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SHORT RESEARCH COMMUNICATION

ON GEOMETRICAL INTERPRETATION OF THE FRACTIONAL STRAIN CONCEPT

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In this paper, for the first time, the geometrical interpretation of fractional strain tensor components is presented. In this sense, previous considerations by this author are shown in a new light. The fractional material and spatial line elements concept play a crucial role in the interpretation.

Keywords: fractional strain, fractional calculus, non-local models

1. Introduction

The fractional strain is a generalisation of the classical strain measure utilising the fractional calculus (the branch of mathematical analysis which deals with differential equations of an arbitrary order (Podlubny, 2002)). Such defined strain is non-local because of the fractional derivative definition.

In the literature, there exist a few concepts of fractional strain. One can mention here those by Klimek (2001), Lazopoulos (2006), equivalent concepts of Atanackovic and Stankovic (2009) and Carpinteri *et al.* (2011) or, finally, that by Drapaca and Sivaloganathan (2012). It is important that except for the concept presented in Drapaca and Sivaloganathan (2012), the previous ones were defined for 1D problems and small strains. Of fundamental meaning is also the fact that these authors consider different physical units of fractional strain tensor components, e.g. in Klimek (2001), Atanackovic and Stankovic (2009), Carpinteri *et al.* (2011) we have $[\mathrm{m}^{1-\alpha}]$, in Lazopoulos (2006) $[\mathrm{m}^{-\alpha}]$, or in (Drapaca and Sivaloganathan, 2012) $[\mathrm{m}^{3-\alpha_{1k}-\alpha_{2k}-\alpha_{3k}}]$ $k=1,2,3$, where m denotes meter, and the parameter α is in general different than 1.

In the paper by Sumelka (2014c) a different concept of fractional strain was presented. In that version, the fractional strain is without physical unit, as in the classical continuum mechanics, and the length scale parameter is given explicitly and simultaneously related to the terminals of the fractional differential operator.

In this paper, we follow the fundamental results given in the above mentioned paper (Sumelka, 2014c), giving finally the geometrical interpretation of fractional strain tensor components.

2. Geometrical interpretation of fractional strain

The description is given in the Euclidean space in Cartesian coordinates. We refer to $\mathcal{B}$ as the reference configuration of the continuum body while $\mathcal{S}$ denotes its current configuration. Points in $\mathcal{B}$ are denoted by $\mathbf{X}$ and in $\mathcal{S}$ by $\mathbf{x}$.

The regular motion of the material body $\mathcal{B}$ can be written as

$$\mathbf{x} = \phi(\mathbf{X}, t) \tag{2.1}$$

thus $\phi_t : \mathcal{B} \to \mathcal{S}$ is a C^1 actual configuration of $\mathcal{B}$ in $\mathcal{S}$, at time t.

Taking the Taylor expansion of motion for $\mathrm{d}\mathbf{X}$, we have

$$\phi(\mathbf{X}+\mathrm{d}\mathbf{X},t)=\phi(\mathbf{X},t)+\frac{\partial\phi(\mathbf{X},t)}{\partial\mathbf{X}}\mathrm{d}\mathbf{X}+|\mathrm{d}\mathbf{X}|\mathbf{r}(\mathbf{X},t,\mathrm{d}\mathbf{X}) \tag{2.2}$$

with the property of the residuum that $\lim_{|\mathrm{d}\mathbf{X}|\to 0}|\mathbf{r}(\mathbf{X},t,\mathrm{d}\mathbf{X})|=0$. Denoting $\mathrm{d}\mathbf{x}=\phi(\mathbf{X}+\mathrm{d}\mathbf{X},t)-\phi(\mathbf{X},t)$ and omitting higher order terms, one gets

$$\mathrm{d}\mathbf{x}=\mathbf{F}\mathrm{d}\mathbf{X} \tag{2.3}$$

and

$$\mathrm{d}\mathbf{X}=\mathbf{F}^{-1}\mathrm{d}\mathbf{x} \tag{2.4}$$

where $\mathbf{F}(\mathbf{X},t)=\partial\phi(\mathbf{X},t)/\partial\mathbf{X}$ denotes the deformation gradient, and $\mathbf{F}^{-1}(\mathbf{x},t)=\partial\varphi(\mathbf{x},t)/\partial\mathbf{x}$. We introduce non-local effects through multiplication of Eq. (2.3) (left sided) by $\underset{X}{\overset{\alpha}{\mathbf{F}}}$ and Eq. (2.4) (left sided) by $\underset{x}{\overset{\alpha}{\mathbf{F}}}$, thus

$$\mathrm{d}\widetilde{\mathbf{x}}=\underset{X}{\widetilde{\mathbf{F}}}\mathrm{d}\mathbf{X} \tag{2.5}$$

and

$$\mathrm{d}\widetilde{\mathbf{X}}=\underset{x}{\widetilde{\mathbf{F}}}\mathrm{d}\mathbf{x} \tag{2.6}$$

where (following the notation in (Sumelka, 2014c)), $\mathrm{d}\widetilde{\mathbf{x}}=\underset{X}{\overset{\alpha}{\mathbf{F}}}\mathrm{d}\mathbf{x}$ is a fractional spatial line element, $\mathrm{d}\widetilde{\mathbf{X}}=\underset{x}{\overset{\alpha}{\mathbf{F}}}\mathrm{d}\mathbf{X}$ is a fractional material line element, while $\underset{X}{\widetilde{\mathbf{F}}}=\underset{X}{\overset{\alpha}{\mathbf{F}}}\mathbf{F}$ and $\underset{x}{\widetilde{\mathbf{F}}}=\underset{x}{\overset{\alpha}{\mathbf{F}}}\mathbf{F}^{-1}$ are fractional deformation gradients defined as follows

$$\underset{X}{\widetilde{\mathbf{F}}}(\mathbf{X},t)=\ell_X^{\alpha-1}\underset{X}{D}^{\alpha}\phi(\mathbf{X},t) \tag{2.7}$$

and

$$\underset{x}{\widetilde{\mathbf{F}}}(\mathbf{x},t)=\ell_x^{\alpha-1}\underset{x}{D}^{\alpha}\varphi(\mathbf{x},t) \tag{2.8}$$

where ℓ_X and ℓ_x are length scales in $\mathcal{B}$ and $\mathcal{S}$, respectively. In Eqs. (2.7) and (2.8), D^{α} is the Riesz-Caputo fractional differential operator while α denotes the order of differentiation, cf. Sumelka (2014c). Comparing Eq. (2.3) and Eq. (2.5) (or Eq. (2.4) and Eq. (2.6)), one can also interpret such an assumption (by analogy to (Drapaca and Sivaloganathan, 2012)) as the existence of motion of the order α, which means the motion accounting for non-local effects. The situation is summarised in Fig. 1.

Notice that the length scales ℓ_X, ℓ_x preserve classical physical unit [m], and together with α, they are additional material parameters. As an example, for metallic materials, they can be identified as distances connected with non-homogeneous distribution of dislocations and cell structures (Pecherski, 1983; Sumelka, 2014b).

We have now four ways to define the strain tensor (cf. Fig. 1). Denoting by $\overset{\diamond}{\mathbf{F}}$ deformation gradients $\mathbf{F}$ or $\underset{X}{\widetilde{\mathbf{F}}}$ or $\underset{x}{\widetilde{\mathbf{F}}}$ or $\overset{\alpha}{\mathbf{F}}$, one can obtain local/non-local classical/fractional strain tensors through classical rules, namely

$$\mathbf{E}=\frac{1}{2}\Big(\overset{\diamond}{\mathbf{F}}{}^{\mathrm{T}}\overset{\diamond}{\mathbf{F}}-\mathbf{I}\Big)\qquad\qquad \mathbf{e}=\frac{1}{2}\Big(\mathbf{i}-\overset{\diamond}{\mathbf{F}}{}^{-\mathrm{T}}\overset{\diamond}{\mathbf{F}}{}^{-1}\Big) \tag{2.9}$$

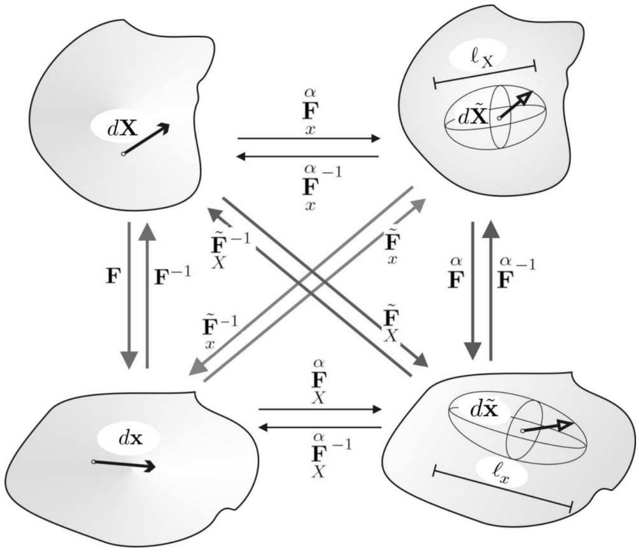

Fig. 1. Relations between the material and spatial line elements with their fractional counterparts $(\overset{\alpha}{\mathbf{F}} = \underset{X}{\widetilde{\mathbf{F}}}\mathbf{F}^{-1}\underset{x}{\widetilde{\mathbf{F}}}{}^{-1}$, $\overset{\alpha}{\underset{x}{\mathbf{F}}} = \underset{x}{\widetilde{\mathbf{F}}}\mathbf{F}$ and $\overset{\alpha}{\underset{X}{\mathbf{F}}} = \underset{X}{\widetilde{\mathbf{F}}}\mathbf{F}^{-1})$

where $\mathbf{E}$ is the classical Green-Lagrange strain tensor or its fractional counterpart, and $\mathbf{e}$ is the classical Euler-Almansi strain tensor or its fractional counterpart.

It should be emphasised that appropriate mapping of terminals from a material to spatial description (or inversely – cf. analogy in Sumelka (2014a)) that fulfil

$$\overset{\alpha}{\underset{X}{\mathbf{F}}} = \mathbf{F}\overset{\alpha}{\underset{x}{\mathbf{F}}}{}^{-1}\mathbf{F}^{-1} \qquad \text{or} \qquad \overset{\alpha}{\underset{x}{\mathbf{F}}} = \mathbf{F}^{-1}\overset{\alpha}{\underset{X}{\mathbf{F}}}{}^{-1}\mathbf{F} \tag{2.10}$$

assures that $\overset{\alpha}{\underset{X}{\mathbf{F}}} = \overset{\alpha}{\underset{x}{\mathbf{F}}}{}^{-1}$, so then the operating on the pair $\mathrm{d}\widetilde{\mathbf{X}} \to \mathrm{d}\mathbf{x}$ or $\mathrm{d}\mathbf{X} \to \mathrm{d}\widetilde{\mathbf{x}}$ is equivalent.

We can now draw a picture showing the geometrical meaning of fractional strain components – cf. Fig. 2 ($\overset{\diamond}{(\cdot)}$ stands for classical or fractional line elements). It is clear that extension (normal strain) of a (fractional) material line element $\mathrm{d}\overset{\diamond}{\mathbf{X}} = |\mathrm{d}\overset{\diamond}{\mathbf{X}}|e$ is defined as

$$\overset{\diamond}{\varepsilon} = \frac{|\mathrm{d}\overset{\diamond}{\mathbf{x}}| - |\mathrm{d}\overset{\diamond}{\mathbf{X}}|}{|\mathrm{d}\overset{\diamond}{\mathbf{X}}|} \qquad \text{or} \qquad \overset{\diamond}{\varepsilon} = \sqrt{1 + 2\mathbf{e}\cdot\overset{\diamond}{\mathbf{E}}\mathbf{e}} \Leftrightarrow \mathbf{e}\cdot\overset{\diamond}{\mathbf{E}}\mathbf{e} = \overset{\diamond}{\varepsilon} + \frac{\overset{\diamond}{\varepsilon}{}^2}{2} \tag{2.11}$$

where $\mathbf{e}$ is a unit vector along the fibre direction.

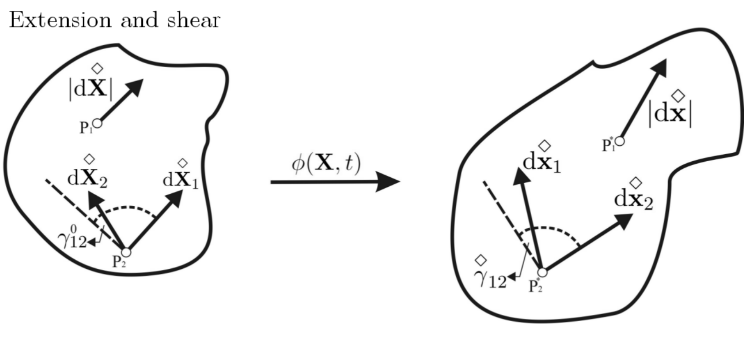

Fig. 2. Geometrical interpretation of the fractional extension and shear

The shear (shear strain) is defined by the deviation from orthogonality of two (fractional) material line elements $\mathrm{d}\overset{\diamond}{\mathbf{X}}_1 = |\mathrm{d}\overset{\diamond}{\mathbf{X}}_1|\mathbf{e}_1$ and $\mathrm{d}\overset{\diamond}{\mathbf{X}}_2 = |\mathrm{d}\overset{\diamond}{\mathbf{X}}_2|\mathbf{e}_2$, namely (cf. Fig. 2)

$$\sin \overset{\diamond}{\gamma}_{12} = \frac{\mathrm{d}\overset{\diamond}{\mathbf{x}}_1 \cdot \mathrm{d}\overset{\diamond}{\mathbf{x}}_2}{|\mathrm{d}\overset{\diamond}{\mathbf{x}}_1||\mathrm{d}\overset{\diamond}{\mathbf{x}}_2|} \qquad \text{or} \qquad \sin \overset{\diamond}{\gamma}_{12} = \frac{2\mathbf{e}_1 \cdot \overset{\diamond}{\mathbf{E}}\mathbf{e}_2 + \mathbf{e}_1 \cdot \mathbf{e}_2}{\sqrt{1 + 2\mathbf{e}_1 \cdot \overset{\diamond}{\mathbf{E}}\mathbf{e}_1}\sqrt{1 + 2\mathbf{e}_2 \cdot \overset{\diamond}{\mathbf{E}}\mathbf{e}_2}} \tag{2.12}$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are unit vectors along the fibres directions. In the case when initially the material line elements are perpendicular, $\mathbf{e}_1 \cdot \mathbf{e}_2 = 0$.

3. Conclusions

Geometrical interpretation of the fractional strain components is the same as that for classical strain. It is because of its analogical definition which is based on fractional ('scaled') material and spatial line elements. Hence, the extension is the ratio of the difference of squares of current and initial elemental lengths and squared initial elemental length. At the same time, shear defines the deviation from orthogonality of two elemental line elements (in fractional picture they must not be initially perpendicular).

It is important that an analogous geometrical interpretation can also be applied to other competitive formulations known in the literature (cf. Section 1 and paper (Sumelka *et al.*, 2015)), where similarities between formulations are shortly listed) – however one should remember that they operate on different physical units.

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