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ANALYSIS OF THE PARAMETERS OF A SPHERICAL STRESS WAVE EXPANDING IN A LINEAR ISOTROPIC ELASTIC MEDIUM

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The vast qualitative and quantitative analysis of the characteristics of a spherical stress wave expanding in a linear-elastic medium was made. The wave was generated by pressure $p_0 = \text{const}$ suddenly created in a spherical cavity of initial radius r_0. From the analytical form of the solution to the problem it results that displacement and stresses decrease approximately in inverse proportion to the square and cube of the distance from cavity center. It was found that the cavity surface and successive spherical sections of the compressible medium move in the course of time with damped vibrating motion around their static positions. The remaining characteristics of the wave behave analogously. Material compressibility, represented by Poisson's ratio ν in this paper, has the main influence on vibration damping. The increase of the parameter ν over 0.4 causes an intense decrease of the damping, and in the limiting case $\nu = 0.5$, i.e. in the incompressible material the damping vanishes completely. The incompressible medium vibrates like a conservative mechanical system of one degree of freedom.

Key words: expanding spherical stress wave, isotropic elastic medium, dynamic load

1. Introduction

The problem of propagation of the spherical stress wave expanding in a compressible isotropic elastic medium was solved in Włodarczyk and Zielenkiewicz (2009). Linear elasticity theory was used (Achenbach, 1975; Nowacki, 1970), i.e.

$$
\begin{aligned}
&\varepsilon_r = \frac{\partial u}{\partial r} && \varepsilon_\varphi = \varepsilon_\theta = \frac{u}{r} \qquad v = \frac{\partial u}{\partial t} \\
&\sigma_r = (2\mu + \lambda)\frac{\partial u}{\partial r} + 2\lambda\frac{u}{r} && \sigma_\varphi = 2(\mu + \lambda)\frac{u}{r} + \lambda\frac{\partial u}{\partial r} \\
&\frac{\rho_0}{\rho} = 1 + \frac{\partial u}{\partial r} + 2\frac{u}{r}
\end{aligned}
\tag{1.1}
$$

where

$$
\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \qquad \mu = \frac{E}{2(1+\nu)}
$$

u	–	radial displacement of elements of the medium
v	–	radial velocity of the elements
σ_r	–	radial stress
$\sigma_\varphi = \sigma_\theta$	–	circumferential (tangential) stresses
ε_r	–	radial strain
$\varepsilon_\varphi = \varepsilon_\theta$	–	circumferential (tangential) strains
ρ_0, ρ	–	medium densities: initial and disturbed
E	–	Young's modulus
ν	–	Poisson's ratio
r, t	–	Lagrangian coordinates.

The wave was generated by pressure $p(t)$ dynamically created inside the spherical cavity of initial radius r_0 (Fig. 1).

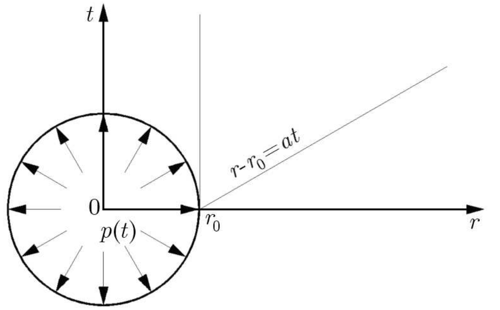

Fig. 1. A scheme of the boundary value problem

All wave parameters were determined with the use of a scalar potential $\varphi(x)$ and its derivatives, namely

$$
\begin{aligned}
& u(r,t) = \frac{\varphi'(r-r_0-at)}{r} - \frac{\varphi(r-r_0-at)}{r^2} \\
& \varepsilon_r = \frac{\varphi''}{r} - 2\frac{\varphi'}{r^2} + 2\frac{\varphi}{r^3} \qquad \varepsilon_\varphi = \frac{\varphi'}{r^2} - \frac{\varphi}{r^3} \qquad \frac{\rho_0}{\rho} = 1 + \frac{\varphi''}{r} \\
& \sigma_r = (2\mu+\lambda)\frac{\varphi''}{r} - 4\mu\frac{\varphi'}{r^2} + 4\mu\frac{\varphi}{r^3} \\
& \sigma_\varphi = \lambda\frac{\varphi''}{r} + 2\mu\frac{\varphi'}{r^2} - 2\mu\frac{\varphi}{r^3} \\
& \sigma_\varphi - \sigma_r = -2\mu\frac{\varphi''}{r} + 6\mu\frac{\varphi'}{r^2} - 6\mu\frac{\varphi}{r^3}
\end{aligned} \tag{1.2}
$$

where a denotes the velocity of elastic stress wave propagation

$$
a = \sqrt{\frac{1-\nu}{(1+\nu)(1-2\nu)}}\, a_0 \qquad\qquad a_0 = \sqrt{\frac{E}{\rho_0}} \tag{1.3}
$$

The apostrophes at the symbol φ denote derivatives of this function with respect to its argument.

The potential φ for an arbitrary pressure $p(t)$ has the following form

$$
\varphi(x) = -\frac{(1+\nu)\sqrt{1-2\nu}}{E} r_0^2 \int\limits_0^x p\Big(\frac{y-x_0}{a}\Big) \mathrm{e}^{hy} \sin\omega y \, dy \tag{1.4}
$$

where

$$
h = \frac{1-2\nu}{1-\nu}\frac{1}{r_0} \geqslant 0 \qquad\qquad \omega = \frac{\sqrt{1-2\nu}}{(1-\nu)r_0} \tag{1.5}
$$

For a constant pressure suddenly created in the spherical cavity, i.e. $p(t) \equiv p_0 = \text{const}$, from expression (1.4), we obtain

$$
\begin{aligned}
& \varphi(x) = -\frac{1+\nu}{2} r_0^3 \frac{p_0}{E}\Big[1 + \mathrm{e}^{hx}(\sqrt{1-2\nu}\sin\omega x - \cos\omega x)\Big] \\
& \varphi'(x) = -(1+\nu)\sqrt{1-2\nu} r_0^2 \frac{p_0}{E} \mathrm{e}^{hx}\sin\omega x \\
& \varphi''(x) = -\frac{(1+\nu)(1-2\nu)}{1-\nu} r_0 \frac{p_0}{E} \mathrm{e}^{hx}(\sqrt{1-2\nu}\sin\omega x + \cos\omega x)
\end{aligned} \tag{1.6}
$$

where

$$
\begin{aligned}
& \omega x = \frac{\sqrt{1-2\nu}}{1-\nu}\Big(\frac{r}{r_0} - 1 - \sqrt{\frac{1-\nu}{(1+\nu)(1-2\nu)}}\frac{a_0 t}{r_0}\Big) \\
& hx = \frac{1-2\nu}{1-\nu}\Big(\frac{r}{r_0} - 1 - \sqrt{\frac{1-\nu}{(1+\nu)(1-2\nu)}}\frac{a_0 t}{r_0}\Big)
\end{aligned} \tag{1.7}
$$

The static parameters of the problem, generated by the pressure p_0 statically created on the inside of the cavity, can be determined by the following formulae

$$
\begin{aligned}
& u_s(r) = \frac{1+\nu}{2}\frac{p_0}{E} r_0 \Big(\frac{r_0}{r}\Big)^2 \\
& \varepsilon_{rs}(r) = -(1+\nu)\frac{p_0}{E}\Big(\frac{r_0}{r}\Big)^3 \qquad && \varepsilon_{\varphi s}(r) = \frac{1+\nu}{2}\frac{p_0}{E}\Big(\frac{r_0}{r}\Big)^3 \\
& \sigma_{rs}(r) = -p_0\Big(\frac{r_0}{r}\Big)^3 \qquad && \sigma_{\varphi s}(r) = \frac{p_0}{2}\Big(\frac{r_0}{r}\Big)^3
\end{aligned}
\tag{1.8}
$$

In order to simplify the quantitative analysis of the particular parameters of the expanding stress wave, the following dimensionless quantities have been introduced

$$
\begin{array}{lll}
\xi = \dfrac{r}{r_0} & \eta = \dfrac{a_0 t}{r_0} & U = \dfrac{u}{r_0} \\
U_s = \dfrac{u_s}{r_0} & V = \dfrac{v}{a_0} & P = \frac{p_0}{E} \\
R = \dfrac{\rho}{\rho_0} & S_r = \dfrac{\sigma_r}{p_0} & S_{rs} = \dfrac{\sigma_{rs}}{p_0} \\
S_\varphi = \dfrac{\sigma_\varphi}{p_0} & S_{\varphi s} = \dfrac{\sigma_{\varphi s}}{p_0} & S_z = \dfrac{\sigma_\varphi - \sigma_r}{p_0} \\
S_0 = \dfrac{\sigma_0}{p_0} & S_{zs} = \dfrac{\sigma_{\varphi s} - \sigma_{rs}}{p_0} &
\end{array}
\tag{1.9}
$$

The dimensionless independent variables ξ and η are contained within the following intervals

$$
1 \leqslant \xi \leqslant \infty \qquad\qquad \eta \geqslant \sqrt{\frac{(1+\nu)(1-2\nu)}{1-\nu}}(\xi - 1)
\tag{1.10}
$$

Using expressions (1.2) and (1.6), the parameters of the expanding stress wave generated in the linear elastic medium by pressure $p_0 = \mathrm{const}$ suddenly created on the inside of the spherical cavity, can be found with the use of dimensionless quantities (1.9) by means of the following expressions

$$
\begin{aligned}
U(\xi,\eta) &= \frac{1+\nu}{2}\frac{P}{\xi^2}\{1-[\sqrt{1-2\nu}(2\xi-1)\sin\omega x+\cos\omega x]\mathrm{e}^{hx}\}\\
V(\xi,\eta) &= \frac{P}{\xi}\Big\{\Big[(1-2\nu)\sqrt{\frac{1+\nu}{1-\nu}}-\frac{\sqrt{1-\nu^2}}{\xi}\Big]\sin\omega x+\\
&\quad+\sqrt{\frac{(1+\nu)(1-2\nu)}{1-\nu}}\cos\omega x\Big\}\mathrm{e}^{hx}\\
\varepsilon_r(\xi,\eta) &= -(1+\nu)P\Big\{\frac{1}{\xi^3}+\Big[\sqrt{1-2\nu}\Big(\frac{1-2\nu}{1-\nu}\frac{1}{\xi}-\frac{2}{\xi^2}+\frac{1}{\xi^3}\Big)\sin\omega x+\\
&\quad-\Big(\frac{1}{\xi^3}-\frac{1-2\nu}{1-\nu}\frac{1}{\xi}\Big)\cos\omega x\Big]\mathrm{e}^{hx}\Big\} \qquad (1.11)\\
\varepsilon_\varphi(\xi,\eta) &= \frac{1+\nu}{2}\frac{P}{\xi^3}\{1-[\sqrt{1-2\nu}(2\xi-1)\sin\omega x+\cos\omega x]\mathrm{e}^{hx}\}\\
R(\xi,\eta) &= [1+\varepsilon_r(\xi,\eta)+2\varepsilon_\varphi(\xi,\eta)]^{-1}\\
S_r(\xi,\eta) &= -\frac{1}{\xi^3}\{1+(\xi-1)[\sqrt{1-2\nu}(\xi-1)\sin\omega x+(\xi+1)\cos\omega x]\mathrm{e}^{hx}\}\\
S_\varphi(\xi,\eta) &= \frac{1}{2\xi^3}\Big\{1+\Big[\sqrt{1-2\nu}\Big(-\frac{2\nu}{1-\nu}\xi^2-2\xi+1\Big)\sin\omega x+\\
&\quad-\Big(\frac{2\nu}{1-\nu}\xi^2+1\Big)\cos\omega x\Big]\mathrm{e}^{hx}\Big\}
\end{aligned}
$$

and

$$
\begin{aligned}
S_z(\xi,\eta) &= S_\varphi(\xi,\eta)-S_r(\xi,\eta)=\\
&=\frac{1}{\xi^3}\Big\{\frac{3}{2}+\Big[\sqrt{1-2\nu}\Big(\frac{1-2\nu}{1-\nu}\xi^2-3\xi+\frac{3}{2}\Big)\sin\omega x+\\
&\quad+\Big(\frac{1-2\nu}{1-\nu}\xi^2-\frac{3}{2}\Big)\cos\omega x\Big]\mathrm{e}^{hx}\Big\}
\end{aligned} \qquad (1.12)
$$

where

$$
\begin{aligned}
\omega x &= \frac{\sqrt{1-2\nu}}{1-\nu}(\xi-1)-\frac{1}{\sqrt{1-\nu^2}}\eta\\
hx &= \frac{1-2\nu}{1-\nu}(\xi-1)-\sqrt{\frac{1-2\nu}{1-\nu^2}}\eta
\end{aligned} \qquad (1.13)
$$

According to (1.8) and (1.9), the dimensionless values of static parameters can be expressed with the following formulae

$$\begin{aligned} &U_s(\xi) = \frac{1+\nu}{2} P\xi^{-2} && \varepsilon_{rs}(\xi) = -(1+\nu)P\xi^{-3} \\ &\varepsilon_{\varphi s}(\xi) = \frac{1+\nu}{2} P\xi^{-3} && S_{rs}(\xi) = -\xi^{-3} \\ &S_{\varphi s}(\xi) = \frac{1}{2}\xi^{-3} && S_{\varphi s}(\xi) - S_{rs}(\xi) = \frac{3}{2}\xi^{-3} \end{aligned} \tag{1.14}$$

The closed analytical formulae presented above were derived in Włodarczyk and Zielenkiewicz (2009). They are the basis of quantitative analysis of parameters of the spherical stress wave expanding in the linear isotropic elastic medium presented below.

From the introductory analysis of the quoted formulae, it follows that the dynamic values of mechanical parameters of the expanding stress wave generated by constant pressure p_0 suddenly created in the spherical cavity in the linear elastic medium, intensively decrease in space with an increase of distance from the system center. On the other hand, in particular spherical sections of the medium, the parameters oscillate around their static values. In a compressible medium, these oscillations decay in the course of time. This results from the spherical divergence of the expanding stress wave. The decay of oscillations is caused by the transport of mechanical energy by the stress wave propagating to successively disturbed regions of the medium.

Now we will perform the detailed analysis of wave parameters. The medium displacement will be considered first.

2. Analysis of the displacement for an infinite pressure impulse of the intensity $p_0 = \text{const}$

According to the general remarks presented above, the maximum displacement values represented in the dimensionless form by function $U(\xi,\eta)$ occur on the cavity surface, i.e. for $\xi = 1$. In this case, expression $(1.11)_1$ can be reduced to the form

$$U(1,\eta) = \frac{1+\nu}{2} P\Big[1+\Big(\sqrt{1-2\nu}\sin\frac{\eta}{\sqrt{1-\nu^2}} - \cos\frac{\eta}{\sqrt{1-2\nu}}\Big)\exp\Big(-\sqrt{\frac{1-2\nu}{1-\nu^2}}\eta\Big)\Big] \tag{2.1}$$

From expression (2.1), it follows that function $U(1,\eta)$ oscillates with damped motion versus dimensionless time η, and has relative extrema. The absolute maximum occurs at $\eta = \eta_e$ determined by the first positive solution to the trigonometric equation, namely

$$\tan\Big(\pi - \frac{\eta_e}{\sqrt{1-\nu^2}}\Big) = \frac{\sqrt{1-2\nu}}{\nu} \tag{2.2}$$

As can be directly seen from relations (2.1) and (2.2), for every value of Poisson's ratio ν, the corresponding maximum displacement of cavity surface $U_{max} = U(1,\eta_e)$ can be found. Therefore, alike as in oscillating mechanical systems of one degree of freedom, the dynamic coefficient of loading characterising an expanding stress wave can be introduced, namely

$$\varPsi(\nu) = \frac{U(1,\eta_e)}{U_s} \tag{2.3}$$

Values of the coefficient $\varPsi(\nu)$ for selected Poisson's ratios ν, are presented in Table 1 and their course versus ν is depicted in Fig. 2.

Table 1. Values of the dynamic coefficient of loading, $\varPsi(\nu)$

ν	0.1	0.2	0.3	0.4	0.49	0.5
η_e	1.67	1.79	1.92	2.11	2.49	2.72
$U(1,\eta_e)$	0.672	0.746	0.832	0.950	1.242	1.500
U_s	0.55	0.60	0.65	0.70	0.745	0.75
$\varPsi(\nu)$	1.222	1.244	1.280	1.357	1.667	2.000

As can be seen, the maximum effect of the dynamic load occurs in the incompressible medium, i.e. for $\nu = 0.5$. The medium compressibility intensively cushions the effect of dynamic load, which can be seen in Fig. 2.

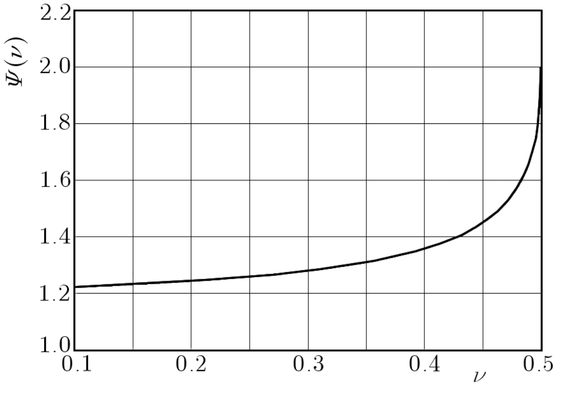

Fig. 2. Variation of coefficient $\varPsi(\nu)$ versus Poisson's ratio ν

The exemplary variations of relative displacement (U/P) of the cavity surface $(\xi = 1)$ versus $\eta = a_0 t/r_0$ for several values of Poisson's ratio ν, are depicted in Fig. 3. As it turns out, the parameter ν, which is the measure of medium compressibility, has the significant influence on the course of quantity $U(1,\eta)/P$ versus η.

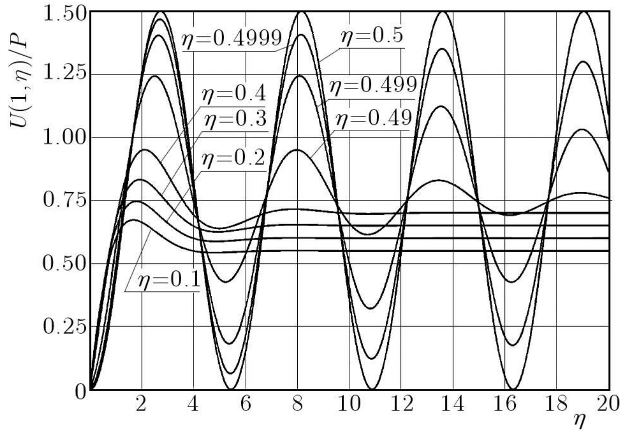

Fig. 3. Variation of relative displacement (U/P) of cavity surface $(\xi = 1)$ versus dimensionless time η for selected values of Poisson's ratio ν

We can mark out two ranges of ν in which the vibration of cavity surface is damped in a different manner. Thus, Poisson's ratio in the range below about 0.4 (media compressibility increase) causes an intense decay of the cavity surface oscillations. For this values of ν, the displacement of cavity surface approaches its static value, i.e. $(U_s/P) = (1+\nu)/2$ already in the first cycle of vibration (Fig. 3). On the other hand, in the range $0.4 < \nu < 0.5$, that is in quasi-compressible media, the damping of vibrations is very low. In the limiting case, $\nu = 0.5$, i.e. in the incompressible material the damping vanishes completely and the cavity surface oscillates harmonically around its static position with the constant amplitude $(U/P) = 0.75$ (Fig. 3).

Note the abnormal behaviour of the media in the range $0.4 < \nu \leqslant 0.5$. In this case, insignificant increments $\Delta\nu$ cause a considerable increase in the vibration amplitude of the cavity surface (Fig. 4). For example, for the increment $\Delta\nu = 0.5 - 0.4 = 0.1$, the maximum relative displacement increment is $(\Delta U/P) = 1.5 - 0.7 = 0.8$.

The above-presented graphic analysis of the ratio $U(1,\eta)/P$ concerns the movement of spherical cavity surface $(\xi = 1)$. The particular spherical sections for $\xi > 1$ oscillate analogously with adequately smaller values resulting from the spatial divergence of the stress wave (Fig. 5). From this graph, it directly

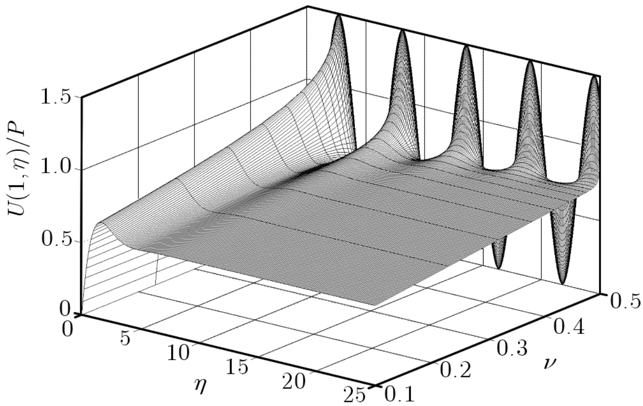

Fig. 4. Influence of parameter ν on variation of function $U(1,\eta)/P$

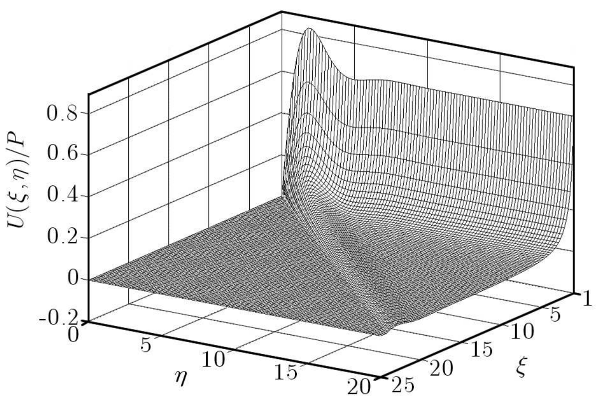

Fig. 5. Spatial graph of function $U(\xi,\eta)/P$ for $\nu = 0.3$

follows that the maximum displacements of particular sections for $\nu < 0.5$ occur in the neighbourhood of the expanding wave front, i.e. near the line

$$\xi - 1 = \sqrt{\frac{1-\nu}{(1+\nu)(1-2\nu)}}\eta$$

3. Analysis of the stress field for an infinite pressure impulse of the intensity $p_0 = \text{const}$

In real media (metals, rocks and the like), the limit of elasticity is always finite. The solution obtained for a studied problem is valid only in the elastic range. From this fact results the limitation of the maximum value of pressure created inside the cavity, i.e. $p_0 \leqslant p_{max}$. If $p_0 > p_{max}$, then in the direct neighbourhood of cavity surface the elastoplastic strains occur in elastoplastic metals or cracks in brittle media (cast iron, rocks). In this range of pressure, the solution quoted in this paper loses its physical sense. Bearing in mind this limitation, we will perform a thorough analysis of the stress field in the studied medium.

As it is known, plastic strains in metals are caused by the components of stress deviator. Therefore, it can be assumed that the condition of material plastic flow depends only on the difference of stresses $\sigma_\varphi - \sigma_r$. Indeed, the expression $(\sigma_\varphi - \sigma_r)/2$ determines the maximum value of tangential stress. So, according to Tresca's plasticity condition, and in the case of spherical symmetry – also Huber-Mises-Hencky's condition, we have

$$\sigma_z = \sigma_\varphi - \sigma_r = \sigma_0 \tag{3.1}$$

where σ_0 is the dynamic yield point obtained in the tension test of a given material.

According to the above-mentioned remarks, in further considerations we will concentrate the main effort on the analysis of relative reduced stress, i.e. $S_z = \sigma_z/p_0 = (\sigma_\varphi - \sigma_r)/p_0$.

As was mentioned, function $S_z(\xi, \eta)$ reaches its maximum on the cavity surface, i.e. for $\xi = 1$. According to expressions (1.12) and (1.13) for $\xi = 1$, function $S_z(\xi, \eta)$ can be reduced to the form

$$S_z(1,\eta) = \frac{3}{2} + \frac{1+\nu}{2(1-\nu)}\Big[\sqrt{1-2\nu}\sin\frac{\eta}{\sqrt{1-\nu^2}} - \cos\frac{\eta}{\sqrt{1-\nu^2}}\Big]\exp\Big(-\sqrt{\frac{1-2\nu}{1-\nu^2}}\eta\Big) \tag{3.2}$$

From comparison of expressions (2.1) and (3.2), it follows that functions $U(1, \eta)$ and $S_z(1, \eta)$ have analogous forms, so $S_z(1, \eta)$ reaches its absolute maximum for $\eta = \eta_e$, too. The value of η_e is determined by trigonometric equation (2.2). This means that despite rapid pressure rise in the spherical cavity, the reduced stress $S_z(1, \eta)$ monotonically increases to its maximum value and reaches it after the finite time $t_e = (r_0/a_0)\eta_e \neq 0$. As can be seen, some "inertia" occurs in the increasing of stress $S_z(1, \eta)$ to its maximum value in comparison with the pressure p_0 (Fig. 6).

The variation of quantity $S_z(1, \eta)$ versus η for selected values of Poisson's ratio ν is shown in Fig. 6. From comparison of graphs depicted in Fig. 3 and Fig. 6, it follows that the courses of functions $U(1, \eta)$ and $S_z(1, \eta)$ are similar. Analogously to $U(1, \eta)$, the significant influence of medium compres-

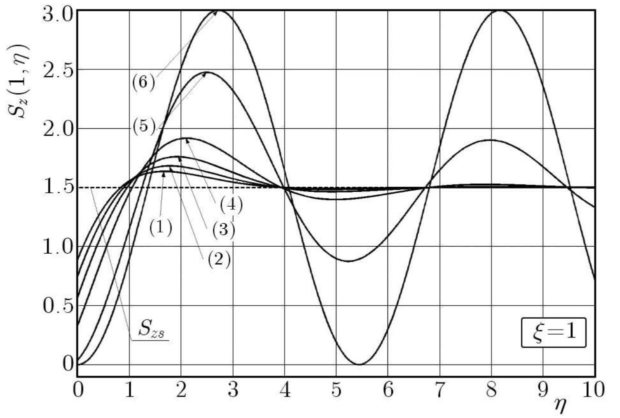

Fig. 6. Variation of relative reduced stress $S_z(1,\eta)$ versus dimensionless time η for selected values of Poisson's ratio ν; (1) $S_z = 1.64$, $\eta_e = 1.67$, (2) $S_z = 1.68$, $\eta_e = 1.79$, (3) $S_z = 1.76$, $\eta_e = 1.92$, (4) $S_z = 1.92$, $\eta_e = 2.11$, (5) $S_z = 2.47$, $\eta_e = 2.49$, (6) $S_z = 3.00$, $\eta_e = 2.72$, S_{zs} – static value

sibility (ν parameter) on the pulsating variation of $S_z(1,\eta)$ versus η is noteworthy. In the compressible media for $\nu < 0.4$, the oscillation of function $S_z(1,\eta)$ is intensively damped to its static value $S_{zs} = 1.5$. For example, for $\nu < 0.3$ already after time $t = 4r_0/a_0$ we have $S_z(1.4) \approx S_{zs} = 1.5$. On the contrary, for $\nu = 0.5$, i.e. in the incompressible medium, the damping vanishes completely and stress $S_z(1,\eta)$ harmonically pulsates around its static value $S_{zs} = 1.5$ with a constant amplitude equal to 1.5. The abnormal influence of ν on the course of function $S_z(1,\eta)$ is depicted spatially in Fig. 7.

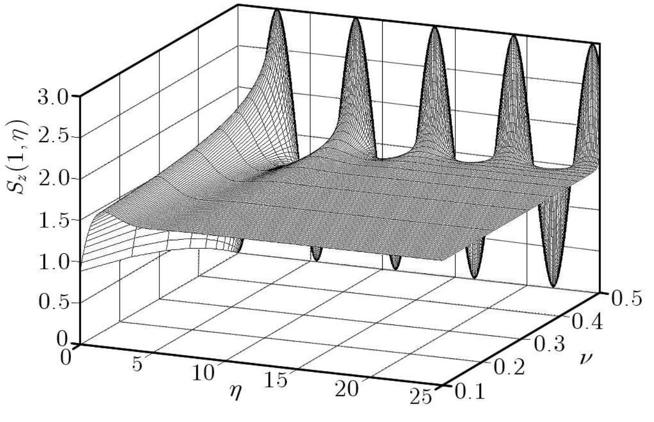

Fig. 7. Influence of parameter ν on variation of function $S_z(1,\eta)$

In turn, the variation of stress $S_z(\xi,\eta_i)$ versus ξ for three selected values of η and $\nu = 0.3$ is shown in Fig. 8. The intense decrease of stress S_z with an increase in the distance from cavity center is apparent.

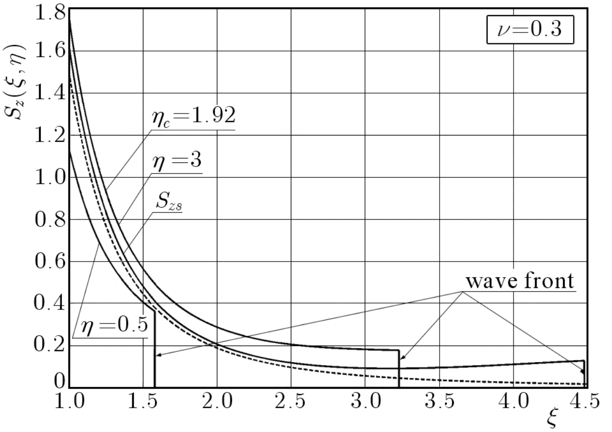

Fig. 8. Variation of relative reduced stress $S_z(\xi,\eta)$ versus dimensionless radius ξ for three selected values of dimensionless time η and for $\nu = 0.3$; S_{zs} – static value

Full spatial graphs of function $S_z(\xi,\eta)$ for $\nu = 0.3$ and $\nu = 0.49$ are depicted in Fig. 9 and Fig. 10. The significant influence of ν on the values of stress $S_z(\xi,\eta)$ is noticeable there.

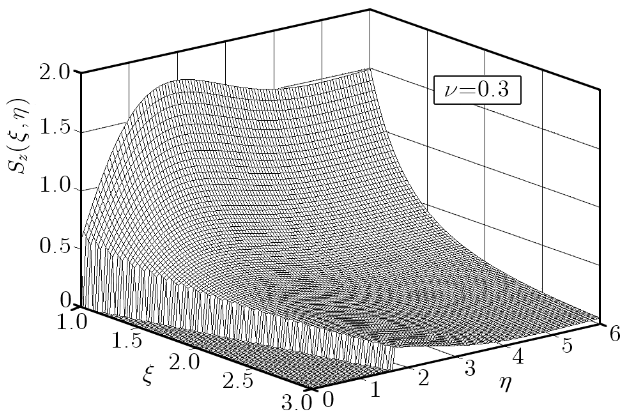

Fig. 9. Spatial graph of function $S_z(\xi,\eta)$ for $\nu = 0.3$

From the analysis of the variations of stress S_z, it results that in the direct neighbourhood of cavity surface, with a pressure p_0 high enough, the medium yield point will be exceeded. In this range, the solution described in this paper

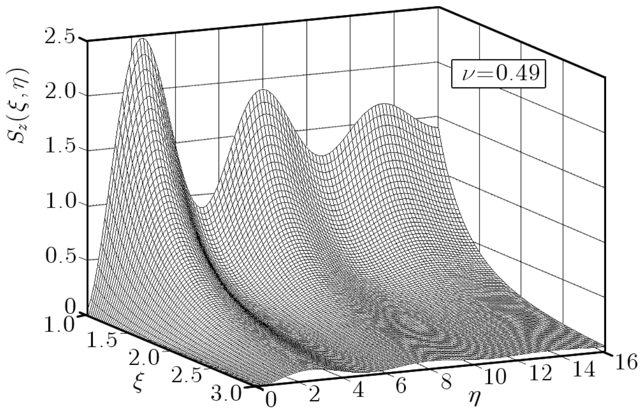

Fig. 10. Spatial graph of function $S_z(\xi,\eta)$ for $\nu = 0.49$

loses its physical sense. The maximum pressure by which the dynamic yield point is not exceeded in the medium layer directly surrounding the cavity can be determined from the following expression

$$p_{max} = \frac{\sigma_0}{S_z(1,\eta_e)} \qquad \text{or} \qquad \frac{p_{max}}{\sigma_0} = \frac{1}{S_z(1,\eta_e)} \tag{3.3}$$

The values of p_{max}/σ_0 ratio for selected values of parameter ν, are presented in Table 2.

Table 2. Values of p_{max}/σ_0 ratio for selected ν

ν	0.1	0.2	0.3	0.4	0.49	0.5
η_e	1.67	1.79	1.92	2.11	2.49	2.72
$S_z(1,\eta_e)$	1.64	1.68	1.76	1.92	2.47	3.00
p_{max}/σ_0	0.61	0.60	0.57	0.52	0.40	0.33

The variation of relative circumferential stress $S_\varphi(1,\eta) = \sigma_\varphi/p_0$ on the cavity surface versus η for selected values of the parameter ν is shown in Fig. 11. The courses are analogous to function $S_z(1,\eta)$ (Fig. 6). Note the fact that in the initial period of cavity expansion in the compressible medium ($\nu < 0.5$) the stress $S_\varphi(1,\eta)$ is negative. The negative value of stress S_φ propagates at the front of the wave and in its direct neighbourhood, which can be seen in Fig. 12. It is the dynamic effect of inertial action of the medium in the direct neighbourhood of strong discontinuity wave front.

The course of relative radial stress $S_r(\xi,\eta)$ versus η for selected values of ξ and $\nu = 0.3$ is presented in Fig. 13. Stress $S_r(\xi,\eta)$ reaches its maximum

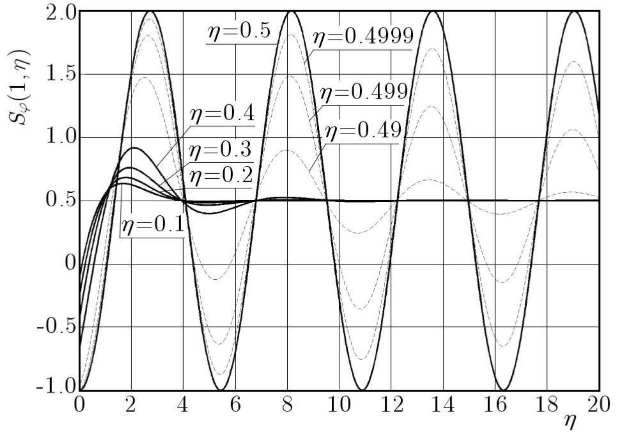

Fig. 11. Variation of relative circumferential stress $S_\varphi(1,\eta)$ versus dimensionless time η for selected values of Poisson's ratio ν

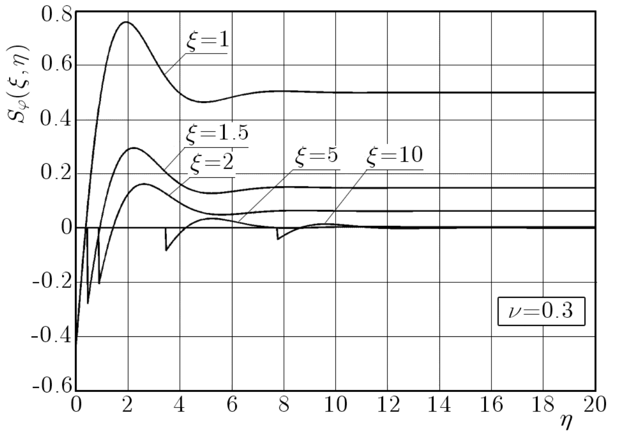

Fig. 12. Variation of relative circumferential stress $S_\varphi(\xi,\eta)$ versus dimensionless time η for selected values of dimensionless radius for $\xi =$ const and $\nu = 0.3$

absolute values on the wave front. This value intensively decreases with the increase of the distance from cavity center. Note the fact that in the medium sections sufficiently distant from the cavity surface, behind the front of wave, regions occur in which the medium is radially stretched $(S_r(\xi,\eta) > 0)$. It is the result of vibrating movement of the compressible medium.

At the end of the presented analysis, it can be stated that for the entire range of the parameter ν the material density varies insignificantly (Fig. 14). The maximum increments ΔR do not exceed a few tens per cent.

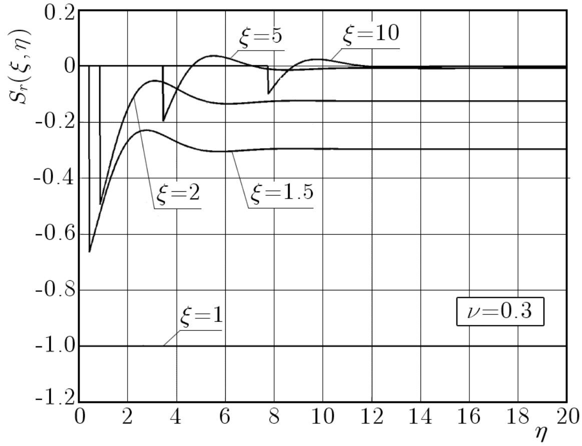

Fig. 13. Variation of relative radial stress $S_r(\xi,\eta)$ versus dimensionless time η for selected values of dimensionless radius for $\xi =$ const and $\nu = 0.3$

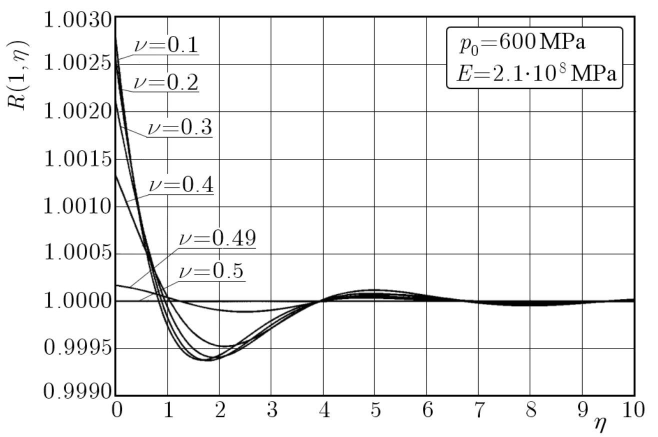

Fig. 14. Variation of relative medium density $R(1,\eta)$ versus dimensionless time η for $\xi = 1$ and selected values of Poisson's ratio ν

4. Final conclusions

From the analysis of the studied problem, the following conclusions can be drawn:

- From the form of the obtained analytical solution, it results that displacements and stresses decrease approximately in inverse proportion to the square and cube of the distance from the cavity center. Therefore, the maximum absolute values of stress wave parameters occur on the cavity surface. It is the result of the spherical divergence of the expanding wave.

- The cavity surface and successive spherical sections of the compressible medium move in the course of time with damped vibrating motion around their static positions caused by the pressure p_0 created statically inside the cavity. The remaining characteristics of the wave behave analogously.
- Material compressibility, represented in this paper by Poisson's ratio ν, has the main influence on the vibration damping. Poisson's ratio in the range below about 0.4 (media compressibility increase) causes an intense decay of the wave oscillations. For such values of ν, the wave parameters approach their static values already in the first cycle of vibration (Fig. 3 and Fig. 6). On the other hand, in the range $0.4 < \nu < 0.5$ (quasi-compressible media) the damping of vibrations is very low. In the limiting case $\nu = 0.5$, i.e. in the incompressible material, the damping vanishes completely and the cavity surface pulsates harmonically around its static position with the constant amplitude $(U/P) = 0.75$ (Fig. 3).
- The velocity of propagating spherical stress wave in a compressible linearly elastic medium is an increasing function of Poisson's ratio ν

$$a = \sqrt{\frac{(1-\nu)E}{(1+\nu)(1-2\nu)\rho_0}}$$

This velocity determines the rate of energy transfer to successive layers of the medium from the loaded cavity surface. In the limiting case, the medium becomes incompressible and vibrates like a conservative mechanical system with one degree of freedom and the natural frequency

$$\omega_0 = \frac{2}{r_0}\sqrt{\frac{E}{3\rho_0}}$$

Note that the frequency varies in inverse proportion to the radius of cavity.
- In the incompressible medium $(\nu = 0.5)$, the parameters of the problem are determined by the following formulae

$$U(\xi,\eta) = \frac{3}{4}P\frac{1}{\xi^2}\Big(1 - \cos\frac{2}{\sqrt{3}}\eta\Big)$$

$$V(\xi,\eta) = \frac{\sqrt{3}}{2}P\frac{1}{\xi^2}\sin\frac{2}{\sqrt{3}}\eta$$

$$\varepsilon_r(\xi,\eta) = -\frac{3}{2}P\frac{1}{\xi^3}\Big(1 - \cos\frac{2}{\sqrt{3}}\eta\Big)$$

$$\begin{aligned}
&\varepsilon_\varphi(\xi,\eta) = \frac{3}{4}P\frac{1}{\xi^3}\Big(1-\cos\frac{2}{\sqrt{3}}\eta\Big)\\
&R(\xi,\eta) = 1\\
&S_r(\xi,\eta) = -\frac{1}{\xi^3}\Big[1+(\xi^2-1)\cos\frac{2}{\sqrt{3}}\eta\Big]\\
&S_\varphi(\xi,\eta) = \frac{1}{2\xi^3}\Big[1-(2\xi^2+1)\cos\frac{2}{\sqrt{3}}\eta\Big]\\
&S_z(\xi,\eta) = S_\varphi(\xi,\eta) - S_r(\xi,\eta) = \frac{3}{2\xi^3}\Big(1-\cos\frac{2}{\sqrt{3}}\eta\Big)
\end{aligned}$$

- The circumferential stress $S_\varphi(1,\eta)$ on the cavity surface during its movement increases from the initial negative value. From this it follows that the maximum value of reduced stress $S_{z\,max}$ depends on the pressure pulse duration. We will consider this problem in a separate paper.
- The results of analyses presented in this paper can be used, among others to investigate spherical ballistic casings. In addition, from our point of view, the results of this analysis are a modest contribution of knowledge to the theory of stress waves propagation in elastic media.

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Analiza parametrów kulistej fali naprężenia ekspandującej w liniowym sprężystym ośrodku izotropowym

Streszczenie

Dokonano obszernej jakościowej i ilościowej analizy charakterystyk ekspandującej kulistej fali naprężenia w liniowym sprężystym ośrodku izotropowym. Falę wygenerowano nagle wytworzonym w kulistej kawernie o początkowym promieniu r_0 stałym ciśnieniem $p_0 = \text{const}$. Z postaci analitycznego rozwiązania problemu wynika,

że przemieszczenie i naprężenia maleją w przybliżeniu odwrotnie proporcjonalnie do kwadratu i sześcianu odległości od centrum kawerny. Stwierdzono, że powierzchnia kawerny i kolejne przekroje sferyczne ośrodka ściśliwego przemieszczają się z upływem czasu tłumionym ruchem drgającym wokół przemieszczenia statycznego. Podobnie zachowują się pozostałe charakterystyki fali. Na tłumienie drgań decydujący wpływ ma ściśliwość ośrodka, reprezentowana w pracy przez liczbę Poissona ν. Wzrost parametru ν ponad 0.4 powoduje gwałtowny spadek intensywności tłumienia, a w granicznym przypadku dla $\nu = 0.5$, tj. w ośrodku nieściśliwym, tłumienie całkowicie zanika. Ośrodek nieściśliwy drga, jak zachowawczy układ mechaniczny o jednym stopniu swobody.

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