Example II

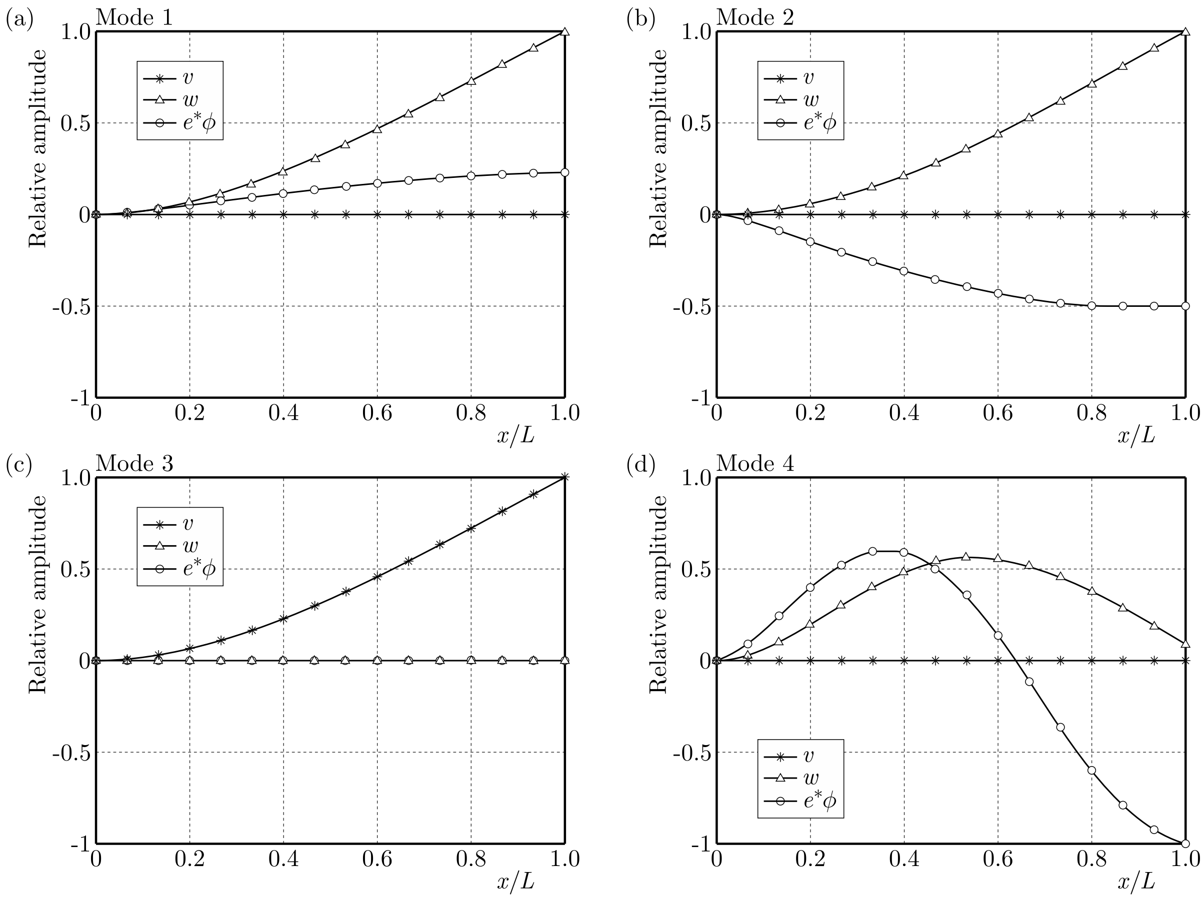

Fig. 5. Mode shapes in Example I with the speed parameter for $\Omega^* = 0$

4.4. Mode shapes

The first four normalized modal shape functions in Example I are illustrated in Figs. 5 and 6. It is concluded that in any case, the transverse deflection along the z-axis and torsional motion about the x-axis are coupled. The first two modes are coupled vibration modes of z-axial bending and x-axial torsion, while the third mode is the y-axial bending mode. When the spinning speed parameter is set to 2.0, all the modes become strongly coupled. Moreover, the speed has caused significant changes to the relative amplitudes between the z-axial displacement and x-axial torsional angle, especially for lower modes. The beams with unsymmetrical cross-sections show different characteristics compared with symmetric cross-sectional beams, for which the effects of spinning speed on mode shapes are marginal, as declared by Banerjee and Su (2004).

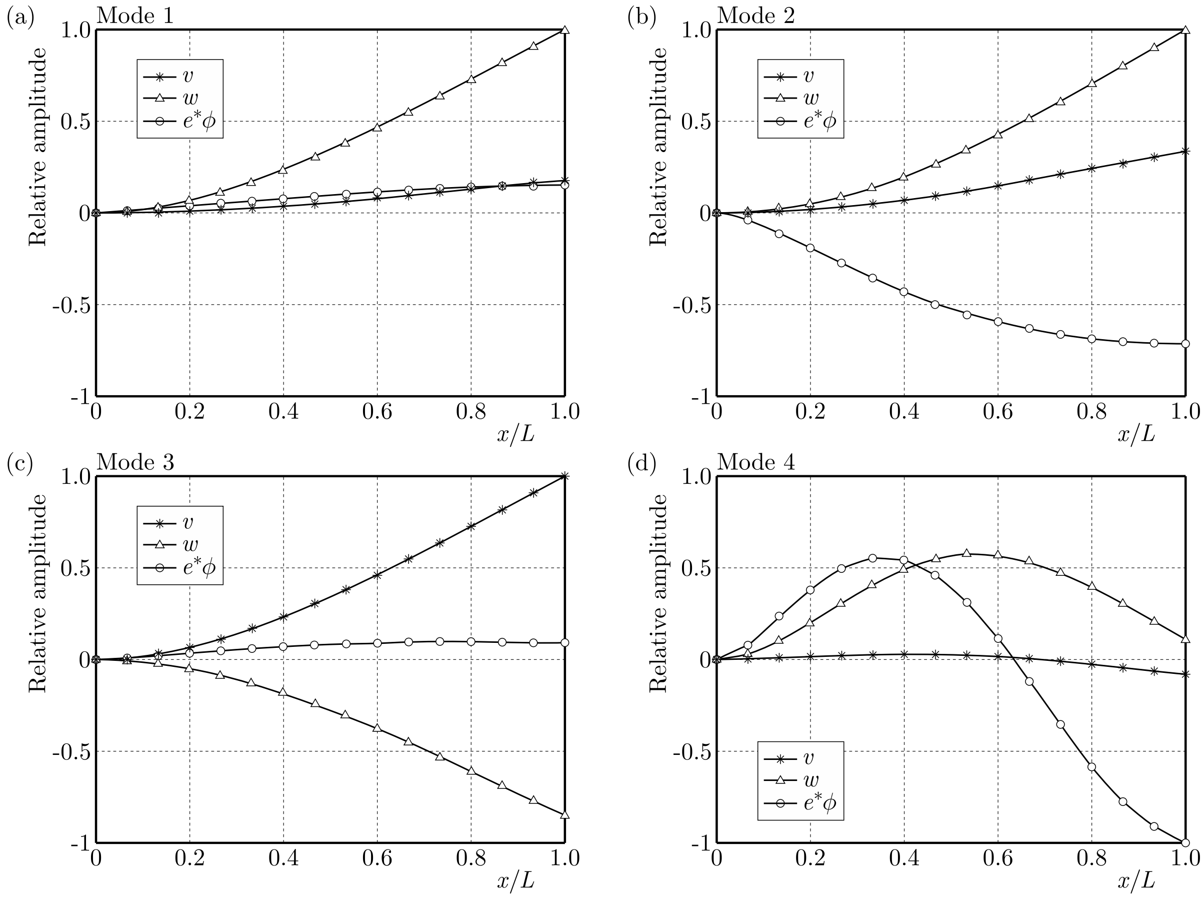

Fig. 6. Mode shapes in Example I with the speed parameter for $\Omega^* = 2$

5. Conclusions

This paper presents dynamic analysis of a spinning beam with an unsymmetrical cross section. The governing equations are formulated based on the Euler-Bernoulli beam theory and the Hamilton principle, and then natural frequencies and mode shapes are derived by the assumed mode method. Effects of the spinning speed and warping on natural frequencies are investigated. Numerical simulations are conducted in order to validate the present method and some main conclusions are derived as follows.

The noncoincidence of centroid and shear center of the unsymmetrical cross section induces the coupling of transverse deflection and torsional motion. The spinning speed induces coupling between transverse deflections along two orthogonal axes. The value of the spinning speed is critical to natural frequencies of the system. The mode shapes are notably changed due to the spinning speed which is different compared to the beam with the symmetric cross section. Also the critical speed increases for the spinning beam with unsymmetrical cross sections. It has also shown that the warping effect has a significant influence on the natural frequencies. Moreover, the effects of warping on the natural frequencies become increasingly large when the proportion of warping rigidity to bending rigidity is notable.

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Manuscript received September 18, 2015; accepted for print July 14, 2016