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P-WAVE INTERACTION WITH A PAIR OF RIGID STRIPS EMBEDDED IN AN ORTHOTROPIC STRIP

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The present paper is concerned with the problem of scattering of the P-wave by two co--planer finite rigid strips placed symmetrically in an infinitely long orthotropic strip. Using the Hilbert transform technique, the mixed boundary value problem has been reduced to the solution of dual integral equations which has finally been reduced to the solution of a Fredholm integral equation of the second kind. Solving this integral equation numerically, stress intensity factors have been calculated at the inner and outer edges of the rigid strips, and the vertical displacement outside the strips has been calculated and plotted graphically to show the effect of material orthotropy.

Keywords: P-wave, Fourier transform, Hilbert transform, Fredholm integral equation, stress intensity factor

1. Introduction

The dynamic interaction of rigid strips with an elastic isotropic or orthotropic medium is a subject of considerable interest in mechanics. Dynamical analysis of this kind is of importance to earth-quake engineering, machine, vibrations and seismology. The performance of engineered systems is affected by inhomogeneities such as cracks and inclusions present in the material. Cracks and rigid inclusions in an elastic material have become the subject of investigations. Presently, the use of anisotropic materials is increasing due to their strength. The increasing use of anisotropic media demands that the study should be extensive. A detailed reference of work done on the determination of the dynamic stress field around a crack or inclusion in an elastic solid was given by Sih (1977), Sih and Chen (1981), Chen (1978), Cinar (1983). However, in the presence of finite boundaries, the problem becomes complicated since they involve additional geometric parameters, describing the dimension of the solids. Forced vertical vibration of a single strip was treated by Wickham (1977). Singh *et al.* (1983) solved the problem of diffraction of a torsional wave by a circular rigid disc at the interface of two bonded dissimilar elastic solids. In that paper, they discussed an iterative method to solve the Fredholm integral equation of the second kind and described the stress intensity factor with the wave number. Mandal *et al.* (1997, 1998) solved the problem of forced vibration of two and four rigid strips on a semi-infinite elastic medium. Mandal *et al.* (1998) also treated the diffraction problem by four rigid strips in an orthotropic medium. Interaction of elastic waves with a periodic array of the coplanar Griffith crack in an orthotropic medium was discussed by Mandal *et al.* (1994). Das *et al.* (1998) solved the problem of determining the stress intensity factor for an interfacial crack between two orthotropic half planes bonded to a dissimilar orthotropic layer with a punch. They reduced the problem to a system of simultaneous integral equations which were solved by Chebyshev polynomials. The problem of two perfectly bonded dissimilar orthotropic strips with an interfacial crack was studied by Li (2005). He derived the analytical expression for the stress intensity factor. Sarkar *et al.* (1995) solved the problem of diffraction of elastic waves by three coplanar Griffith cracks in an orthotropic medium. Das (2002) solved the problem of interaction between line cracks in an orthotropic layer. An elastostatic problem of an infinite row of parallel cracks in an orthotropic medium was analyzed by Sinharoy (2013). Monfared and Ayatollahi (2013) investigated the problem of determining the dynamic stress intensity factors of multiple cracks in an orthotropic strip with a functionally graded materials coating. They solved the problem by reducing it to a singular integral equation of the Cauchy type. The problem of interaction of three interfacial Griffith cracks between bonded dissimilar orthotropic half planes was studied by Mukherjee and Das (2007). Das *et al.* (2008) solved the problem of determining the stress intensity factors due to symmetric edge cracks in an orthotropic strip under normal loading. They derived an analytical expression for the stress intensity factor at the crack tip. The problem of finding the stress intensity factors for two parallel interface cracks between a nonhomogeneous bonding layer and two dissimilar orthotropic half-planes under tension was studied by Itou (2012). Shear wave interaction with a pair of rigid strips in elastic strip was analyzed by Pramanick *et al.* (1999). WU Da-zhi *et al.* (2006) considered the torsional vibration problem of a rigid circular plate on a transversely isotropic saturated soil. Very recently Morteza *et al.* (2010a,b) considered the vibration problem of a rigid circular disc on transversely isotropic media. Diffraction of elastic waves by two parallel rigid strips in an infinite orthotropic medium was analyzed by Sarkar *et al.* (1995).

In this paper, the diffraction of the elastic P-wave by two rigid strips embedded in an infinite orthotropic strip is analyzed. Using the Hilbert transform technique, the mixed boundary value problem has been reduced to the Fredholm integral equation of the second kind which has been solved numerically by the Fox and Goodwin method (1953). Stress intensity factors at both the edges of the strips have been calculated and shown graphically for different parameters and materials. Finally, vertical displacement has been calculated outside the strips and shown by 3D-graphs.

2. Formulation of the problem

Let us consider an infinitely long orthotropic elastic strip of width $2h$ containing two coplanar rigid strips embedded in it. The location of the strips are $b \leqslant |X| \leqslant a$, $Y = 0$, $|Z| < \infty$, with reference to the cartesian co-ordinate axes (X, Y, Z). Normalizing all lengths with respect to a and putting $X/a = x$, $Y/a = y$, $Z/a = z$, $b/a = c$, the locations of the rigid strips are defined by $c \leqslant |x| \leqslant 1$, $y = 0$, $|z| < \infty$ (Fig. 1).

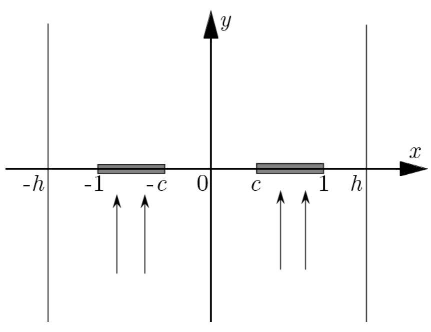

Fig. 1. Geometry of the strips

Let a time harmonic wave given by $u = 0$ and $v = v_0 e^{i(ky-\omega t)}$, where $k = a\omega/(c_s\sqrt{c_{22}})$, $c_s = \sqrt{\mu_{12}/\rho}$ with ρ being the density of the material, ω the circular frequency and v_0 a constant, travelling in the direction of the positive y-axis and be incident normally on the strips.

The non-zero stress components τ_{yy}, τ_{xy} and τ_{xx} are given by

$$\frac{\tau_{yy}}{\mu_{12}} = c_{12}\frac{\partial u}{\partial x} + c_{22}\frac{\partial v}{\partial y} \qquad \frac{\tau_{xy}}{\mu_{12}} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad \frac{\tau_{xx}}{\mu_{12}} = c_{11}\frac{\partial u}{\partial x} + c_{12}\frac{\partial v}{\partial y} \tag{2.1}$$

where u and v are displacement components and c_{ij} $(i,j = 1,2)$ are non-dimensional parameters related to the engineering elastic constants E_i, μ_{ij} and ν_{ij} $(i,j = 1,2,3)$ by the relations

$$\begin{aligned} c_{11} &= \frac{E_1}{\mu_{12}}\Big(1 - \frac{\nu_{12}^2 E_2}{E_1}\Big) \qquad c_{22} = \frac{E_2}{\mu_{12}}\Big(1 - \frac{\nu_{12}^2 E_2}{E_1}\Big) = c_{11}\frac{E_2}{E_1} \\ c_{12} &= \frac{\nu_{12}E_2}{\mu_{12}}\Big(1 - \frac{\nu_{12}^2 E_2}{E_1}\Big) = \nu_{12}c_{22} = \nu_{21}c_{11} \end{aligned} \tag{2.2}$$

for the generalized plane stress and

$$\begin{aligned} c_{11} &= \frac{E_1}{\Delta\mu_{12}}(1 - \nu_{23}\nu_{32}) \qquad c_{22} = \frac{E_2}{\Delta\mu_{12}}(1 - \nu_{13}\nu_{31}) \\ c_{12} &= \frac{E_1}{\Delta\mu_{12}}\Big(\nu_{21} + \frac{\nu_{13}\nu_{32}E_2}{E_1}\Big) = \frac{E_2}{\Delta\mu_{12}}\Big(\nu_{12} + \frac{\nu_{23}\nu_{31}E_1}{E_2}\Big) \end{aligned} \tag{2.3}$$

where

$$\Delta = 1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - \nu_{12}\nu_{23}\nu_{31} - \nu_{13}\nu_{21}\nu_{32}$$

for the plane strain. The constants E_i and ν_{ij} satisfy Maxwell's relation

$$\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j} \tag{2.4}$$

Therefore, substituting $u(x,y,t) = u(x,y)\mathrm{e}^{-\mathrm{i}\omega t}$ and $v(x,y,t) = v(x,y)\mathrm{e}^{-\mathrm{i}\omega t}$, our problem reduces to the solution of the equations

$$\begin{aligned} c_{11}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + (1 + c_{12})\frac{\partial^2 v}{\partial x \partial y} + k_s^2 u = 0 \\ c_{22}\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} + (1 + c_{12})\frac{\partial^2 u}{\partial x \partial y} + k_s^2 v = 0 \end{aligned} \tag{2.5}$$

where $k_s^2 = a^2\omega^2/c_s^2$.

Thus the problem is to find the stress distribution near the edges of the strips subject to the following boundary conditions

$$v(x,0+) = v(x,0-) = -v_0 \qquad c \leqslant |x| \leqslant 1 \tag{2.6}$$

$$\tau_{yy}(x,0) = 0 \qquad |x| < c \qquad 1 < |x| < h \tag{2.7}$$

$$u(x,0) = 0 \qquad |x| < h \tag{2.8}$$

$$\tau_{xx}(\pm h, y) = 0 \qquad \tau_{xy}(\pm h, y) = 0 \tag{2.9}$$

Henceforth, the time factor $\mathrm{e}^{-\mathrm{i}\omega t}$ which is common to all field variables will be omitted in the sequel.

The solution to equations (2.5) can be taken as

$$\begin{aligned} u(x,y) &= \frac{2}{\pi}\int\limits_0^\infty [A_1(\xi)\mathrm{e}^{-\nu_1|y|} + A_2(\xi)\mathrm{e}^{-\nu_2|y|}]\sin(\xi x)\,d\xi \\ &\quad + \frac{2}{\pi}\int\limits_0^\infty [A_3(\zeta)\sinh(\nu_3 x) + A_4(\zeta)\sinh(\nu_4 x)]\sin(\zeta y)\,d\zeta \\ v(x,y) &= \frac{2}{\pi}\int\limits_0^\infty \frac{1}{\xi}[\alpha_1 A_1(\xi)\mathrm{e}^{-\nu_1|y|} + \alpha_2 A_2(\xi)\mathrm{e}^{-\nu_2|y|}]\cos(\xi x)\,d\xi \\ &\quad + \frac{2}{\pi}\int\limits_0^\infty \frac{1}{\zeta}[\alpha_3 A_3(\zeta)\cosh(\nu_3 x) + \alpha_4 A_4(\zeta)\cosh(\nu_4 x)]\cos(\zeta y)\,d\zeta \end{aligned} \tag{2.10}$$

where $A_i(\xi)$ $(i = 1,2,3,4)$ are the unknown functions to be determined, ν_1^2 and ν_2^2 are the roots of the equation

$$c_{22}\nu^4 + \{(c_{12}^2 + 2c_{12} - c_{11}c_{22})\xi^2 + (1 + c_{22})k_s^2\}\nu^2 + (c_{11}\xi^2 - k_s^2)(\xi^2 - k_s^2) = 0 \tag{2.11}$$

and ν_3^2, ν_4^2 are the roots of the equation

$$c_{11}\nu^4 + \{(c_{12}^2 + 2c_{12} - c_{11}c_{22})\zeta^2 + (1 + c_{11})k_s^2\}\nu^2 + (c_{22}\zeta^2 - k_s^2)(\zeta^2 - k_s^2) = 0 \tag{2.12}$$

where

$$\alpha_i = \begin{cases} \dfrac{c_{11}\xi^2 - k_s^2 - \nu_i^2}{(1 + c_{12})\nu_i} & i = 1,2 \\ \dfrac{\zeta^2 - k_s^2 - c_{11}\nu_i^2}{(1 + c_{12})\nu_i} & i = 3,4 \end{cases} \tag{2.13}$$

From boundary condition (2.11), it is found that

$$A_2(\xi) = -A_1(\xi) \tag{2.14}$$

Therefore, the displacements u, v and stresses τ_{yy}, τ_{xy}, τ_{xx} can be finally written as

$$\begin{aligned} u(x,y) &= \frac{2}{\pi}\int\limits_0^\infty [\mathrm{e}^{-\nu_1|y|} - \mathrm{e}^{-\nu_2|y|}]A_1(\xi)\sin(\xi x)\,d\xi \\ &\quad + \frac{2}{\pi}\int\limits_0^\infty [A_3(\zeta)\sinh(\nu_3 x) + A_4(\zeta)\sinh(\nu_4 x)]\sin(\zeta y)\,d\zeta \\ v(x,y) &= \frac{2}{\pi}\int\limits_0^\infty \frac{1}{\xi}[\alpha_1\mathrm{e}^{-\nu_1|y|} - \alpha_2\mathrm{e}^{-\nu_2|y|}]A_1(\xi)\cos(\xi x)\,d\xi \\ &\quad + \frac{2}{\pi}\int\limits_0^\infty \frac{1}{\zeta}[\alpha_3 A_3(\zeta)\cosh(\nu_3 x) + \alpha_4 A_4(\zeta)\cosh(\nu_4 x)]\cos(\zeta y)\,d\zeta \end{aligned} \tag{2.15}$$

and

$$
\begin{aligned}
\frac{\tau_{yy}}{\mu_{12}} &= \frac{2}{\pi}\Big\{ \int\limits_0^\infty \Big[\Big(c_{12}\xi - \operatorname{sgn}(y)\frac{c_{22}\alpha_1\nu_1}{\xi}\Big)\mathrm{e}^{-\nu_1|y|} - \Big(c_{12}\xi - \operatorname{sgn}(y)\frac{c_{22}\alpha_2\nu_2}{\xi}\Big)\mathrm{e}^{-\nu_2|y|}\Big] \\
&\quad \cdot A_1(\xi)\cos(\xi x)\;d\xi + \int\limits_0^\infty [(c_{12}\nu_3 - c_{22}\alpha_3)A_3(\zeta)\cosh(\nu_3 x) \\
&\quad + (c_{12}\nu_4 - c_{22}\alpha_4)A_4(\zeta)\cosh(\nu_4 x)]\sin(\zeta y)\;d\zeta\Big\} \\
\frac{\tau_{xy}}{\mu_{12}} &= -\frac{2}{\pi}\Big\{ \int\limits_0^\infty [(\nu_1+\alpha_1)\mathrm{e}^{-\nu_1 y} - (\nu_2+\alpha_2)\mathrm{e}^{-\nu_2 y}]A_1(\xi)\sin(\xi x)\;d\xi \\
&\quad + \int\limits_0^\infty \Big[\Big(\zeta + \frac{\nu_3\alpha_3}{\zeta}\Big)A_3(\zeta)\sinh(\nu_3 x) \\
&\quad + \Big(\zeta + \frac{\nu_4\alpha_4}{\zeta}\Big)A_4(\zeta)\sinh(\nu_4 x)\Big]\cos(\zeta y)\;d\zeta\Big\} \qquad y>0 \\
\frac{\tau_{xx}}{\mu_{12}} &= \frac{2}{\pi}\Big\{ \int\limits_0^\infty \Big[\Big(c_{11}\xi - \frac{c_{12}\alpha_1\nu_1}{\xi}\Big)\mathrm{e}^{-\nu_1|y|} - \Big(c_{11}\xi - \frac{c_{12}\alpha_2\nu_2}{\xi}\Big)\mathrm{e}^{-\nu_2|y|}\Big]A_1(\xi)\cos(\xi x)\;d\xi \\
&\quad + \int\limits_0^\infty [(c_{11}\nu_3 - c_{12}\alpha_3)A_3(\zeta)\cosh(\nu_3 x) \\
&\quad + (c_{11}\nu_4 - c_{12}\alpha_4)A_4(\zeta)\cosh(\nu_4 x)]\sin(\zeta y)\;d\zeta\Big\} \qquad y>0
\end{aligned}
\tag{2.16}
$$

Boundary conditions (2.6) and (2.7) yield the following pair of dual integral equations

$$
\begin{aligned}
&\int\limits_0^\infty \frac{1}{\xi}[1+H(\xi)]A(\xi)\cos(\xi x)\;d\xi = p(x) \qquad c \leqslant |x| \leqslant 1 \\
&\int\limits_0^\infty A(\xi)\cos(\xi x)\;d\xi = 0 \qquad |x|<c \qquad 1<|x|<h
\end{aligned}
\tag{2.17}
$$

where

$$
\begin{aligned}
A(\xi) &= \frac{\alpha_1\nu_1 - \alpha_2\nu_2}{\xi}A_1(\xi) \\
H(\xi) &= \Big(\frac{\alpha_1-\alpha_2}{\alpha_1\nu_1 - \alpha_2\nu_2}\Big)\frac{\xi}{d} - 1 \quad \to \quad 0 \qquad \text{as} \quad \xi \to \infty \\
p(x) &= -\frac{\pi}{2c}v_0 - \frac{1}{c}\int\limits_0^\infty \frac{1}{\zeta}[\alpha_3 A_3(\zeta)\cosh(\nu_3 x) + \alpha_4 A_4(\zeta)\cosh(\nu_4 x)]\;d\zeta \\
d &= \frac{c_{11} + N_1 N_2}{N_1 N_2(N_1+N_2)}
\end{aligned}
\tag{2.18}
$$

and

$$
\begin{aligned}
N_1^2 &= \frac{1}{2c_{22}}\Big[-(c_{12}^2 + 2c_{12} - c_{11}c_{22}) + \sqrt{(c_{12}^2 + 2c_{12} - c_{11}c_{22})^2 - 4c_{11}c_{22}}\Big] \\
N_2^2 &= \frac{1}{2c_{22}}\Big[-(c_{12}^2 + 2c_{12} - c_{11}c_{22}) - \sqrt{(c_{12}^2 + 2c_{12} - c_{11}c_{22})^2 - 4c_{11}c_{22}}\Big]
\end{aligned}
\tag{2.19}
$$

Using boundary conditions (2.9), $A_3(\zeta)$ and $A_4(\zeta)$ are expressed in terms of the function $A(\xi)$ as

$$\begin{aligned} M(\zeta)A_3(\zeta) &= \Big(\zeta + \frac{\alpha_4\nu_4}{\zeta}\Big)i_1(\zeta)\sinh(\nu_4 h) - (c_{11}\nu_4 - c_{12}\alpha_4)i_2(\zeta)\cosh(\nu_4 h) \\ M(\zeta)A_4(\zeta) &= -\Big(\zeta + \frac{\alpha_3\nu_3}{\zeta}\Big)i_1(\zeta)\sinh(\nu_3 h) + (c_{11}\nu_3 - c_{12}\alpha_3)i_2(\zeta)\cosh(\nu_3 h) \end{aligned} \tag{2.20}$$

where

$$\begin{aligned} M(\zeta) = & \Big(\zeta + \frac{\alpha_4\nu_4}{\zeta}\Big)(c_{11}\nu_3 - c_{12}\alpha_3)\cosh(\nu_3 h)\sinh(\nu_4 h) \\ & - \Big(\zeta + \frac{\alpha_3\nu_3}{\zeta}\Big)(c_{11}\nu_4 - c_{12}\alpha_4)\sinh(\nu_3 h)\cosh(\nu_4 h) \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} i_1(\zeta) &= \frac{2}{\pi}\int\limits_0^\infty \Big\{\frac{\zeta[c_{11}\xi^2 + c_{12}(k_s^2 + \nu_1^2)]}{\nu_1^2 + \zeta^2} - \frac{\zeta[c_{11}\xi^2 + c_{12}(k_s^2 + \nu_2^2)]}{\nu_2^2 + \zeta^2}\Big\}\frac{A(\xi)\cos(\xi h)}{\nu_1^2 - \nu_2^2}\,d\xi \\ i_2(\zeta) &= -\frac{2}{\pi}\int\limits_0^\infty \Big(\frac{c_{12}\nu_1^2 + c_{11}\xi^2 - k_s^2}{\nu_1^2 + \zeta^2} - \frac{c_{12}\nu_2^2 + c_{11}\xi^2 - k_s^2}{\nu_2^2 + \zeta^2}\Big)\frac{\xi A(\xi)\sin(\xi h)}{\nu_1^2 - \nu_2^2}\,d\xi \end{aligned} \tag{2.22}$$

3. Method of solution

In order to reduce dual integral equations (2.17) to a single Fredholm integral equation, let us assume that

$$A(\xi) = \int\limits_c^1 \frac{h(t^2)}{t}[1 - \cos(\xi t)]\,dt \tag{3.1}$$

where the unknown function $h(t^2)$ is to be determined.

Substituting $A(\xi)$ from (3.1) into equations $(2.17)_2$, we note that

$$\int\limits_0^\infty A(\xi)\cos(\xi x)\,d\xi = \pi\int\limits_c^1 \frac{h(t^2)}{t}\Big[\delta(x) - \frac{1}{2}\delta(x+t) - \frac{1}{2}\delta(|x-t|)\Big]\,dt$$

so that equation $(2.17)_2$ is automatically satisfied.

Again, the substitution of the value of $A(\xi)$ from (3.1) into equation $(2.17)_1$ yields

$$\frac{1}{2}\int\limits_c^1 \frac{h(t^2)}{t}\log\Big|\frac{x^2 - t^2}{x^2}\Big|\,dt = p(x) - \int\limits_c^1 \frac{h(t^2)}{t}\,dt\int\limits_0^\infty \xi^{-1}H(\xi)\cos(\xi x)[1 - \cos(\xi t)]\,d\xi \tag{3.2}$$

Differentiating both sides of equation (3.2) with respect to x and subsequently multiplying by $(-2x/\pi)$, we obtain

$$\begin{aligned} & \frac{2}{\pi}\int\limits_c^1 \frac{th(t^2)}{t^2 - x^2}\,dt \\ & \quad = \frac{2x}{\pi}\int\limits_c^1 \frac{h(t^2)}{t}\,dt\Big\{\frac{1}{d}\int\limits_0^\infty \frac{1}{\zeta}[\alpha_3\nu_3 A_5(\zeta)\sinh(\nu_3 x) + \alpha_4\nu_4 A_6(\zeta)\sinh(\nu_4 x)]\,d\zeta \\ & \quad - \int\limits_0^\infty H(\xi)\sin(\xi x)[1 - \cos(\xi t)]\,d\xi\Big\} \qquad c \leqslant |x| \leqslant 1 \end{aligned} \tag{3.3}$$

Using the Hilbert transform technique, the solution to integral equation (3.3) is given by

$$h(u^2) + \int\limits_c^1 \frac{h(t^2)}{t}[k_1(u^2,t^2) + k_2(u^2,t^2)]\,dt = \frac{D}{\sqrt{(u^2-c^2)(1-u^2)}} \tag{3.4}$$

where

$$\begin{aligned}
k_1(u^2,t^2) &= \frac{4}{\pi^2 d}\sqrt{\frac{u^2-c^2}{1-u^2}}\int\limits_c^1 \sqrt{\frac{1-x^2}{x^2-c^2}}\frac{x^2}{x^2-u^2}\,dx \\
&\quad\cdot\Big\{\int\limits_0^\infty \frac{1}{\zeta}[\alpha_3\nu_3 A_5(\zeta)\sinh(\nu_3 x) + \alpha_4\nu_4 A_6(\zeta)\sinh(\nu_4 x)]\,d\zeta\Big\} \\
k_2(u^2,t^2) &= -\frac{4}{\pi^2}\sqrt{\frac{u^2-c^2}{1-u^2}}\int\limits_c^1 \sqrt{\frac{1-x^2}{x^2-c^2}}\frac{x^2\,dx}{x^2-u^2}\int\limits_0^\infty H(\xi)\sin(\xi x)[1-\cos(\xi t)]\,d\xi \\
A_5(\zeta) &= \frac{1}{M(\zeta)}\Big[\Big(\zeta + \frac{\alpha_4\nu_4}{\zeta}\Big)i_3(\zeta)\sinh(\nu_4 h) - (c_{11}\nu_4 - c_{12}\alpha_4)i_4(\zeta)\cosh(\nu_4 h)\Big] \\
A_6(\zeta) &= -\frac{1}{M(\zeta)}\Big[\Big(\zeta + \frac{\alpha_3\nu_3}{\zeta}\Big)i_3(\zeta)\sinh(\nu_3 h) + (c_{11}\nu_3 - c_{12}\alpha_3)i_4(\zeta)\cosh(\nu_3 h)\Big]
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
i_3(\zeta) &= \frac{2}{\pi}\int\limits_0^\infty \Big\{\frac{\zeta[c_{11}\xi^2 + c_{12}(k_s^2+\nu_1^2)]}{\nu_1^2+\zeta^2} - \frac{\zeta[c_{11}\xi^2 + c_{12}(k_s^2+\nu_2^2)]}{\nu_2^2+\zeta^2}\Big\}\frac{[1-\cos(\xi t)]\cos(\xi h)}{\nu_1^2-\nu_2^2}\,d\xi \\
i_4(\zeta) &= -\frac{2}{\pi}\int\limits_0^\infty \Big(\frac{c_{12}\nu_1^2 + c_{11}\xi^2 - k_s^2}{\nu_1^2+\zeta^2} - \frac{c_{12}\nu_2^2 + c_{11}\xi^2 - k_s^2}{\nu_2^2+\zeta^2}\Big)\frac{\xi[1-\cos(\xi t)]}{\nu_1^2-\nu_2^2}\sin(\xi h)\,d\xi
\end{aligned} \tag{3.6}$$

In order to determine the arbitrary constant D, multiplying equation (3.2) by $x/\sqrt{(x^2-c^2)(1-x^2)}$ and integrating from c to 1 with respect to x, we obtain

$$\begin{aligned}
\int\limits_c^1 \frac{h(u^2)}{u}\,du &= -\frac{\pi v_0}{c\log\left|\frac{1-c}{1+c}\right|} - \frac{4}{\pi\log\left|\frac{1-c}{1+c}\right|}\Big[\int\limits_c^1 \frac{xB_1(x,t^2)}{\sqrt{(x^2-c^2)(1-x^2)}}\,dx \\
&\quad + \int\limits_c^1 \frac{h(t^2)}{t}\,dt\int\limits_c^1 \frac{xB_2(x,t^2)}{\sqrt{(x^2-c^2)(1-x^2)}}\,dx\Big]
\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}
B_1(x,t^2) &= \frac{1}{d}\int\limits_0^\infty \frac{1}{\zeta}[\alpha_3 A_5(\zeta)\cosh(\nu_3 x) + \alpha_4 A_6(\zeta)\cosh(\nu_4 x)]\,d\zeta \\
B_2(x,t^2) &= \int\limits_0^\infty \frac{1}{\xi}H(\xi)\cos(\xi x)[1-\cos(\xi t)]\,d\xi
\end{aligned} \tag{3.8}$$

Again, substituting $h(u^2)$ from equation (3.4) into equation (3.7) and simplifying, we obtain

$$\begin{aligned}
D &= -\frac{2v_0 c}{d\log\left|\frac{1-c}{1+c}\right|} - \frac{8c}{\pi^2\log\left|\frac{1-c}{1+c}\right|}\int\limits_c^1 \frac{h(t^2)}{t}\,dt\int\limits_c^1 \frac{x[B_1(x,t^2)+B_2(x,t^2)]}{\sqrt{(x^2-c^2)(1-x^2)}}\,dx \\
&\quad + \frac{2c}{\pi}\int\limits_c^1 \frac{h(t^2)}{t}\,dt\int\limits_c^1 \frac{1}{u}[k_1(u^2,t^2) + k_2(u^2,t^2)]\,du
\end{aligned} \tag{3.9}$$

Eliminating D from equations (3.4) and (3.9) and simplifying, we obtain

$$\begin{aligned}&\sqrt{(u^2-c^2)(1-u^2)}h(u^2)+\int\limits_c^1 \frac{h(t^2)}{t}[k_a(u^2,t^2)+k_b(u^2,t^2)+k_c(u^2,t^2)]\;dt\\&\quad=-\frac{2v_0c}{d\log\Big|\frac{1-c}{1+c}\Big|}\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}k_a(u^2,t^2)&=\frac{4}{\pi^2}(u^2-c^2)\int\limits_c^1\sqrt{\frac{1-x^2}{x^2-c^2}}\frac{x^2}{x^2-u^2}\Big\{\frac{\partial}{\partial x}[B_1(x,t^2)+B_2(x,t^2)]\Big\}\;dx\\k_b(u^2,t^2)&=\frac{8c}{\pi^2\log\Big|\frac{1-c}{1+c}\Big|}\int\limits_c^1\frac{x[B_1(x,t^2)+B_2(x,t^2)]}{\sqrt{(x^2-c^2)(1-x^2)}}\;dx\\k_c(u^2,t^2)&=-\frac{8c}{\pi^3u}\sqrt{\frac{u^2-c^2}{1-u^2}}\int\limits_c^1\sqrt{\frac{1-x^2}{x^2-c^2}}\frac{x^2}{x^2-u^2}\Big\{\frac{\partial}{\partial x}[B_1(x,t^2)+B_2(x,t^2)]\Big\}\;dx\end{aligned} \tag{3.11}$$

Next, for further simplification, we put

$$\sqrt{(u^2-c^2)(1-u^2)}h(u^2)=H(u^2)$$

and make the substitution

$$u^2=c^2\cos^2\phi+\sin^2\phi \qquad\qquad t^2=c^2\cos^2\theta+\sin^2\theta$$

into equation (3.10) which then reduces to the form

$$G(\phi)+\int\limits_0^{\frac{\pi}{2}}\frac{G(\theta)}{c^2\cos^2\theta+\sin^2\theta}[k'_a(\phi,\theta)+k'_b(\phi,\theta)+k'_c(\phi,\theta)]\;d\theta=-\frac{2v_0c}{d\log\Big|\frac{1-c}{1+c}\Big|} \tag{3.12}$$

where

$$\begin{aligned}G(\phi)&=H(c^2\cos^2\phi+\sin^2\phi)\\G(\theta)&=H(c^2\cos^2\theta+\sin^2\theta)\\k'_a(\phi,\theta)&=k_a(c^2\cos^2\phi+\sin^2\phi,c^2\cos^2\theta+\sin^2\theta)\\k'_b(\phi,\theta)&=k_b(c^2\cos^2\phi+\sin^2\phi,c^2\cos^2\theta+\sin^2\theta)\\k'_c(\phi,\theta)&=k_c(c^2\cos^2\phi+\sin^2\phi,c^2\cos^2\theta+\sin^2\theta)\end{aligned} \tag{3.13}$$

When h tends to infinity ($h\to\infty$), the medium becomes infinite. In this case, the expression for $p(x)$ given by equation $(2.18)_3$ becomes $p(x)=-(\pi/2c)v_0$, since $A_3(\zeta)$ and $A_4(\zeta)$ given by equations (2.20)-(2.22) become zero.

$A_3(\zeta)$ can be written as

$$A_3(\zeta)=\frac{1}{2M(\zeta)}\Big[\Big(\zeta+\frac{\alpha_4\nu_4}{\zeta}\Big)i_1(\zeta)(\mathrm{e}^{\nu_4h}-\mathrm{e}^{-\nu_4h})-(c_{11}\nu_4-c_{12}\alpha_4)i_2(\zeta)(\mathrm{e}^{\nu_4h}+\mathrm{e}^{-\nu_4h})\Big]$$

where

$$\begin{aligned}M(\zeta)=\frac{1}{4}\Big[&\Big(\zeta+\frac{\alpha_4\nu_4}{\zeta}\Big)(c_{11}\nu_3-c_{12}\alpha_3)(\mathrm{e}^{\nu_3h}+\mathrm{e}^{-\nu_3h})(\mathrm{e}^{\nu_4h}-\mathrm{e}^{-\nu_4h})\\&-\Big(\zeta+\frac{\alpha_3\nu_3}{\zeta}\Big)(c_{11}\nu_4-c_{12}\alpha_4)(\mathrm{e}^{\nu_3h}-\mathrm{e}^{-\nu_3h})(\mathrm{e}^{\nu_4h}+\mathrm{e}^{-\nu_4h})\Big]\end{aligned}$$

Therefore,

$$A_3(\zeta) = \frac{1}{M_1(\zeta)}\Big[\Big(\zeta + \frac{\alpha_4\nu_4}{\zeta}\Big)i_1(\zeta)(1-\mathrm{e}^{-2\nu_4 h}) - (c_{11}\nu_4 - c_{12}\alpha_4)i_2(\zeta)(1+\mathrm{e}^{-2\nu_4 h})\Big]$$

and

$$M_1(\zeta) = \frac{\mathrm{e}^{\nu_3 h}}{2}\Big[\Big(\zeta + \frac{\alpha_4\nu_4}{\zeta}\Big)(c_{11}\nu_3 - c_{12}\alpha_3)(1+\mathrm{e}^{-2\nu_3 h})(1-\mathrm{e}^{-2\nu_4 h})$$
$$-\Big(\zeta + \frac{\alpha_3\nu_3}{\zeta}\Big)(c_{11}\nu_4 - c_{12}\alpha_4)(1-\mathrm{e}^{-2\nu_3 h})(1+\mathrm{e}^{-2\nu_4 h})\Big]$$

As $h \to \infty$, $M_1(\zeta) \to \infty$ and therefore $A_3(\zeta) \to 0$. Similarly, $A_4(\zeta) \to 0$.

So in this case, dual integral equations $(2.17)_1$ and $(2.17)_2$ become

$$\int\limits_0^\infty \frac{1}{\xi}[1+H(\xi)]A(\xi)\cos(\xi x)\;d\xi = -\frac{\pi}{2c}v_0 \qquad c \leqslant |x| \leqslant 1$$
$$\int\limits_0^\infty A(\xi)\cos(\xi x)\;d\xi = 0 \qquad |x| < c \qquad |x| > 1$$

This problem has been analyzed in detail by Sarkar *et al.* (1995).

4. Quantities of physical interest

The stress $\tau_{yy}(x,y)$ for $y \to 0$ in the neighbourhood of the strip can be found from equation $(2.16)_1$, and is given by

$$\tau_{yy}(x,0\pm) = \mp\frac{2\mu_{12}c_{22}}{\pi}\int\limits_0^\infty A(\xi)\cos(\xi x)\;d\xi \qquad c \leqslant |x| \leqslant 1 \tag{4.1}$$

Now

$$\varDelta\tau_{yy}(x,0) = \tau_{yy}(x,0+) - \tau_{yy}(x,0-) \tag{4.2}$$

then

$$\varDelta\tau_{yy}(x,0) = -\frac{4}{\pi}\mu_{12}c_{22}\int\limits_0^\infty A(\xi)\cos(\xi x)\;d\xi \tag{4.3}$$

Substituting the value of $A(\xi)$ from equation (3.1) into equation (4.3), we get

$$\varDelta\tau_{yy}(x,0) = 2\mu_{12}c_{22}\frac{h(x^2)}{x} \tag{4.4}$$

Since

$$h(x^2) = \frac{1}{\sqrt{(x^2-c^2)(1-x^2)}}H(x^2) \qquad x^2 = c^2\cos^2\phi + \sin^2\phi$$

equation (4.4) becomes

$$\varDelta\tau_{yy}(x,0) = \frac{2\mu_{12}c_{22}\,G(\phi)}{x\sqrt{(x^2-c^2)(1-x^2)}} \tag{4.5}$$

So the stress intensity factors N_c and N_1 at the two tips of the strip can be expressed as

$$N_c = \lim_{x\to c+}\Big[\frac{\Delta\tau_{yy}(x,0)}{\pi c_{22}\mu_{12}}\sqrt{x-c}\Big] = \frac{2}{\pi}\frac{G(0)}{c\sqrt{2c(1-c^2)}} \tag{4.6}$$

and

$$N_1 = \lim_{x\to 1-}\Big[\frac{\Delta\tau_{yy}(x,0)}{\pi c_{22}\mu_{12}}\sqrt{1-x}\Big] = \frac{2}{\pi}\frac{G\Big(\frac{\pi}{2}\Big)}{\sqrt{2(1-c^2)}} \tag{4.7}$$

Making c tend to zero, the two strips merge into one, and in that case

$$N_1 = \frac{\sqrt{2}}{\pi}G\Big(\frac{\pi}{2}\Big)$$

Now from equation $(2.15)_2$ after substituting the value of $A_1(\xi)$ and using equation (3.1), we get the vertical displacement outside the strip as

$$\begin{aligned} v(x,y) = \frac{2}{\pi}\int_c^1 \frac{h(t^2)}{t}\,dt\Big\{\int_0^\infty (\alpha_1 \mathrm{e}^{-\nu_1 y} - \alpha_2 \mathrm{e}^{-\nu_2 y})\frac{[1-\cos(\xi t)]\cos(\xi x)}{\alpha_1\nu_1 - \alpha_2\nu_2}\,d\xi \\ + \int_0^\infty \frac{1}{\zeta}[\alpha_3 A_5(\zeta)\cosh(\nu_3 x) + \alpha_4 A_6(\zeta)\cosh(\nu_4 x)]\cos(\zeta y)\,d\zeta\Big\} \end{aligned} \tag{4.8}$$

5. Numerical calculations and discussions

It is important to choose a numerical method of solving the Fredholm integral equation. The Fox and Goodwin methods require that the definite integrals should be calculable by numerical quadrature, using known formulae in the theory of finite differences, and Fredholm equations are conveniently treated by solving simultaneous equations. The methods enable accurate solutions to be obtained without a prohibitive expenditure of time and energy. The choice of an interval is of course rather arbitrary. We want to keep to a minimum number of linear equations, but the interval must not be large that the finite-difference equations are meaningless. Since the differences are examined, the method guards against the possibility of obtaining wrong results from this case. It also ensures that neither too few nor too many differences are retained in the quadrature formulae.

The method of Fox and Goodwin (1953) has been used to solve integral equation (3.12) numerically for different values of the dimensionless frequency k_s, material strip width $2h$ and separating distance of the strips c. The integral in (3.12) has been represented by a quadrature formula involving values of the desired function G at pivotal points in the range of integration, which leads to a set of algebraic linear simultaneous equations. The solution of the set of linear algebraic equations gives the first approximation of the required pivotal values of G which has been improved by the use of the difference correction technique. After solving integral equation (3.12) for different values of engineering elastic constants of several orthotropic materials listed in Table 1, the stress intensity factors (SIF), k_c and k_1 at both ends of the strip given by equations (4.6) and (4.7) has been plotted against k_s for different values of h and c and for different materials. Instead of the real part of SIF, its mod value is taken because both shows the same type of results.

In Fig. 2a and 4a, N_c (SIF, at the inner edge of the strip) and N_1 (SIF, at the outer edge of the strip) have been plotted against k_s for $h = 2.0$ and $h = 2.5$ and for different strip lengths ($c = 0.2$, 0.4, 0.6) for material type I. In Fig. 3a and 5a, N_c and N_1 have been plotted against

Table 1. Engineering elastic constants

		E_1 [Pa]	E_2 [Pa]	μ_{12} [Pa]	ν_{12}
Type I	E-type glass-epoxi composite	$9.79 \cdot 10^9$	$42.3 \cdot 10^9$	$3.66 \cdot 10^9$	0.063
Type II	Stainless steel-aluminium composite	$79.76 \cdot 10^9$	$85.91 \cdot 10^9$	$30.02 \cdot 10^9$	0.31

k_s for $c = 0.4$ and $c = 0.6$ and for different material strip widths ($h = 2.0$, 2.5, 3.0) for material type I. The same set of parameters stated above for the graphs of N_c and N_1 have been plotted in Figs. 2b, 4b, 3b, 5b for material type II. For a particular value of material strip width h (=2.0, 2.5), the value of N_c decreases initially and, after increasing again, it decreases with an increase in k_s for material type I (Fig. 2a), whereas for material type II, it is slowly decreasing with an increase in k_s (Fig. 2b) for different values of strip length c (=0.2, 0.4, 0.6). It is also observed that with an increase in c, the value of N_c increases. When strip length c is fixed, the value of N_c is higher for higher values of h (=2.0, 2.5, 3.0) (Fig. 3a and Fig. 3b) for both types of materials. Figure 4 and 5 show that N_1 has initial decreasing tendency and then increases with an increase in k_s for both the materials. For fixed c, N_c is higher when material strip width h is higher. In all the cases, it is seen that as the length of the strip increases the value of N_1 decreases.

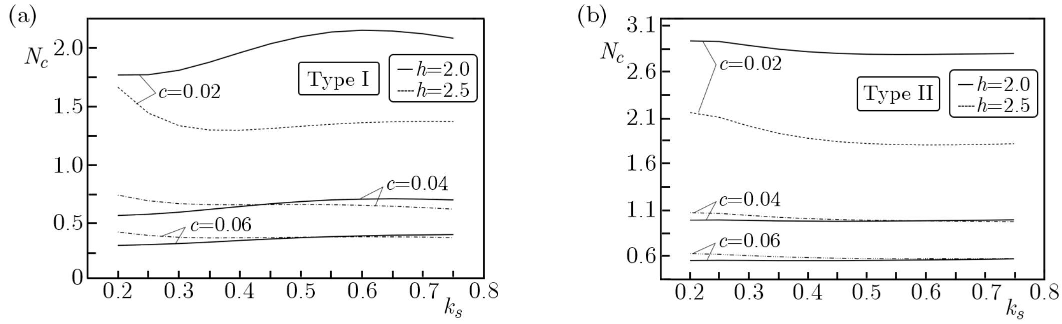

Fig. 2. Stress intensity factor N_c verses frequency k_s

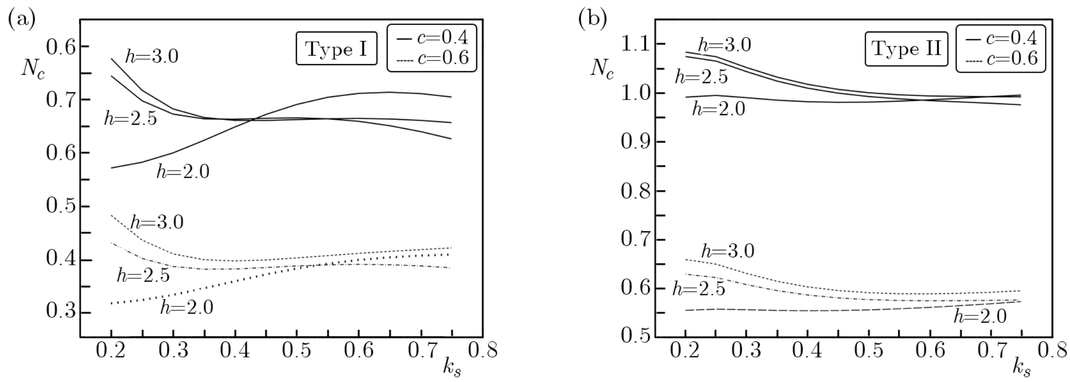

Fig. 3. Stress intensity factor N_c verses frequency k_s

Finally, in Fig. 6 and 7 the vertical displacement $v(x, y)$ has been plotted outside the strips ($0 < x < c$, $1 < x < h$) for fixed values of $h = 2.5$, $k_s = 0.4$ and $c = 0.6$ for both the materials. In Fig. 6, $v(x, y)$ has been plotted for the inner side of the strip ($0 < x < c$) and in Fig. 7 for the outer side of the strip ($1 < x < h$). In Fig. 6a and 7a, it is observed that the vertical displacement $v(x, y)$ increases initially with an increment of the values of x and y, then it decreases for material I. But in the case of Fig. 6b and 7b, it is seen that the vertical displacement $v(x, y)$ increases slowly with an increase in the values of x and y, then it decreases

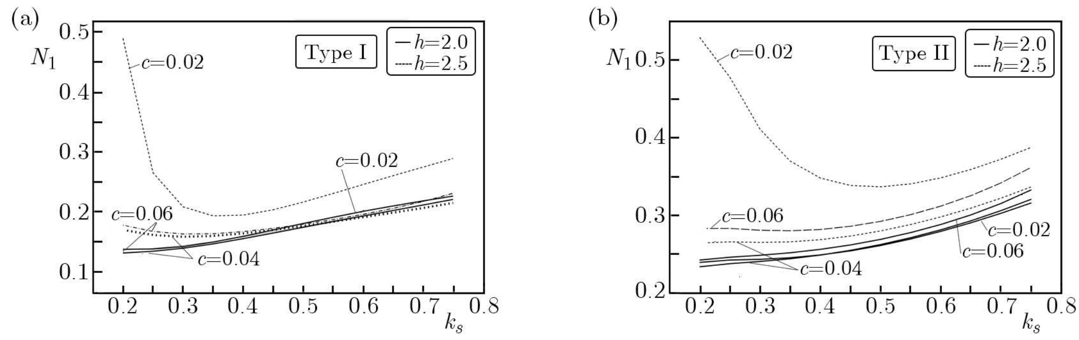

Fig. 4. Stress intensity factor N_1 versus frequency k_s

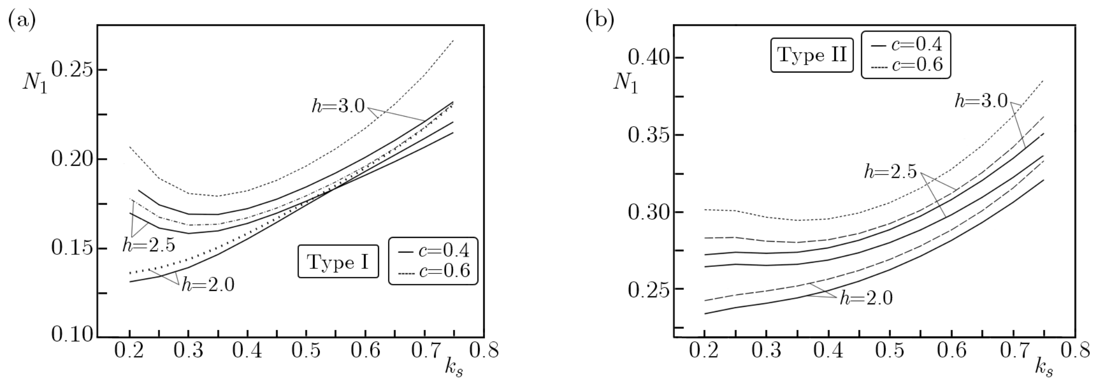

Fig. 5. Stress intensity factor N_1 versus frequency k_s

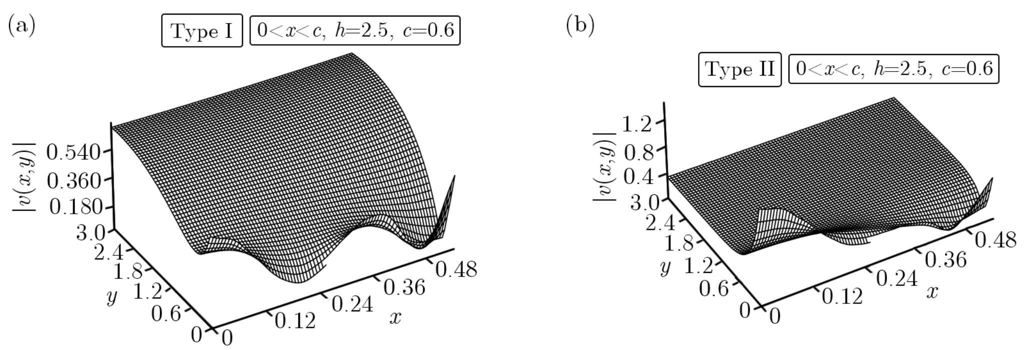

Fig. 6. Displacement $|v(x, y)|$ versus distances (x, y)

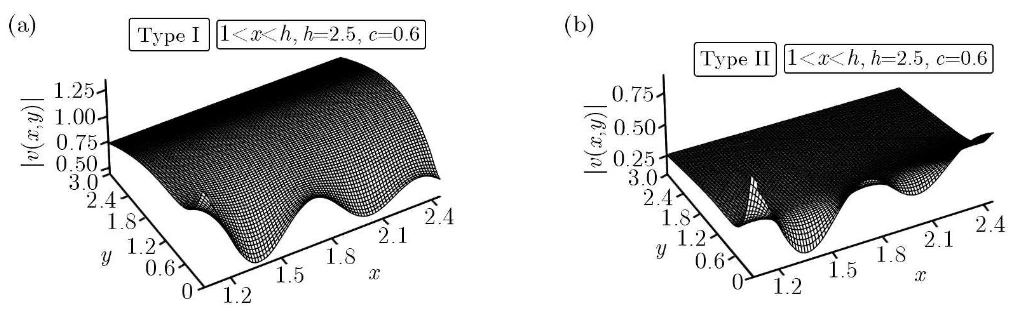

Fig. 7. Displacement $|v(x, y)|$ versus distances (x, y)

for material II. In all cases, the wave like nature has been observed, and finally the displacement tends to zero as $(x, y) \to \infty$, which satisfies the radiation condition.

6. Conclusions

The diffraction of the elastic P-wave by two rigid strips embedded in an infinite orthotropic strip is investigated on two types of materials by using the integral equation technique. The governing differential equation with constant coefficients with the boundary conditions becomes a mixed boundary value problem. Then, the mixed boundary value problem is transformed into a pair of dual integral equations with an unknown constant $A(\xi)$. To reduce the dual integral equations $(2.17)_1$ and $(2.17)_2$ to a single Fredholm integral equation, we assume the unknown constant $A(\xi)$ in the form of equation (3.1), so that equation $(2.17)_2$ can be automatically satisfied. Also, it has been found that the normal stress component $\tau_{yy}(x, 0)$ at the two tips of the strip has a square root singularity at $x = c$ and $x = 1$. The form of (3.1) has a square root type singularity in it, which can be utilized to find stress singularities at the tips of the strips.

From all the graphs of SIF, it can be concluded that the SIF decreases gradually with an increment of the frequency (k_s), after reaching the minimum value, it increases slowly. In all suggested cases, it is noted that the maximum value of the SIF at both tips of the strip for material II is little higher than that for material I. The SIF can be arrested within a certain range, which is very important with respect to growth of the crack. Finally, the vertical displacement $v(x, y)$ has been calculated outside the strips for both the materials. It has been observed the wave like nature from all the 3D figures, which finally decreases as the distance increases.

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