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SOLUTION OF INFINITE SLAB STATIC PROBLEM USING THE FINITE STRIP METHOD BY DIFFERENCE EQUATIONS

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An analytical approach to the solution of an infinite slab static problem using the finite strip method is presented. The structure simply supported on its opposite edges is treated as a discrete one. A regular mesh of identical finite strips approximates the continuous structure. This regular slab discretization enables one to derive a fundamental solution for the two-dimensional discrete strip structure in an analytical, closed form. Equilibrium conditions are derived from the finite element method formulation. The set of the infinite number of equilibrium conditions is replaced by one equivalent difference equation. The solution to this equation is the fundamental function, i.e. Green's function for considered slab strip.

Key words: finite element method, finite strips, difference equations, plane slab

1. Introduction

The static problem of an infinite plane slab simply supported on its opposite edges is considered (see Fig. 1). We assume that the longitudinal displacements and transverse stresses are equal to zero at both edges. The slab strips with such boundary conditions are commonly used at bridge structures.

In the finite strip method (Loo and Cusens, 1978) the structure is divided into a set of finite strips of the length L (L is width of the slab) and an arbitrary width b (see Fig. 2).

The unknowns are displacements u and v (in the x and y directions) along the nodal line r which connects two adjacent strips. It is assumed that two

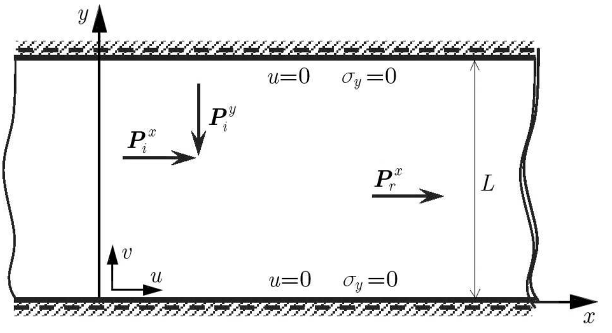

Fig. 1. An infinite slab strip

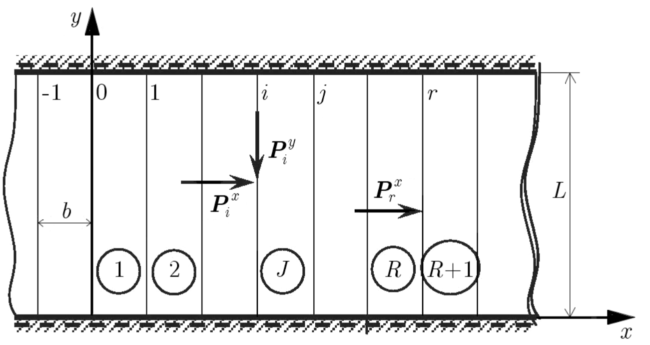

Fig. 2. Finite strip discretization

point forces P^x and P^y concentrated at an arbitrary nodal line load the slab. According to the finite strip procedure, the fields of loading and displacement functions are expressed in the form of harmonic series (Loo and Cusens, 1978)

$$
\begin{aligned}
p(x) &= \sum_{n=1}^{\infty} P^x \sin\frac{n\pi y_o}{L}\\
u(x,y) &= \sum_{n=1}^{\infty} \Big[\Big(1-\frac{x}{b}\Big)u_{i,n} + \frac{x}{b}u_{j,n}\Big] \sin\frac{n\pi y}{L}\\
v(x,y) &= \sum_{n=1}^{\infty} \Big[\Big(1-\frac{x}{b}\Big)v_{i,n} + \frac{x}{b}v_{j,n}\Big] \cos\frac{n\pi y}{L}
\end{aligned}
\tag{1.1}
$$

The four-degree-of-freedom finite strips (see Fig. 3) are used in the calculations.

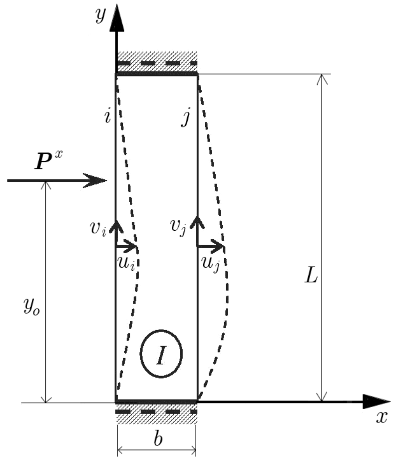

Fig. 3. The finite strip element

The making use of displacement functions (1.1) and the minimization procedure for the potential energy formula

$$U^I = \frac{t}{2}\int_0^L\int_0^b \boldsymbol{\sigma}^\top \boldsymbol{\varepsilon}\; dx\,dy - \int_0^L\int_0^b [u,v]\begin{bmatrix} p(x) \\ p(y) \end{bmatrix}\; dx\,dy \tag{1.2}$$

where

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} \dfrac{E_x}{1-\nu_x\nu_y} & \dfrac{\nu_y E_y}{1-\nu_x\nu_y} & 0 \\ \dfrac{\nu_x E_x}{1-\nu_x\nu_y} & \dfrac{E_y}{1-\nu_x\nu_y} & 0 \\ 0 & 0 & G_{xy} \end{bmatrix}\boldsymbol{\varepsilon}$$

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{bmatrix}$$

leads to a set of infinite number of linear equations

$$\sum_{I=-\infty}^{\infty} \mathbf{K}_n^I \boldsymbol{W}_n^I = \boldsymbol{P}_n^I \tag{1.3}$$

where

$$\begin{aligned} \boldsymbol{W}_n^I &= [u_{i,n}, v_{i,n}, u_{j,n}, v_{j,n}]^\top \\ \boldsymbol{P}_n^I &= [P_{i,n}^x, P_{i,n}^y, P_{j,n}^x, P_{j,n}^y]^\top \end{aligned}$$

The finite strip stiffness matrix (I-th strip) takes the form

$$\begin{aligned} \mathbf{K}_n &= \alpha_1 D_x \mathbf{K}_{Dx} + \alpha_{2n} D_y \mathbf{K}_{Dy} + \alpha_{3n}(D_1 \mathbf{K}_{D1} + D_2 \mathbf{K}_{D2}) + \\ &+ D_{xy}(\alpha_{2n}\mathbf{K}_1 + \alpha_1 \mathbf{K}_2 - \alpha_{3n}\mathbf{K}_3) \end{aligned} \tag{1.4}$$

where

$$\alpha_1 = \frac{L}{2b} \qquad \alpha_{2n} = \frac{bn^2\pi^2}{12L} \qquad \alpha_{3n} = -\frac{n\pi}{4}$$

$$D_{xy} = G_{xy} \qquad D_x = \frac{E_x}{(1-\nu_x\nu_y)} \qquad D_y = \frac{E_y}{(1-\nu_x\nu_y)}$$

$$D_1 = \frac{\nu_y E_y}{(1-\nu_x\nu_y)} \qquad D_2 = \frac{\nu_x E_x}{(1-\nu_x\nu_y)}$$

E_x, E_y are Young's moduli, ν_x, ν_y – Poisson's ratios, G_{xy} – Kirchhoff's modulus, t – thickness of the slab, $\mathbf{K}_i$ – the following number matrices

$$\mathbf{K}_{Dx} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{K}_{Dy} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

$$\mathbf{K}_{D1} = \begin{bmatrix} 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{K}_{D2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{K}_1 = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{K}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\mathbf{K}_3 = \begin{bmatrix} 0 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

2. The difference equations formulation

Assembling two adjacent elements R and $R+1$ (see Fig. 4), one obtains the equilibrium equations of force amplitudes for each n-th harmonic element (Pawlak *et al.*, 1997)

$$\begin{aligned} P^x_{(r,r-1),n} + P^x_{(r,r+1),n} &= P^x_{r,n} \\ P^y_{(r,r-1),n} + P^y_{(r,r+1),n} &= P^y_{r,n} \end{aligned} \tag{2.1}$$

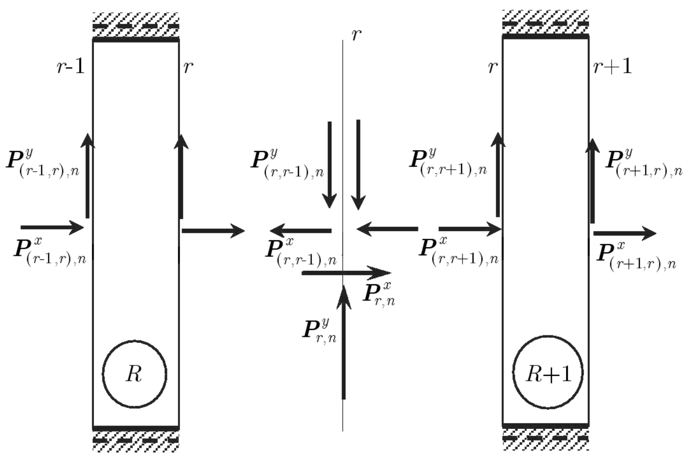

Fig. 4. Forces at the nodal line r

Expression (2.1) can be written for a regular system in the form of difference equations (Pawlak and Rakowski, 1995) equivalent to the FEM matrix formulation

$$\begin{aligned} (\beta_1\varDelta^2 + \beta_0)u_r + \beta_2(E^{-1} - E)v_r &= A_{px}P^x_r \\ \beta_5(E^{-1} - E)u_r + (\beta_3\varDelta^2 + \beta_4)v_r &= A_{py}P^y_r \end{aligned} \tag{2.2}$$

where

$$\beta_0 = 6D_{xy}\alpha_2 \qquad \beta_1 = D_{xy}\alpha_2 - D_x\alpha_1 \qquad \beta_2 = \alpha_3(D_{xy} + D_1)$$

$$\beta_3 = D_y\alpha_2 - D_{xy}\alpha_1 \qquad \beta_4 = 6D_y\alpha_2 \qquad \beta_5 = -\alpha_3(D_{xy} + D_2)$$

$$A_{px} = \sin\frac{n\pi y_{0x}}{L} \qquad A_{py} = \sin\frac{n\pi y_{0y}}{L}$$

$\varDelta^2_r = \varDelta^2 = (E + E^{-1} - 2)$ is the second-order difference operator, $E^n_r = E^n$ is the shifting operator $E^n(f_r) = f_{r+n}$ (Rakowski, 1991), P^x_r and P^y_r are forces acting at the nodal line r (with the co-ordinates y_{0x} and y_{0y}, respectively).

After elimination of the unknown v_r, the equilibrium conditions can be written in the form of one fourth-order difference equation with the unknown u_r (r-th nodal line transverse displacement amplitude for the n-th harmonic)

$$(B_4\Delta^4 + B_2\Delta^2 + B_0)u_r = -A_{px}(\beta_3\Delta^2 + \beta_4)P_r^x + A_{py}\beta_2(E^{-1} - E)P_r^y \tag{2.3}$$

where

$$B_4 = \beta_2\beta_5 - \beta_1\beta_3 \qquad B_2 = 4\beta_2\beta_5 - \beta_0\beta_3 - \beta_1\beta_4$$
$$B_0 = -\beta_0\beta_4$$

Elimination of the unknown u_r leads to the equation with the unknown longitudinal displacement amplitude v_r

$$(B_4\Delta^4 + B_2\Delta^2 + B_0)v_r = -A_{py}(\beta_1\Delta^2 + \beta_0)P_r^y + A_{px}\beta_5(E^{-1} - E)P_r^x \tag{2.4}$$

For a regular infinite plate strip, equations (2.3) and (2.4) are equivalent to the set of infinite number of equilibrium conditions derived using the finite strip methodology (FSM).

3. The fundamental solution

To simplify the calculations, the superposition principle is applied. Static analysis of the structure loaded by the force $P^x = P_r^x\delta_{r,0} = P\delta_{r,0}$ ($P_r^y = 0$, $\delta_{r,0}$ – the Kronecker delta) leads to the fundamental solution for the infinite slab strip (see Fig. 5).

The equilibrium conditions are expressed by means of FEM in the form of a difference equation

$$(B_4\Delta^4 + B_2\Delta^2 + B_0)u_r = -A_{px}(\beta_3\Delta^2 + \beta_4)P^x \tag{3.1}$$

In order to solve difference equilibrium equation (3.1) we use the discrete Fourier transform (Babuška, 1959; Pawlak and Rakowski, 1996) in x-direction (variable r)

$$F[f_r] = \widetilde{f}(\alpha) = \sum_{r=-\infty}^{\infty} f_r \mathrm{e}^{\mathrm{i}r\alpha}$$
$$F^{-1}[\widetilde{f}(\alpha)] = f_r = \frac{1}{2\pi}\int_{-\pi}^{\pi} \widetilde{f}(\alpha)\mathrm{e}^{-\mathrm{i}r\alpha}\, d\alpha \tag{3.2}$$

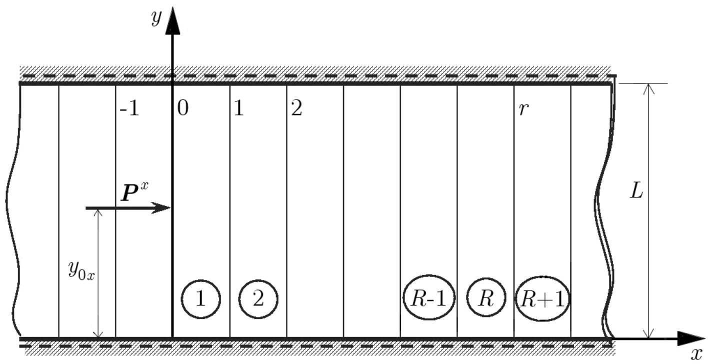

Fig. 5. The infinite strip with a point load

After transformations, one may obtain the solution

$$u_r = \frac{P^x}{\pi} \int\limits_0^{\pi} \frac{(S_1 \cos\alpha + S_2)\cos(r\alpha)}{\cos^2\alpha + B_m \cos\alpha + C_m}\, d\alpha \tag{3.3}$$

where

$$B_m = \frac{B_2 - 4B_4}{2B_4} \qquad C_m = \frac{4B_4 - 2B_2 + B_0}{4B_4}$$

$$S_1 = -\frac{\beta_1 A_{px}}{2B_4} \qquad S_2 = \frac{(2\beta_1 - \beta_0)A_{px}}{4B_4}$$

The function of nodal displacement amplitude (3.3) may be expressed in the form of the following recurrent relation

$$u_r = \frac{P^x}{\pi}[S_1 F_1(r) + S_2 F_2(r)] \tag{3.4}$$

where

$$\begin{aligned} F_1(r) &= 2^{r-1}C(r+1) - \binom{r}{1} 2^{r-3} C(r-1) + \\ &+ \frac{r}{2}\binom{r-3}{1} 2^{r-5} C(r-3) - \frac{r}{3}\binom{r-4}{2} 2^{r-7} C(r-5) + \dots \\ F_2(r) &= 2^{r-1}C(r) - \binom{r}{1} 2^{r-3} C(r-2) + \\ &+ \frac{r}{2}\binom{r-3}{1} 2^{r-5} C(r-4) - \frac{r}{3}\binom{r-4}{2} 2^{r-7} C(r-6) + \dots \end{aligned}$$

The integrals $C(n)$ are analytically calculated coefficients

$$C(n) = \int_0^{\pi} \frac{\cos^n \alpha}{\cos^2 \alpha + B_m \cos \alpha + C_m}\, d\alpha$$

Fundamental solution (3.4) provides the horizontal displacement amplitudes for nodal lines $r \geqslant 0$. When the horizontal force acts, the discrete function u_r has the symmetric form

$$u_{-r} = u_r \tag{3.5}$$

In the considered strip, the vertical amplitudes at nodal line $r = 0$ are equal to zero, $v_0 = 0$. Using equilibrium conditions (2.2), one may determine the vertical displacement amplitude for $r = 1$

$$v_1 = \frac{\beta_1}{\beta_2}(u_1 - u_0) + \frac{\beta_0}{2\beta_2}u_0 - \frac{A_{px}}{2\beta_2}P^x \tag{3.6}$$

and all the following ones for $r > 1$

$$v_r = \frac{\beta_5}{\beta_3}(u_r - u_{r-2}) + 2v_{r-1} - v_{r-2} - \frac{\beta_4}{\beta_3}v_{r-1} \tag{3.7}$$

The vertical displacement functions have the antysymmetric form

$$v_{-r} = -v_r \tag{3.8}$$

The real displacements at the nodal line r (for the co-ordinate y) are in the form of sums

$$u(r, y) = \sum_{n=1}^{N} u_r(n) \sin \frac{n\pi y}{L} \qquad v(r, y) = \sum_{n=1}^{N} v_r(n) \sin \frac{n\pi y}{L} \tag{3.9}$$

where: $u_r(n)$ and $v_r(n)$ are amplitudes obtained from equations (3.4)-(3.8), respectively, N is the number of harmonic elements.

4. Numerical examples

Displacements functions (3.9) yield the fundamental solution for the infinite plane slab simply supported on its opposite edges. The calculations were carried out for the following dimensionless physical parameters: $E_x = 1.0$, $E_y = 0.5$, $\nu_x = 0.25$. The continuous body was divided into a regular set of finite strips. The values of displacements were determined at ten points along the transverse strip dimension L $(y = 0, 1, \ldots, 10)$. The assumption of finite strip width $b = 1.0$ means the discretization ratio $b = L/10$. The results of numerical calculations for the strip subjected to a unit horizontal force acting at the point $y_p = L/2$ are presented in Table 1 and Table 2.

Table 1. Horizontal displacements $u(r, y)$

	$r = 0$	$r = 1$	$r = 2$
$y = 0$	0	0	0
$y = 1$	0.1795	0.1776	0.1597
$y = 2$	0.4590	0.4053	0.3151
$y = 3$	0.9269	0.5868	0.3949
$y = 4$	0.5255	0.4711	0.3795
$y = 5$	0.3155	0.3136	0.2930
$y = 6$	0.2247	0.2209	0.2119
$y = 7$	0.1502	0.1503	0.1474
$y = 8$	0.0960	0.0951	0.0933
$y = 9$	0.0458	0.0458	0.0453
$y = 10$	0	0	0

Table 2. Vertical displacements $v(r, y)$

	$r = 0$	$r = 1$	$r = 2$
$y = 0$	0	−0.1057	−0.1577
$y = 1$	0	−0.0906	−0.1603
$y = 2$	0	−0.1440	−0.1367
$y = 3$	0	−0.0162	−0.0347
$y = 4$	0	0.1016	0.0659
$y = 5$	0	0.0531	0.0832
$y = 6$	0	0.0484	0.0711
$y = 7$	0	0.0301	0.0613
$y = 8$	0	0.0325	0.0540
$y = 9$	0	0.0235	0.0505
$y = 10$	0	0.0288	0.0491

The plot of strip deformation is shown in Fig. 6.

The numerical result, i.e. the obtained field of displacements is treated as the fundamental solution for the considered system. It can be used for solving

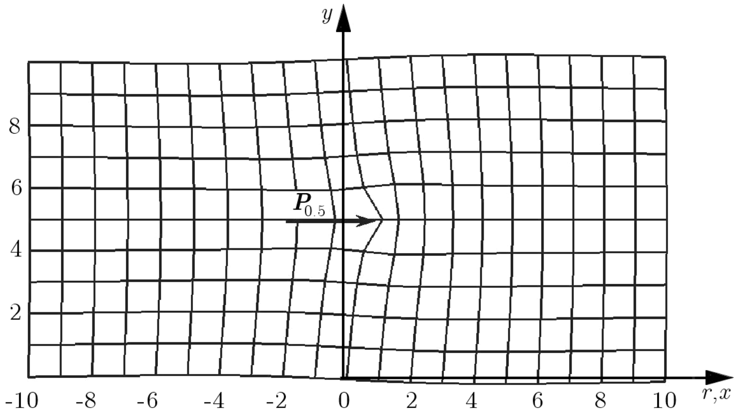

Fig. 6. Deformation of the infinite strip

the static problem of a finite slab in an analogous way as in the boundary element method for continuous systems.

Let us consider an orthotropic, square slab simply supported on two edges and free on the other ones (see Fig. 7). Both slab dimensions are equal to $10b$, the load is distributed in the middle of the slab.

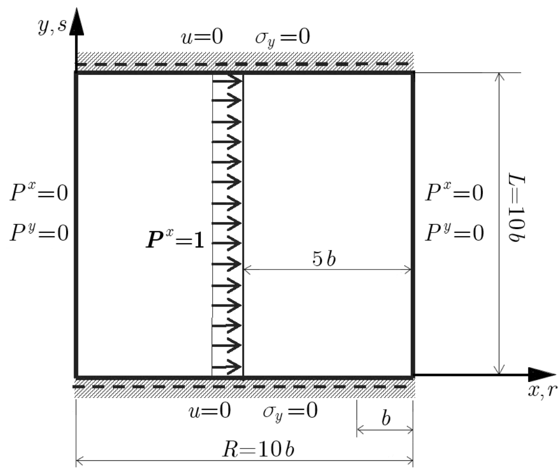

Fig. 7. The finite square slab

We assume that the considered finite slab is a part of the infinite strip. Following the finite strip methodology we approximate the continuous body by the set of identical finite strips of the width b. In order to fulfil the boundary conditions at the nodal lines $r = 0$ and $r = 10$, we introduce an additional loading $X_{i,n}$ (see Fig. 8).

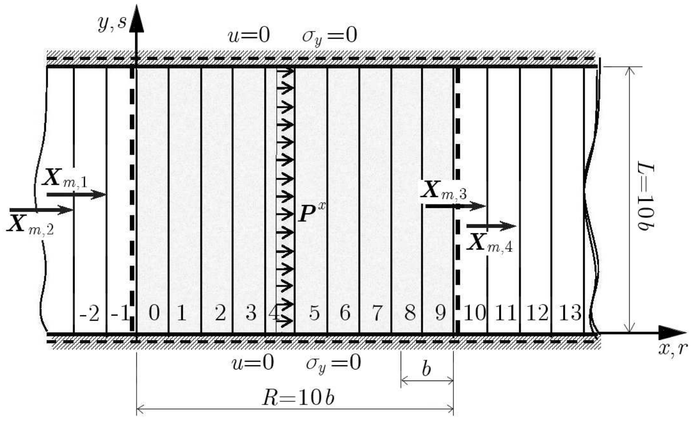

Fig. 8. The strip with additional forces beyond the free edges

The forces $X_{i,n}$ are determined from the boundary equations (indirect boundary element method (Banerjee and Butterfield, 1981)) for free edges of finite slab (see Fig. 7)

$$
\begin{aligned}
&P^x_{0,n}(P_{r,n}, X_{i,n}) = 0 \qquad && P^y_{0,n}(P_{r,n}, X_{i,n}) = 0 \\
&P^x_{10,n}(P_{r,n}, X_{i,n}) = 0 \qquad && P^y_{10,n}(P_{r,n}, X_{i,n}) = 0
\end{aligned}
\tag{4.1}
$$

where n is the number of harmonic elements of sinus series, r is the number of the nodal line with acting forces $(r = 5b)$, i is the number of the additional force $(i = 1, 2, 3, 4)$.

The values of forces $X_{i,n}$ for various numbers n are presented in Table 3.

Table 3. Additional forces $X_{i,n}$

n	$X_{1,n}$	$X_{2,n}$	$X_{3,n}$	$X_{4,n}$
1	18.4906	−18.1107	−18.1107	18.4906
2	0	0	0	0
3	−2.6113	−1.0518	−1.0518	−2.6113
4	0	0	0	0
5	–	–	–	–
6	0	0	0	0
7	0.0001	0.0077	0.0077	0.0001
8	0	0	0	0
9	−0.0007	0.0002	0.0002	−0.0007
10	–	–	–	–

The amplitude of the nodal line r displacement for the n-th harmonic element is found from the expression

$$u_r(n) = \sum_j X_{j,n} U(n, r - x_j) + \sum_p P_{p,n} U(n, r - x_p) \tag{4.2}$$

where $U(n, r)$ is the fundamental solution, the amplitudes are obtained from equations from (3.4)-(3.8) respectively, x_j is the position of the additional force $X_{j,n}$, x_p is the position of the load $P_{p,n}$.

By adding the solutions for all used harmonic elements, one may obtain the total displacement of the finite slab

$$u(r, y) = \sum_n^N u_r(n) \sin \frac{n\pi y}{L} \tag{4.3}$$

where N is the number of used harmonic elements, r, y are the co-ordinates for the point of investigated displacement.

The final results of numerical calculations are symmetric. The horizontal and vertical displacements for one quadrant of the finite strip are presented in Table 4 and Table 5.

Table 4. Horizontal displacements $u(r, y)$

	$r=0$	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$
$y=0$	0.000	0.000	0.000	0.000	0.000	0.000
$y=1$	−0.591	0.802	0.257	0.476	0.469	1.291
$y=2$	−0.106	0.152	0.467	0.836	1.249	1.811
$y=3$	−0.136	0.207	0.605	1.042	1.494	2.065
$y=4$	−0.152	0.239	0.677	1.142	1.608	2.182
$y=5$	−0.156	0.249	0.699	1.171	1.641	2.217

Table 5. Vertical displacements $v(r, y)$

	$r=0$	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$
$y=0$	−0.048	0.183	0.316	0.363	0.268	0.000
$y=1$	−0.044	0.169	0.287	0.317	0.219	0.000
$y=2$	−0.033	0.133	0.217	0.222	0.136	0.000
$y=3$	−0.020	0.086	0.137	0.134	0.078	0.000
$y=4$	−0.009	0.042	0.065	0.062	0.036	0.000
$y=5$	0.000	0.000	0.000	0.000	0.000	0.000

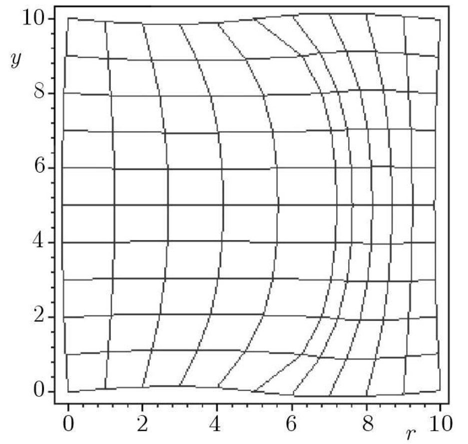

Fig. 9. Deformation of the finite square slab

The plot of finite square slab deformation is shown in Fig. 9.

The analytical closed form of the derived fundamental functions provides the possibility of parametric analysis. The influence of the number N of harmonics used in the calculations on the results accuracy is shown in Fig. 10.

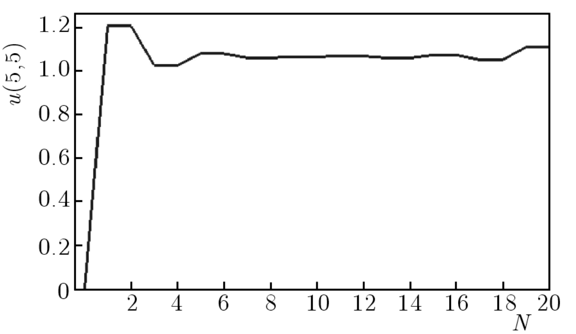

Fig. 10. Accuracy of the results

The relation between the value of the selected displacement and the moduli ratio is shown in Fig. 11.

The next numerical test was carried out for a slab presented in Fig. 7, where the following data were chosen: slab dimensions $L = R = 2.0\,\text{m}$ (the finite strip width $b = 0.2\,\text{m}$), Young's moduli $E_x = E_y = 205\,\text{GPa}$, Poisson's ratios $\nu_x = \nu_y = 0.3$, thickness of the slab $t = 1.0\,\text{cm}$, load distribution in the middle of the slab $P_x = 1.0\,\text{MN/m}$. The result, i.e. the horizontal displacement at the point located in the middle of the slab $(1.0;\ 1.0\,\text{m})$ coincides with the

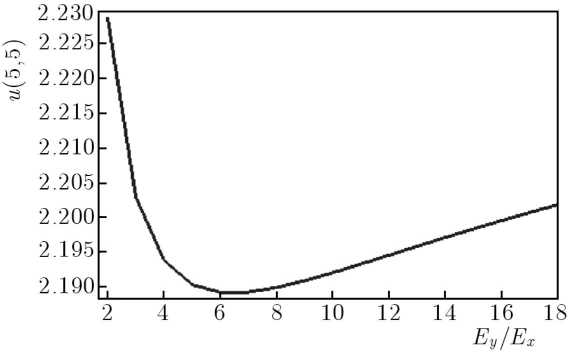

Fig. 11. The influence of moduli ratio on the chosen displacement

one given by the finite element method. The ratio between the displacement obtained here $u_{FSM} = 0.5846$ cm and the displacement found from the finite element method $u_{FEM} = 0.5964$ cm is $u_{FSM}/u_{FEM} = 0.9802$.

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Zastosowanie równań różnicowych w rozwiązaniu problemu statyki nieograniczonej tarczy metodą pasm skończonych

Streszczenie

W pracy zaprezentowano analityczne rozwiązanie problemu statyki nieograniczonej tarczy metodą pasm skończonych. Tarcza swobodnie podparta na przeciwległych krawędziach jest rozwiązywana jako układ dyskretny. Ciągła struktura jest aproksymowana regularną siatką składającą się z identycznych pasm skończonych. Regularny podział pozwala na wyprowadzenie funkcji fundamentalnych dla dwuwymiarowego układu dyskretnego w analitycznej, zamkniętej formie. Warunki równowagi zostały wyprowadzone zgodnie ze sformułowaniem metody elementów skończonych. Układ równań równowagi składający się z nieskończonej liczby równań został zastąpiony jednym, równoważnym równaniem różnicowym. Rozwiązanie tego równania jest funkcją fundamentalną dla rozpatrywanej tarczy.

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