JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
51, 4, pp. 959-968, Warsaw 2013

THE EFFECT OF QUINTIC NONLINEARITY ON THE INVESTIGATION OF TRANSVERSELY VIBRATING BUCKLED EULER-BERNOULLI BEAMS

Hamid M. Sedighi, Arash Reza
Department of Mechanical Engineering, Ahvaz Branch, Islamic Azad University, Ahvaz, Iran
Jamal Zare
National Iranian South Oil Company (Nisoc), Ahvaz, Iran
e-mail: hmsedighi@gmail.com

A new formulation of vibrations of the axially loaded Euler-Bernoulli beam with quintic nonlinearity is investigated in the present study. The beam nonlinear natural frequency as a function of the initial amplitude is obtained. In this direction, modern powerful analytical methods namely He's Max-Min Approach (MMA) and Amplitude-Frequency Formulation (AFF) are employed to approximate the frequency-amplitude relationship of the beam vibrations. Afterwards, it is clearly shown that the first term in the series expansions is sufficient to produce a highly accurate approximation of the nonlinear system. Finally, preciseness of the present analytical procedures is evaluated in contrast with numerical calculation methods.

Key words: quintic nonlinearity, He's max-min approach, amplitude-frequency formulation, nonlinear vibration, buckled beam

1. Introduction

With evolution of technology, accurate comprehending of characteristics of beam vibrations is extremely important to researchers and engineers. The dynamic response of simply supported and clamped-clamped structures at large amplitudes of vibration can be encountered in many engineering applications. In such cases, it is of interest to know how far the characteristics of the dynamic response deviate from those defined via the linear theory. The problem of beam vibrations was recently investigated by many researchers with different boundary conditions and hypotheses. These researches predict the nonlinear frequency of beams which are very important for the design of many engineering structures. However, research on flexible beams has so far been restricted to cubic nonlinearity. Literature which considered higher order of nonlinearities is very limited (Sedighi *et al.*, 2012d).

Nowadays, substantial progresses had been made in analytical solutions to nonlinear equations without small parameters. There have been several classical approaches employed to solve governing nonlinear differential equations to study nonlinear vibrations including perturbation methods, Hamiltonian approach (He, 2010; Sedighi and Shirazi, 2013), He's max-min approach (MMA) (He, 2008b; Sedighi *et al.*, 2012a; Yazdi *et al.*, 2010), HAM (Sedighi and Shirazi, 2011; Sedighi *et al.*, 2012d), parameter expansion method (He and Shou, 2007; Sedighi and Shirazi, 2012; Sedighi *et al.*, 2011, 2012c,e), homotopy perturbation method (HPM) (Shadloo and Kimiaeifar, 2011), multistage adomian decomposition method (Evirgen and Özdemir, 2011), variational iteration method (Khosrozadeh *et al.*, 2013; Sedighi *et al.*, 2012a), modified variational iteration method (Yang *et al.*, 2012), Laplace transform method (Rafieipour *et al.*, 2012), monotone iteration schemes (Hasanov, 2011), multiple scales method (Hammad *et al.*, 2011), and Navier and Levy-type solution (Baferani *et al.*, 2011; Naderi and Saidi, 2011). The application of the new equivalent function to the deadzone, preload and saturation nonlinearities for finding dynamical behavior of beam vibrations using PEM and HA was investigated by Sedighi and

Shirazi (2012, 2013), Sedighi *et al.* (2011, 2012c,e). The MMA and AFF have been shown to solve a large class of nonlinear problems efficiently, accurately and easily, with approximations converging very rapidly to the solution. Usually, few iterations lead to high accuracy of the solution. The min-max approach and amplitude-frequency formulation proposed by He (2008a,b) are proved to be a very effective and convenient way for handling the non-linear problems, One iteration is sufficient to obtain a highly accurate solution.

To develop the comprehensive understanding on nonlinear frequency of beam vibrations, this paper brings quintic nonlinearities into consideration. The analytical solutions for geometrically nonlinear vibration of the Euler-Bernoulli beam including quintic nonlinearity using MMA and AFF is obtained. The nonlinear ordinary differential equation of the beam vibration is extracted from the partial differential equation with first mode approximation, based on the Galerkin theory. The results presented in this paper exhibit that the analytical methods are very effective and convenient for nonlinear the beam vibration for which highly nonlinear governing equations exist.

2. Equation of motion

Consider the Euler- Bernoulli beam of length l, moment of inertia I, mass per unit length m and modulus of elasticity E, which is axially compressed by loading P as shown in Fig. 1.

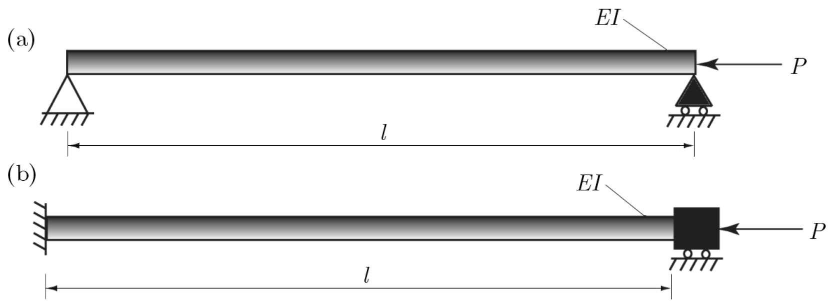

Fig. 1. Configuration of a uniform Euler-Bernoulli beam (a) simply supported beam, (b) clamped-clamped beam

Denoting by w the transverse deflection, the differential equation governing the equilibrium in the deformed situation is derived as

$$\frac{d^2}{dx^2}\left(\frac{EIw''(x,t)}{\sqrt{[1+w'^2(x,t)]^3}}\right)+Pw''(x,t)\Big(1+\frac{3}{2}w'^2\Big)+m\ddot{w}(x,t)=0 \tag{2.1}$$

where $w''(x,t)/\sqrt{[1+w'^2(x,t)]^3}$ is the "exact" expression for the curvature, using the approximation

$$\frac{w''(x,t)}{\sqrt{[1+w'^2(x,t)]^3}}\cong w''(x,t)\Big[1-\frac{3}{2}w'^2(x,t)+\frac{15}{8}w'^4(x,t)\Big] \tag{2.2}$$

where the nonlinear term $Pw''(x,t)[1+3w'^2/2]$ has been extracted from Sedighi *et al.* (2012d). Governing quintic nonlinear equation (2.3) can be expressed as

$$\begin{aligned}&EIw^{(4)}\Big(1-\frac{3}{2}w'^2+\frac{15}{8}w'^4\Big)-9EIw'''w'w''+\frac{45}{2}EIw'''w'^3w''-3EIw''^3\\&+\frac{45}{2}EIw'^2w''^3+Pw''\Big(1+\frac{3}{2}w'^2\Big)+m\ddot{w}=0\end{aligned} \tag{2.3}$$

which is subjected to the following boundary conditions:
— for simply supported (S-S) beam

$$w(0,t) = \frac{\partial^2 w}{\partial x^2}(0,t) = 0 \qquad w(l,t) = \frac{\partial^2 w}{\partial x^2}(l,t) = 0 \tag{2.4}$$

— for clamped-clamped (C-C) beam

$$w(0,t) = w(l,t) = 0 \qquad \frac{\partial w}{\partial x}(0,t) = \frac{\partial w}{\partial x}(l,t) = 0 \tag{2.5}$$

Assuming $w(x,t) = q(t)\phi(x)$, where $\phi(x)$ is the first eigenmode of the beam vibration, it can be expressed as:
— S-S beam

$$\phi(x) = \sin\frac{\pi x}{l} \tag{2.6}$$

— C-C beam

$$\phi(x) = \Big(\frac{x}{l}\Big)^2\Big(1 - \frac{x}{l}\Big)^2 \tag{2.7}$$

applying the Bubnov-Galerkin method yields

$$\begin{aligned}\int\limits_0^l \Big[&EIw^{(4)}\Big(1 - \frac{3}{2}w'^2 + \frac{15}{8}w'^4\Big) - 9EIw'''w'w'' + \frac{45}{2}EIw'''w'^3w'' - 3EIw''^3 \\ &+ \frac{45}{2}EIw'^2w''^3 + Pw''\Big(1 + \frac{3}{2}w'^2\Big) + m\ddot{w}\Big]\phi(x)\;dx = 0\end{aligned} \tag{2.8}$$

By introducing the following non-dimensional variables

$$\tau = \sqrt{\frac{EI}{ml^4}} \qquad \overline{q} = \frac{q}{l} \tag{2.9}$$

the non-dimensional nonlinear equation of motion about its first buckling mode can be written as

$$\frac{d^2\overline{q}(\tau)}{d\tau^2} + \gamma_1\overline{q}(\tau) + \gamma_2[\overline{q}(\tau)]^3 + \gamma_3[\overline{q}(\tau)]^5 = 0 \tag{2.10}$$

where:
— S-S beam

$$\gamma_1 = \pi^4 - \frac{Pl^2\pi^2}{EI} \qquad \gamma_2 = -\frac{3}{8}\pi^6 - \frac{3}{8}\frac{Pl^2\pi^4}{EI} \qquad \gamma_3 = \frac{15}{64}\pi^8 \tag{2.11}$$

– C-C beam

$$\gamma_1 = 500.534 - \frac{12.142Pl^2}{EI} \qquad \gamma_2 = -6.654 - \frac{0.1694Pl^2}{EI} \qquad \gamma_3 = -0.3673 \tag{2.12}$$

3. He's max-min approach

In many engineering problems, it is easy to find maximum/minimum interval of the solution to a nonlinear equation. From this maximum-minimum relationship called "He Chengtian inequality" which has millennia history, He (2008b) introduced approximated solutions for nonlinear vibrating systems.

Consider a generalized nonlinear oscillator in the form

$$\ddot{q}+qf(q,\dot{q},\ddot{q})=0 \qquad q(0)=A \qquad \dot{q}(0)=0 \tag{3.1}$$

According to the idea of the min-max approach, we choose a trial-function in the form

$$q=A\cos(\omega t) \tag{3.2}$$

where ω is an unknown frequency which to be determined. The method implies that the square of the frequency satisfies the following inequality

$$f_{min}\leqslant\omega^2\leqslant f_{max} \tag{3.3}$$

where f_{max} and f_{min} are the maximum and minimum values of the function f, respectively. According to the Chentian interpolation (He, 2008b), we obtain

$$\omega^2=\frac{f_{min}+kf_{max}}{1+k} \tag{3.4}$$

The value of $k$can be approximately determined by various approximate methods. So the solution to Eq. (3.1) can be expressed as

$$q=A\cos\Big(\sqrt{\frac{f_{min}+kf_{max}}{1+k}}t\Big) \tag{3.5}$$

To investigate the min-max procedure, Eq. (2.10) should be rewritten in the following form

$$\frac{d^2q}{dt^2}+(\gamma_1+\gamma_2q^2+\gamma_3q^4)q=0 \tag{3.6}$$

Assuming the solution to the above equation in the form of Eq. (10), yields

$$\omega^2_{min}=\gamma_1 \qquad \omega^2_{max}=\gamma_1+\gamma_2A^2+\gamma_3A^4 \tag{3.7}$$

Therefore, according to Eq. (3.3)

$$\gamma_1\leqslant\omega^2\leqslant\gamma_1+\gamma_2A^2+\gamma_3A^4 \tag{3.8}$$

and on assumption (3.4), we obtain

$$\omega^2=\frac{\gamma_1+k(\gamma_1+\gamma_2A^2+\gamma_3A^4)}{1+k} \tag{3.9}$$

Using the Bubnov-Galerkin procedure and substituting Eqs. (3.9) and (3.5) into Eq. (2.10) results in the following value for the parameter k

$$k=\frac{5\gamma_3A^2+6\gamma_2}{2\gamma_2+3\gamma_3A^2} \tag{3.10}$$

Substituting Eq. (3.10) into Eq. (3.9) yields the nonlinear frequency of the beam as a function of the amplitude, as follows

$$\omega(A)=\sqrt{\gamma_1+\frac{3}{4}\gamma_2A^2+\frac{5}{8}\gamma_3A^4} \tag{3.11}$$

4. Amplitude-frequency formulation

To solve nonlinear problems, an amplitude–frequency formulation for nonlinear oscillators was proposed by He (2008a), which was deduced using an ancient Chinese mathematics method. According to He's amplitude-frequency formulation, $u_1 = A\cos\tau$ and $u_2 = A\cos(\omega\tau)$ serve as the trial functions. Substituting u_1 and u_2 into equation (2.10) results in the following residuals

$$\begin{aligned} R_1 &= -A\cos\tau + \gamma_1 A\cos\tau + \gamma_2 A^3\cos^3\tau + \gamma_3 A^5\cos^5\tau \\ R_2 &= -A\omega^2\cos(\omega\tau) + \gamma_1 A\cos(\omega\tau) + \gamma_2 A^3\cos^3(\omega\tau) + \gamma_3 A^5\cos^5(\omega\tau) \end{aligned} \tag{4.1}$$

According to the amplitude-frequency formulation, the above residuals can be rewritten in the forms of weighted residuals (Khan and Akbarzade, 2012)

$$\begin{aligned} R_{11} &= \frac{4}{T_1}\int\limits_0^{T_1/4} R_1\cos\tau\,d\tau & T_1 &= 2\pi \\ R_{22} &= \frac{4}{T_2}\int\limits_0^{T_2/4} R_2\cos(\omega\tau)\,d\tau & T_2 &= \frac{2\pi}{\omega} \end{aligned} \tag{4.2}$$

Applying He's frequency-amplitude formulation

$$\omega^2 = \frac{\omega_1^2 R_{22} - \omega_2^2 R_{11}}{R_{22} - R_{11}} \tag{4.3}$$

where

$$\omega_1 = 1 \qquad \omega_2 = \omega \tag{4.4}$$

then the approximate frequency can be obtained

$$\omega(A) = \sqrt{\gamma_1 + \frac{3}{4}\gamma_2 A^2 + \frac{5}{8}\gamma_3 A^4} \tag{4.5}$$

5. Results and discussion

To verify the soundness of the proposed solutions by asymptotic approaches, the authors plot the analytical solutions for a simply supported and clamped-clamped beam at the side of corresponding numerical results in Fig. 2, where the first approximated amplitude-time curves of a uniform beam subjected to axial compression is presented for different initial conditions. As can be seen, the first order approximation of $q(\tau)$ from the analytical method is in excellent agreement with numerical results from fourth-order Runge-Kutta method. The exact analytical solutions reveal that the first term in the series expansions is sufficient to result in a highly accurate solution of the problem. Furthermore, these equations provide excellent approximations to the exact period regardless of the oscillation amplitude. The material and geometric properties adopted here have been prepared in the Appendix.

For a vibrating Euler-Bernoulli beam, the Euler-Lagrange equation is as follows

$$\frac{d^2}{dx^2}[EIw''(x,t)] + Pw''(x,t) + m\ddot{w}(x,t) = 0 \tag{5.1}$$

Applying the Bubnov-Galerkin method and using the first eigenmode of the simply supported beam, yields

$$\frac{d^2\overline{q}}{dt^2} + \gamma_1\overline{q}(\tau) = 0 \tag{5.2}$$

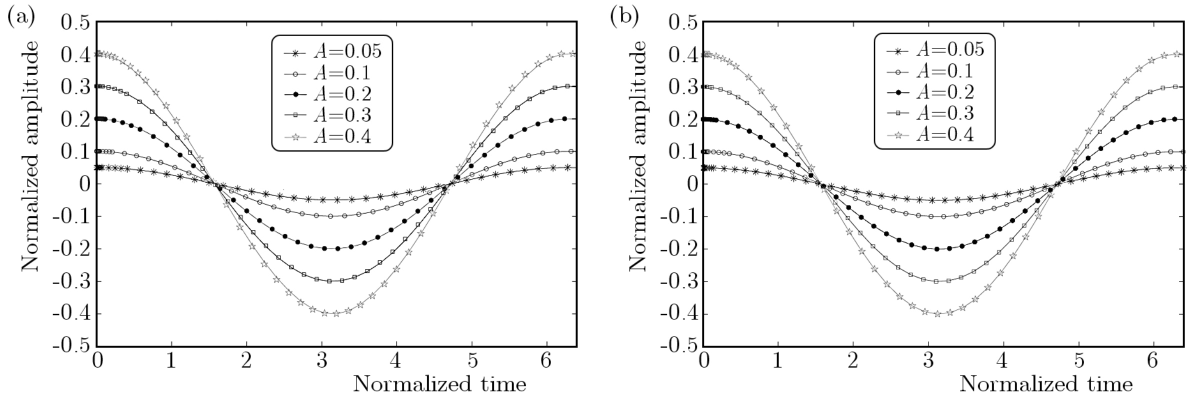

Fig. 2. Comparison between analytical and numerical results for different initial amplitudes, symbols: numerical solution, solid line: analytical solutions, (a) S-S beam, (b) C-C beam

Figures 3 and 4 display the effect of the normalized amplitude on the nonlinear behavior of a vibrating beam. From Eq. (4.5), the nonlinear natural frequency is a function of the amplitude, which means when the oscillation amplitude becomes larger (for both S-S and C-C beam), the accuracy of approximated frequencies in usual beam theory ($\gamma_2 = \gamma_3 = 0$) and cubic nonlinear beam ($\gamma_3 = 0$) decreases. It confirms that the normalized amplitude has a significant effect on nonlinear behavior of the beams. From these figures it is observed that the results from the usual beam theory are incompatible with the quintic nonlinear beam when the initial condition becomes larger.

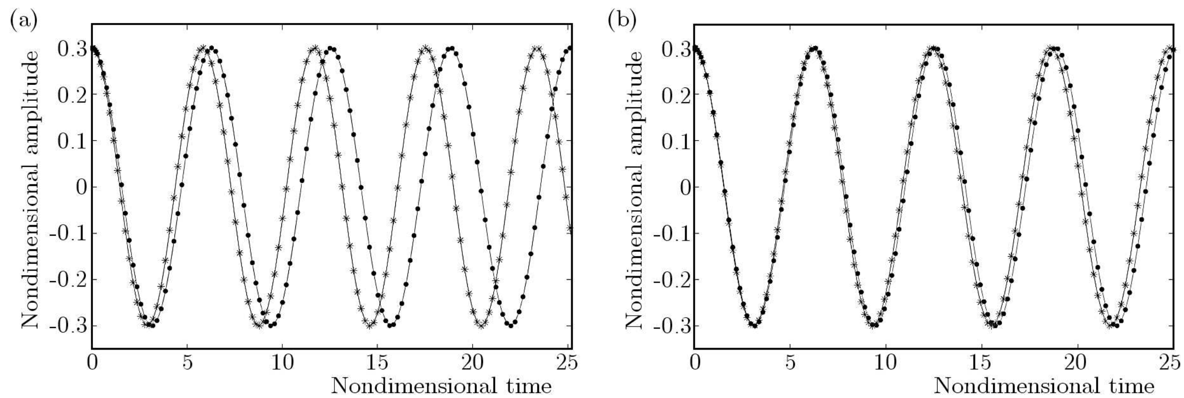

Fig. 3. The impact of nonlinear terms on dynamical behavior of the beam for $A = 0.3$, – * – usual beam theory, – • – quintic nonlinear beam, (a) S-S beam, (b) C-C beam

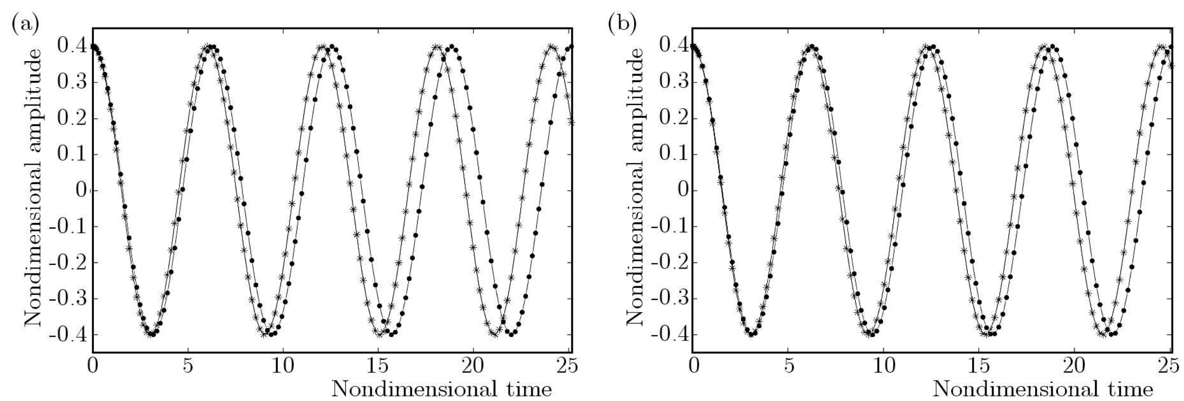

Fig. 4. The impact of nonlinear terms on dynamical behavior of the beam for $A = 0.4$, – * – usual beam theory, – • – quintic nonlinear beam, (a) S-S beam, (b) C-C beam

In order to investigate the effect of parameter γ_1 on the nonlinear behavior of the quintic nonlinear beam, the natural frequency as a function of γ_1 has been illustrated in Fig. 5 for different amplitudes. It is observed that the difference between the nonlinear fundamental frequency and the usual beam frequency increases with the vibration amplitude. Also, the percent of error in approximating the natural frequency as a function of γ_1 for different values of the oscillation amplitude has been depicted in Fig. 6. The relative error of the approximated simple beam theory frequency progressively increases at lower values of γ_1 for all values of vibration amplitudes.

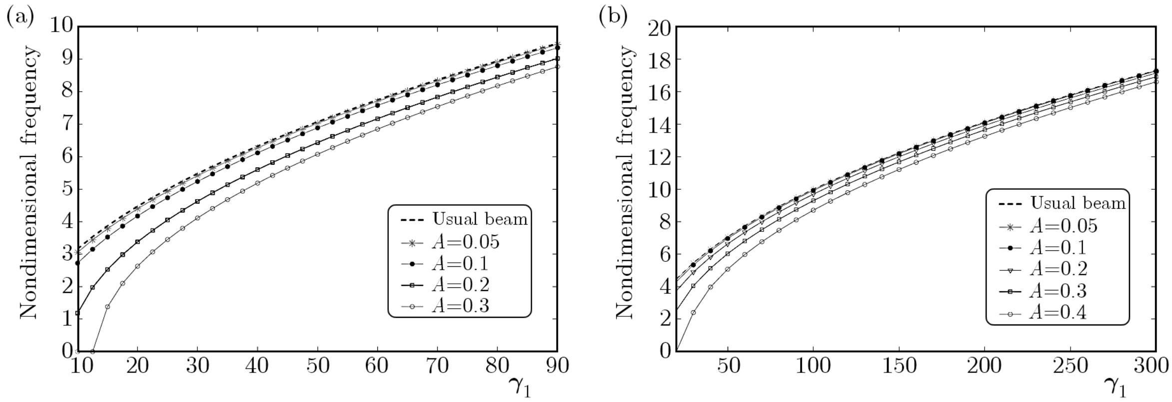

Fig. 5. Comparison of fundamental frequencies of the usual beam and quintic nonlinear beam as a function of γ_1, (a) S-S beam, (b) C-C beam

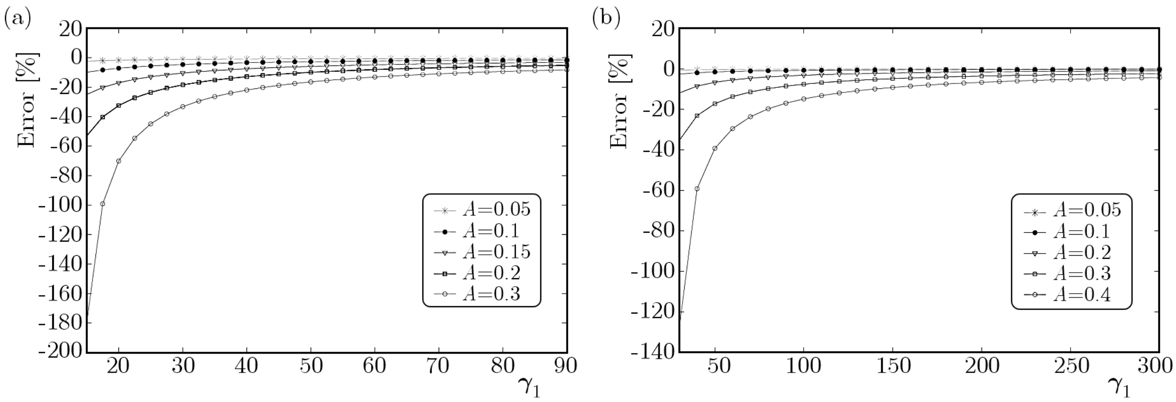

Fig. 6. The percent of error in approximating the natural frequency of the usual beam as a function of γ_1, (a) S-S beam, (b) C-C beam

In order to demonstrate the necessity of quintic nonlinear terms, the percent of error in approximating the cubic natural frequency as a function of γ_1 for different values of the oscillation amplitude has been illustrated in Fig. 7. When the parameter γ_1 decreases, the relative error of the approximated cubic beam frequency increases, especially for the simply supported beam.

6. Conclusion

In the current study, two modern powerful analytical methods called He's max-min approach and amplitude-frequency formulation were employed to solve the governing equation of vibration of quintic nonlinear beams. It demonstrated that the fundamental frequency based upon the linear theory and cubic nonlinear beam can be different from the natural frequency of the quintic

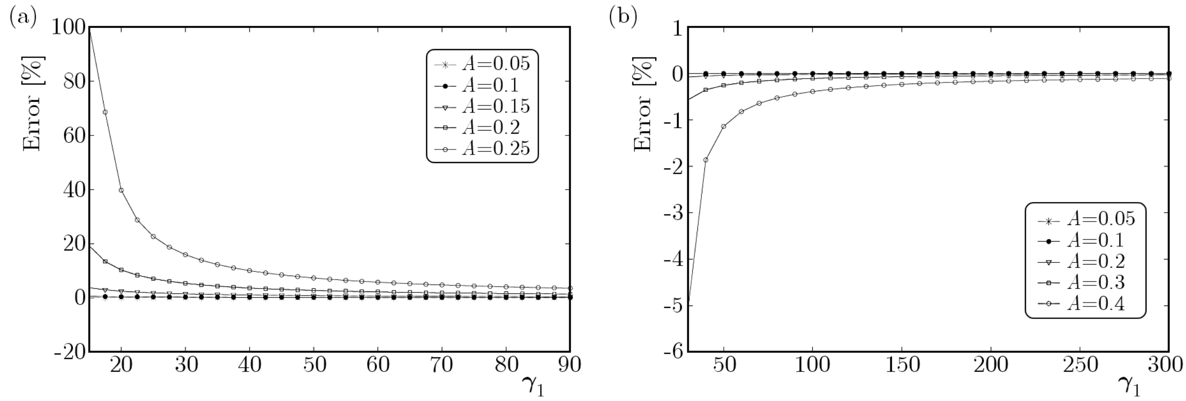

Fig. 7. The percent of error in approximating the natural frequency of the cubic nonlinear beam as a function of γ_1, (a) S-S beam, (b) C-C beam

nonlinear beam at large vibration amplitudes. An excellent first-order analytical solution using modern asymptotic approaches was obtained. The soundness of the obtained analytical solutions was verified by numerical methods.

References

1. Akhtyamov A.M., Il'gamov M.A., 2013, Flexural model for a notched beam: Direct and inverse problems, *Journal of Applied Mechanics and Technical Physics*, **54**, 1, 132-141, DOI: 10.1134/S0021894413010161

2. Andreaus U., Placidi L., Rega G., 2011, Soft impact dynamics of a cantilever beam: equivalent SDOF model versus infinite-dimensional system, *Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science*, **225**, 10, 2444-2456, DOI: 10.1177/09544062114114484

3. Arvin H., Bakhtiari-Nejad F., 2011, Non-linear modal analysis of a rotating beam, *International Journal of Non-Linear Mechanics*, **46**, 877-897

4. Awrejcewicz J., Krysko A.V., Soldatov V., Krysko V.A., 2012, Analysis of the nonlinear dynamics of the Timoshenko flexible beams using wavelets, *Journal of Computational and Nonlinear Dynamics*, **7**, 1, 011005

5. Baferani A.H., Saidi A.R., Jomehzadeh E., 2011, An exact solution for free vibration of thin functionally graded rectangular plates, *Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science*, **225**, 3, 526-536, DOI: 10.1243/09544062JMES2171

6. Barari A., Kaliji H.D., Ghadami M., Domairry G., 2011, Non-linear vibration of Euler-Bernoulli beams, *Latin American Journal of Solids and Structures*, **8**, 139-148

7. Campanile L.F., Jähne R., Hasse H., 2011, Exact analysis of the bending of wide beams by a modified elastica approach, *Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science*, **225**, 11, 2759-2764, DOI: 10.1177/0954406211417753

8. Cha P.D., Rinker J.M., 2012, Enforcing nodes to suppress vibration along a harmonically forced damped Euler-Bernoulli beam, *Journal of Vibration and Acoustics*, **134**, 5, 051010, DOI:10.1115/1.4006375

9. Evirgen, F., Özdemir N., 2011, Multistage adomian decomposition method for solving NLP problems over a nonlinear fractional dynamical system, *Journal of Computational and Nonlinear Dynamics*, **6**, 2, 021003, DOI: 10.1115/1.4002393

10. Hammad B.K., Nayfeh A.H., Abdel-Rahman E.M., 2011, On the use of the subharmonic resonance as a method for filtration, *Journal of Computational and Nonlinear Dynamics*, **6**, 4, 041007, DOI: 10.1115/1.4003031
11. Hasanov A., 2011, Some new classes of inverse coefficient problems in non-linear mechanics and computational material science, *International Journal of Non-Linear Mechanics*, **46**, 5, 667-684
12. He J.H., 2008a, An improved amplitude-frequency formulation for nonlinear oscillators, *International Journal of Nonlinear Sciences and Numerical Simulation*, **9**, 2, 211-212
13. He J.H., 2008b, Max-min approach to nonlinear oscillators, *International Journal of Nonlinear Sciences and Numerical Simulation*, **9**, 2, 207-210
14. He J.H., 2010, Hamiltonian approach to nonlinear oscillators, *Physics Letters A*, **374**, 23, 2312-2314
15. He J.H., Shou D.H., 2007, Application of parameter-expanding method to strongly nonlinear oscillators, *International Journal of Nonlinear Sciences and Numerical Simulation*, **8**, 121-124
16. Jang T.S., Baek H.S., Paik J.K., 2011, A new method for the non-linear deflection analysis of an infinite beam resting on a non-linear elastic foundation, *International Journal of Non-Linear Mechanics*, **46**, 339-346
17. Khan Y., Akbarzade M., 2012, Dynamic analysis of nonlinear oscillator equation arising in double-sided driven clamped microbeam-based electromechanical resonator, *Zeitschrift für Naturforschung*, **67a**, 435-440, DOI: 10.5560/ZNA.2012-0043
18. Khosrozadeh A., Hajabasi M.A., Fahham H.R., 2013, Analytical approximations to conservative oscillators with odd nonlinearity using the variational iteration method, *Journal of Computational and Nonlinear Dynamics*, **8**, 014502, DOI: 10.1115/1.4006789
19. Krys'ko V.A., Koch M.I., Zhigalov M.V., Krys'ko A.V., 2012, Chaotic phase synchronization of vibrations of multilayer beam structures, *Journal of Applied Mechanics and Technical Physics*, **53**, 3, 451-459, DOI: 10.1134/S0021894412030182
20. Kumar S., Kumar R., Sehgal R., 2012, Performance analysis of finite element and energy based analytical methods for modeling of PCLD treated beams, *Journal of Vibration and Acoustics*, **134**, 3, 034501, DOI: 10.1115/1.4006232
21. Naderi A., Saidi A.R., 2011, Buckling analysis of functionally graded annular sector plates resting on elastic foundations, *Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science*, **225**, 2, 312-325
22. Ozturk B., 2011, Free vibration analysis of beam on elastic foundation by the variational iteration method, *International Journal of Nonlinear Sciences and Numerical Simulation*, **10**, 10, 1255-1262, DOI: 10.1515/IJNSNS.2009.10.10.1255
23. Rafieipour H., Lotfavar A., Mansoori M.H., 2012, New analytical approach to nonlinear behavior study of asymmetrically LCBs on nonlinear elastic foundation under steady axial and thermal loading, *Latin American Journal of Solids and Structures*, **9**, 531-545
24. Sedighi H.M., Reza A., Zare J., 2011, Dynamic analysis of preload nonlinearity in nonlinear beam vibration, *Journal of Vibroengineering*, **13**, 778-787
25. Sedighi H.M., Shirazi K.H., 2011, Using homotopy analysis method to determine profile for disk cam by means of optimization of dissipated energy, *International Review of Mechanical Engineering*, **5**, 941-946
26. Sedighi H.M., Shirazi K.H., 2012, A new approach to analytical solution of cantilever beam vibration with nonlinear boundary condition, *Journal of Computational and Nonlinear Dynamics*, **7**, 034502 DOI: 10.1115/1.4005924
27. Sedighi H.M., Shirazi K.H., 2013, Asymptotic approach for nonlinear vibrating beams with saturation type boundary condition, *Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science*, in press, DOI: 10.1177/0954406213475561

28. Sedighi H.M., Shirazi K.H., Noghrehabadi A., 2012a, Application of Recent Powerful Analytical Approaches on the Non-Linear Vibration of Cantilever Beams, *International Journal of Nonlinear Sciences and Numerical Simulation*, **13**, 7/8, 487-494, DOI: 10.1515/ijnsns-2012-0030

29. Sedighi H.M., Shirazi K.H., Noghrehabadi A.R., Yildirim A., 2012b, Asymptotic investigation of buckled beam nonlinear vibration, *Iranian Journal of Science and Technology, Transactions of Mechanical Engineering*, **36**, M2, 107-116

30. Sedighi H.M., Shirazi K.H., Reza A., Zare J., 2012c, Accurate modeling of preload discontinuity in the analytical approach of the nonlinear free vibration of beams, *Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science*, **226**, 10, 2474-2484, DOI: 10.1177/0954406211435196

31. Sedighi H.M., Shirazi K.H., Zare J., 2012d, An analytic solution of transversal oscillation of quintic nonlinear beam with homotopy analysis method, *International Journal of Non-Linear Mechanics*, **47**, 777-784, DOI: 10.1016/j.ijnonlinmec.2012.04.008

32. Sedighi H.M., Shirazi K.H., Zare J., 2012e, Novel equivalent function for deadzone nonlinearity: applied to analytical solution of beam vibration using He's parameter expanding method, *Latin American Journal of Solids and Structures*, **9**, 443-451

33. Shadloo M.S., Kimiaeifar A., Application of homotopy perturbation method to find an analytical solution for magneto hydrodynamic flows of viscoelastic fluids in converging/diverging channels, *Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science*, **225**, 347-353

34. Yang Q.W., Chen Y.M., Liu J.K., Zhao W., 2012, A Modified Variational Iteration Method for Nonlinear Oscillators, *International Journal of Nonlinear Sciences and Numerical Simulation*, **13**, DOI: 10.1515/ijnsns.2011.045

35. Yazdi M.K., Ahmadian H., Mirzabeigy A., Yildirim A., 2012, Dynamic analysis of vibrating systems with nonlinearities, *Communications in Theoretical Physics*, **57**, 2, 183-187

Manuscript received October 15, 2012; accepted for print April 9, 2013