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METHODS BASED ON THE DIFFERENTIAL QUADRATURE IN VIBRATION ANALYSIS OF PLATES

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The paper deals with the methods based on the differential quadrature and their application to the free vibration analysis of plates. The spline-based differential quadrature Method (SDQM) is presented as an alternative to known methods based on the interpolation polynomial (PDQM). The SDQM uses a polynomial piecewise function to approximate the wanted solution of a governing equation. The way of determining the spline functions as well as the way of computing weighting coefficients for the method are presented in the paper. Then the SDQM is applied to determine natural frequencies of plates. The influence of the spline degree, number of nodes and grid point distribution on the accuracy, convergence and stability is investigated in an example. All results are compared with values obtained by the conventional differential quadrature method (PDQM).

Key words: Differential quadrature method, spline interpolation, free vibration

1. Introduction

Most engineering problems are described by partial differential equations. It is very difficult, if possible at all, to find closed-form solutions to them. A great increase in the computational power in recent years enables one to use numerical methods for solving very complex physical tasks. It is the reason that stimulates the development of known methods and the search for new, more efficient ones. Most of these methods rely on the conversion of a physical model of a system from continuous to discrete one. The approximate solution is searched at some special points in the domain. Among currently used discretisation methods, the most popular are: finite difference method, finite element method and finite volume method. These discretisation techniques use a low degree interpolation to determine function values at a node and, therefore, they are numbered among low order methods. They allow one to achieve high accuracy by using a large number of nodes. In many practical engineering problems, the numerical solution is required only at a few discrete points in the domain. The modal analysis could be an example. By discretizing a partial differential equation, an algebraic eigenvalue problem can be obtained. Next, the latter is solved giving approximation values of natural frequencies (or/and natural modes) of a continuous system. The number of obtained frequencies corresponds to the number of nodes of the imposed mesh. Among these frequencies, only a few first are interesting from the practical point of view. However, in order to achieve high accuracy for these values by low order methods, one has to use a large number of nodes. It requires much virtual storage and computational effort. This drawback can be overcome by using so-called global methods, which take much more information than only local neighborhood of a node to approximate the function. It makes the rate of convergence of these methods much higher than low order methods and allows one to achieve very accurate results with only a few discrete points.

The differential quadrature method (DQM) falls under this category. Theoretical foundations of the DQM were given by Bellman and Casti (1971). In recent years, one has been able to notice significant development of the method and its application in many fields of mechanics. Some main works are quoted by Bert and Malik (1996). Higher efficiency of the DQM for linear problems than the finite element and finite difference method was proved by Bert *et al.* (1988, 1993), Wang and Bert (1993), Malik and Civen (1994). In addition, the DQM is much more effective for nonlinear problems comparing with low order methods, which was presented by Bellman and Casti (1971), Bellman *et al.* (1972), Bert *et al.* (1989), Feng and Bert (1992), Malik and Civen (1994).

The idea of the DQM is based on the approximation of spatial derivatives of a function at each node by a linear combination of function values at all discrete points in the domain along the coordinate lines. Considering a two-dimensional case, where the domain of the function $f(x,y)$ is a rectangular area, partial derivatives with respect to spatial variables at each point (x_i, y_i) can be expressed in the method as

$$\begin{aligned}
\frac{\partial^n f}{\partial x^n}\Big|_{\substack{x=x_i\\ y=y_j}} &= \sum_{k=1}^{N_x} a_k^{(n)}(x_i) f(x_k, y_j) = \sum_{k=1}^{N_x} a_{ik}^{(n)} f(x_k, y_j) \\
\frac{\partial^m f}{\partial y^m}\Big|_{\substack{x=x_i\\ y=y_j}} &= \sum_{k=1}^{N_y} b_k^{(m)}(y_j) f(x_i, y_k) = \sum_{k=1}^{N_y} b_{jk}^{(m)} f(x_i, y_k)
\end{aligned} \tag{1.1}$$

for $i = 1,\dots,N_x$, $j = 1,\dots,N_y$, where N_x, N_y are the numbers of nodes in the x and y directions, and $a_{ik}^{(n)}$, $b_{jk}^{(m)}$ are the weighting coefficients for the nth and mth order derivatives with respect to appropriate variables. In order to approximate the mixed derivatives, the weighting coefficient matrices for appropriate derivatives with respect to x and y have to be multiplied. Using formula (1.1), one has to collocate a governing equation at each grid point and impose boundary conditions in order to reduce the partial differential equation to a system of algebraic or ordinary differential equations, respectively of the case under consideration. The main stage of the method is the determination of the weighting coefficients.

2. Polynomial and Spline-based differential quadrature method

Values of the weighting coefficients depend on the way the solution is approximated (selection of trial functions), and they influence the accuracy, convergence and stability of the method. The interpolation polynomial is the most often used to approximate the solution in the differential quadrature method (PDQM). Using Lagrange base functions, the rth order derivatives of the interpolation polynomial at the ith discrete point can be expressed as follows

$$P^{(r)}(x_i) = \sum_{j=1}^{N} l_j^{(r)}(x_i) f_j \tag{2.1}$$

where $l_j(x)$ denotes Lagrange's base polynomial of the $(N-1)$th degree.

It is easy to notice that the derivatives of the appropriate Lagrange base functions are the weighting coefficients for the polynomial based differential quadrature method.

The analytical formulas for the coefficients of the first order derivative were given by Quan and Chang (1989) and have following forms

$$\begin{aligned} a_{ij}^{(1)} &= \frac{1}{x_j - x_i} \prod_{k=1,k\neq i,j}^{N} \frac{x_i - x_k}{x_j - x_k} \qquad \text{for} \quad j \neq i \\ a_{ii}^{(1)} &= \sum_{k=1,k\neq i}^{N} \frac{1}{x_i - x_k} \end{aligned} \tag{2.2}$$

Due to difficulties with derivation of explicit formulas for higher order derivatives of Lagrange's base functions, Shu and Richards (1992) gave recurrence relationship (2.3) to overcome the problem, $i, j = 1, 2, \dots, N$, $r = 2, 3, \dots, N-1$

$$\begin{aligned} a_{ij}^{(r)} &= r\Big(a_{ij}^{(1)} a_{ii}^{(r-1)} - \frac{a_{ij}^{(r-1)}}{x_i - x_j}\Big) \qquad \text{for} \quad i \neq j \\ a_{ii}^{(r)} &= -\sum_{j=1, j\neq i}^{N} a_{ij}^{(r)} \end{aligned} \tag{2.3}$$

There are also other ways based on the Fourier series expansion or B-spline functions to approximate the wanted solution in the DQM. But especially PDQM allows one to achieve very high accuracy by using only a few nodes. However, this method is very sensitive to the type of imposed mesh and the number of nodes. When a uniform grid distribution is imposed and too many sampling points are used, the results are inaccurate or the method is not convergent. The computational instability of the method is the result from the manner of the interpolation polynomial. This polynomial oscillates much when its degree $N-1$ (N – number of nodes) is too high, particularly when the uniform grid is imposed. In most problems of computational mechanics, the only way to estimate the accuracy of results is to carry out the computation again using larger number of nodes. Therefore, the computational stability should be the important feature of a numerical method. To improve stability of the DQM, Zhong (2004) used quintic B-spline as trial functions to determine the weighting coefficients. It considerably improves stability of the method but the convergence rate is less comparing to the PDQM.

In the present work, another way of the approximation of the solution is shown. In the presented method, the solution of a partial differential equation is approximated by the nth degree polynomial piecewise function (SDQM). The approximation was first applied in the DQM by Krowiak (2006). The work, that has been done so far, indicates that using a suitably high spline degree the convergence rate of the method is similar to PDQM and the computational stability is much better even when using uniformly spaced nodes.

When the degree n of the spline is odd then the function is approximated in the following way

$$f(x) \approx \{s_i(x),\ x \in [x_i, x_{i+1}],\ i = 1, \dots, N-1\} \tag{2.4}$$

where N is the number of nodes.

When n is even, the auxiliary spline knots are defined at the midpoints of the nodes in order to be able to meet the conditions for the determination of the spline function

$$z_1 = x_1 \qquad z_{i+1} = \frac{1}{2}[x_i + x_{i+1}] \qquad i = 1, \dots, N-1 \qquad z_{N+1} = x_N \tag{2.5}$$

and the interpolation function has the form

$$f(x) \approx \{s_i(x),\ x \in [z_i, z_{i+1}],\ i = 1, \dots, N\} \tag{2.6}$$

In Equations (2.4) and (2.6), the ith spline section is defined as

$$s_i(x) = \sum_{j=0}^{n} c_{ij} x^j \tag{2.7}$$

The coefficients c_{ij} in Equation (2.7) are determined using the interpolation conditions, the continuity conditions of the derivatives at the nodes and the so-called natural end conditions at the domain boundaries. In the case of an odd degree, these conditions have the following form

$$\begin{aligned} &s_i(x_i) = f_i \qquad s_i(x_{i+1}) = f_{i+1} && i = 1, \dots, N-1 \\ &s_i^{(k)}(x_{i+1}) = s_{i+1}^{(k)}(x_{i+1}) && i = 1, \dots, N-2, \quad k = 1, \dots, n-1 \\ &s_1^{(k)}(x_1) = 0 \qquad s_{N-1}^{(k)}(x_N) = 0 && k = \frac{n+1}{2}, \dots, n-1 \end{aligned} \tag{2.8}$$

Their number is $(n+1)(N-1)$, which corresponds to the number of coefficients c_{ij}. In the case of an even spline degree, where the number of unknown coefficients is $(n+1)N$, the auxiliary knots are also used in interpolation conditions (2.9) and the continuity conditions for derivatives (2.10)

$$\begin{aligned} &s_i(x_i) = f_i && i = 1, \dots, N \\ &s_i(z_{i+1}) = s_{i+1}(z_{i+1}) && i = 1, \dots, N-1 \end{aligned} \tag{2.9}$$

and

$$s_i^{(k)}(z_{i+1}) = s_{i+1}^{(k)}(z_{i+1}) \qquad i = 1, \dots, N-1, \quad k = 1, \dots, n-1 \tag{2.10}$$

To complete the set of equations, the natural end conditions are introduced at the end points

$$s_1^{(k)}(x_1) = 0 \qquad s_N^{(k)}(x_N) = 0 \qquad k = \frac{n}{2}, \dots, n-1 \tag{2.11}$$

It insures that in every case of an even spline degree the number of conditions matches the number of coefficients c_{ij}. These coefficients depend on the grid distribution and unknown function values according to the formula

$$c_{ij} = \sum_{k=1}^{N} C_{ijk}(x_1, \dots, x_N) f_k \qquad i = 1, \dots, \overline{N}, \quad j = 0, \dots, n \tag{2.12}$$

where $\overline{N} = N - 1$ when n is odd and $\overline{N} = N$ when n is even.

Taking advantage of the symbolic computation system, one can easily determine the weighting coefficients for the differential quadrature method based on the approximation given by Eqs. (2.4) and (2.6). By manipulating the expressions in which the unknown function values $f(x_i)$ are defined as symbols f_i, it is possible to compute derivatives of an arbitrary degree r $(r < n-1)$ for function (2.4) or (2.6) at the ith discrete point

$$f^{(r)}(x_i) \approx \begin{cases} s_i^{(r)}(x_i) & i = 1, \dots, \overline{N} \\ s_{i-1}^{(r)}(x_i) & i = N \quad \text{when } n \text{ is odd} \end{cases} \tag{2.13}$$

In Eq. (2.13), $s_i^{(r)}(x_i)$ and $s_{N-1}^{(r)}(x_N)$ have the forms

$$\begin{aligned} s_i^{(r)}(x_i) &= \sum_{j=r}^{n} \Big[\Big(\sum_{k=1}^{N} C_{ijk} f_k \Big) x_i^{j-r} \prod_{l=j-r+1}^{j} l \Big] \qquad i = 1, \dots, \overline{N} \\ s_{N-1}^{(r)}(x_N) &= \sum_{j=r}^{n} \Big[\Big(\sum_{k=1}^{N} C_{N-1jk} f_k \Big) x_N^{j-r} \prod_{l=j-r+1}^{j} l \Big] \end{aligned} \tag{2.14}$$

where the advantage has been taken of Equation (2.12).

Equations (2.14) can be expressed in other forms

$$\begin{aligned} s_i^{(r)}(x_i) &= \sum_{k=1}^{N} \Big[\sum_{j=r}^{n} \Big(C_{ijk} x_i^{j-r} \prod_{l=j-r+1}^{j} l \Big) \Big] f_k \qquad i = 1, \dots, \overline{N} \\ s_{N-1}^{(r)}(x_N) &= \sum_{k=1}^{N} \Big[\sum_{j=r}^{n} \Big(C_{N-1jk} x_N^{j-r} \prod_{l=j-r+1}^{j} l \Big) \Big] f_k \end{aligned} \tag{2.15}$$

Comparing Eq. (1.1) to (2.15), it is clear that the expressions in square brackets in the above formula are the weighting coefficients for the rth order derivative in the differential quadrature method based on the spline functions which can be written as follows

$$
\begin{aligned}
a_{ik}^{(r)} &= \sum_{j=r}^{n}\Big(C_{ijk}x_i^{j-r}\prod_{l=j-r+1}^{j} l\Big) && i=1,\dots,\overline{N} \\
a_{Nk}^{(r)} &= \sum_{j=r}^{n}\Big(C_{N-1jk}x_N^{j-r}\prod_{l=j-r+1}^{j} l\Big) && \text{when } n \text{ is odd}
\end{aligned}
\tag{2.16}
$$

3. Free vibration analysis of rectangular plates

Plates belong to basic structural elements in civil and mechanical engineering and, therefore, they are often subjects of static and dynamic research. Many numerical methods have been used in dynamical analysis of plates with various boundary conditions. The obtained results have been used in real structures or as the comparison of accuracy and convergence for applied methods. The conventional differential quadrature method has also been applied to the vibration analysis of plates (Shu and Du, 1997; Wang and Bert, 1993). The results show that the convergence rate of the PDQM is very high. Very accurate results can be obtained applying a grid with points densely concentrated near boundaries. The use of an arbitrary grid, for example a uniform one, makes the results inaccurate or the method not convergent. This drawback can be overcome by using the method presented in this paper.

Since the results and conclusions coming from application of the PDQM to vibration analysis of plates are common and well know, the SDQM has been applied to check its versatility in using various grid point distributions. In the paper, the SDQM has been applied to determine natural frequencies of a thin, isotropic, rectangular plate. The dimensionless governing equation for free vibration of the plate is as follows

$$
\frac{\partial^4 W}{\partial X^4} + 2\lambda^2\frac{\partial^4 W}{\partial X^2 \partial Y^2} + \lambda^4\frac{\partial^4 W}{\partial Y^4} = \Omega^2 W \tag{3.1}
$$

In the above equation W denotes dimensionless mode shape function, $X = x/a$ and $Y = y/b$ are dimensionless coordinates, a and b are lengths of the plate edges, $\lambda = a/b$ is the aspect ratio and Ω is the dimensionless frequency. Its relation with the dimensional circular frequency is following

$$
\Omega = \omega a^2\sqrt{\frac{\rho}{D}} \tag{3.2}
$$

where ρ is the density of the plate material and $D = Eh^3/[12(1-\nu^2)]$ is the flexural rigidity (E, ν, h are Young's modulus, Poisson's ratio and the plate thickness, respectively). Calculations have been done for square plates ($\lambda = 1$) with following configurations of boundary conditions:

(a) SS-F-SS-F

— for $X = 0$ and $X = 1$

$$W = 0 \qquad \frac{\partial^2 W}{\partial X^2} = 0 \tag{3.3}$$

— for $Y = 0$ and $Y = 1$

$$\lambda^2 \frac{\partial^2 W}{\partial Y^2} + \nu \frac{\partial^2 W}{\partial X^2} = 0 \qquad \lambda^2 \frac{\partial^3 W}{\partial Y^3} + (2-\nu) \frac{\partial^3 W}{\partial X^2 \partial Y} = 0 \tag{3.4}$$

(b) C-F-SS-F

— for $X = 0$

$$W = 0 \qquad \frac{\partial W}{\partial X} = 0 \tag{3.5}$$

— for $X = 1$

$$W = 0 \qquad \frac{\partial^2 W}{\partial X^2} = 0 \tag{3.6}$$

— for $Y = 0$ and $Y = 1$

$$\lambda^2 \frac{\partial^2 W}{\partial Y^2} + \nu \frac{\partial^2 W}{\partial X^2} = 0 \qquad \lambda^2 \frac{\partial^3 W}{\partial Y^3} + (2-\nu) \frac{\partial^3 W}{\partial X^2 \partial Y} = 0 \tag{3.7}$$

(c) C-C-C-C

— for $X = 0$ and $X = 1$

$$W = 0 \qquad \frac{\partial W}{\partial X} = 0 \tag{3.8}$$

— for $Y = 0$ and $Y = 1$

$$W = 0 \qquad \frac{\partial W}{\partial Y} = 0 \tag{3.9}$$

where **C** denotes the clamped edge, **SS** – simply supported edge and **F** – free edge.

Chebyshev-Gauss-Lobatto grid (3.10) and a uniform grid have been used to discretize the area $0 \leqslant X \leqslant 1$, $0 \leqslant Y \leqslant 1$. In both cases, the same number of points has been used in the X and Y directions

$$X_i = Y_i = \frac{1}{2}\Big[1 - \cos\Big(\frac{i-1}{N-1}\pi\Big)\Big] \qquad i = 1, \dots, N \tag{3.10}$$

Equation (3.1), written in a discrete form following from the use of the method, is as follows

$$\sum_{k=1}^{N} a_{ik}^{(4)} W_{kj} + 2\lambda^2 \sum_{k_1=1}^{N} \sum_{k_2=1}^{N} a_{ik_1}^{(2)} a_{jk_2}^{(2)} W_{k_1k_2} + \lambda^4 \sum_{k=1}^{N} a_{jk}^{(4)} W_{ik} = \Omega^2 W_{ij} \tag{3.11}$$

where $a_{ik}^{(r)}$ denote the weighting coefficients for the rth order derivative in the SDQM and W_{ij} are unknown nodal values. Identical coefficients $a_{ik}^{(r)}$ approximate the derivatives in both directions due to the use of the same grid and number of nodes in these directions.

The implementation of boundary conditions is the important stage of the method. Various approaches to this problem were presented by Shu and Du (1997). In the present study, the boundary conditions have been used to calculate function values at the boundary points and points adjacent to the boundaries as a linear combination of the values at the interior points. Details are presented for the **SS-F-SS-F** plate configuration. According to the DQM, the discrete form of the boundary conditions described by Eq. (3.3) and (3.4) is following

$$\begin{aligned} & W_{1j} = 0 && j = 1,\dots,N \\ & \sum_{k=1}^{N} a_{1k}^{(2)} W_{kj} = 0 && j = 2,\dots,N-1 \end{aligned} \tag{3.12}$$

$$\begin{aligned} & W_{Nj} = 0 && j = 1,\dots,N \\ & \sum_{k=1}^{N} a_{Nk}^{(2)} W_{kj} = 0 && j = 2,\dots,N-1 \end{aligned} \tag{3.13}$$

$$\begin{aligned} & \lambda^2 \sum_{k=1}^{N} a_{1k}^{(2)} W_{ik} + \nu \sum_{k=1}^{N} a_{ik}^{(2)} W_{k1} = 0 && i = 2,\dots,N-1 \\ & \lambda^2 \sum_{k=1}^{N} a_{1k}^{(3)} W_{ik} + (2-\nu) \sum_{k_1=1}^{N} \sum_{k_2=1}^{N} a_{ik_1}^{(2)} a_{1k_2}^{(1)} W_{k_1k_2} = 0 && i = 3,\dots,N-2 \end{aligned} \tag{3.14}$$

$$\begin{aligned} & \lambda^2 \sum_{k=1}^{N} a_{Nk}^{(2)} W_{ik} + \nu \sum_{k=1}^{N} a_{ik}^{(2)} W_{kN} = 0 && i = 2,\dots,N-1 \\ & \lambda^2 \sum_{k=1}^{N} a_{Nk}^{(3)} W_{ik} + (2-\nu) \sum_{k_1=1}^{N} \sum_{k_2=1}^{N} a_{ik_1}^{(2)} a_{Nk_2}^{(1)} W_{k_1k_2} = 0 && i = 3,\dots,N-2 \end{aligned} \tag{3.15}$$

The function values at the plate edges parallel to the Y axis are defined by the first equation from equation sets (3.12) and (3.13). The rest from (3.12) and (3.13) are used to determine function values at points adjacent to the boundaries in the Y direction as a linear combination of values at the interior points. The discretisation of plate edges parallel to the X axis is similar. Using function values at the boundary points and points adjacent to the boundaries in Eq. (3.1) one have to solve a standard eigenvalue problem to obtain natural frequencies of the plate.

Tabela 1. SS-F-SS-F plate – mesh of the Chebyshev-Gauss-Lobatto type

N	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
$n = 11$					
16	9.634 (0.031%)	16.153 (0.112%)	36.778 (0.142%)	38.987 (0.108%)	46.877 (0.297%)
20	9.632 (0.010%)	16.139 (0.025%)	36.740 (0.038%)	38.957 (0.031%)	46.778 (0.086%)
$n = 14$					
16	9.632 (0.010%)	16.139 (0.025%)	36.741 (0.041%)	38.961 (0.041%)	46.789 (0.109%)
20	9.631 (0.000%)	16.136 (0.006%)	36.729 (0.008%)	38.948 (0.008%)	46.748 (0.021%)
PDQM					
16	9.631 (0.000%)	16.135 (0.000%)	36.726 (0.000%)	38.945 (0.000%)	46.738 (0.000%)
20	9.631 (0.000%)	16.134 (−0.006%)	36.725 (−0.003%)	38.945 (0.000%)	46.738 (0.000%)

The results obtained by the DQM based on spline functions of various degrees n are presented in Table 1, where the grid described by Equation (3.10) has been applied, and in Table 2, where the uniform grid has been used. The tables contain also the results obtained by the conventional differential quadrature method (PDQM). When the uniform grid is used, the PDQM shows computational instability. The application of too many nodes makes the results very inaccurate. The percentage relative error

$$\delta = \frac{\Omega_{SDQM} - \Omega_{reference}}{\Omega_{reference}} \cdot 100\% \tag{3.16}$$

Tabela 2. SS-F-SS-F plate – uniform grid

N	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
$n = 11$					
16	9.666 (0.363%)	16.349 (1.326%)	37.271 (1.484%)	39.201 (0.657%)	47.964 (2.623%)
20	9.651 (0.208%)	16.254 (0 .738%)	37.039 (0.852%)	39.147 (0.519%)	47.508 (1.647%)
24	9.643 (0.125%)	16.208 (0.452%)	36.923 (0.536%)	39.088 (0.367%)	47.242 (1.078%)
$n = 14$					
16	9.642 (0.114%)	16.203 (0.421%)	36.867 (0 .384%)	39.228 (0 .727%)	47.455 (1.534%)
20	9.636 (0.052%)	16.165 (0.186%)	36.809 (0.226%)	39.072 (0.326%)	47.084 (0.740%)
24	9.633 (0.021%)	16.150 (0.093%)	36.775 (0.133%)	39.010 (0.167%)	46.924 (0.398%)
PDQM					
10	9.638 (0.067%)	16.157 (0.136%)	37.740 (2.761%)	38.925 (−0.051%)	47.029 (0.623%)
12	0	0	0	0	9.627
16	0	0	0	0	0

has been calculated for the frequencies on the basis of Leissa's (1973) results, which are exact for the **SS-F-SS-F** plate. The remaining two sets of Leissa's results (**C-F-SS-F**, **C-C-C-C**) were obtained by the Rayleigh-Ritz method with beam functions for the displacement, taking nine terms into account.

The presented results show that the convergence rate of the SDQM is satisfactory when the weighting coefficients determined from the high degree spline functions are used. The accuracy can be improved by increasing the number of grid points.

Analysing the results for the **C-F-SS-F** plate configuration (Table 3 and Table 4), one can notice that the error of calculated frequencies does not decrease monotonically when the spline degree is higher and the number of nodes increases. It is especially noticeable when non-uniform grid (3.6) is applied to insure high rate of convergence. The calculations for another plate with clamped edges (Table 5 and Table 6) confirm that results obtained by the SDQM and PDQM converge to lower values than the reference results obtained by

Leissa (1973). It seems that the results obtained by differential quadrature methods are closer to exact values since the approximate solutions from the Rayleigh-Ritz method are upper bounds on the exact values. It should be noted that in the case of the **SS-F-SS-F** plate, where the reference results are the exact solutions, the values from the differential quadrature methods are in very good agreement.

Tabela 3. C-F-SS-F plate – mesh of the Chebyshev-Gauss-Lobatto type

N	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
			$n = 8$		
16	15.434 (0.975%)	21.682 (4.881%)	43.083 (8.317%)	50.286 (1.118%)	58.753 (3.773%)
20	15.344 (0.386%)	21.259 (2.835%)	41.763 (4.998%)	49.985 (0.513%)	57.833 (2.148%)
24	15.297 (0.079%)	21.043 (1.790%)	41.099 (3.329%)	49.821 (0.183%)	57.346 (1.288%)
			$n = 11$		
16	15.220 (−0.425%)	20.677 (0.019%)	39.885 (0.277%)	49.588 (−0.286%)	56.632 (0.026%)
20	15.201 (−0.550%)	20.617 (−0.271%)	39.793 (0.045%)	49.497 (−0.469%)	56.404 (−0.376%)
24	15.194 (−0.595%)	20.596 (−0.372%)	39.760 (−0.038%)	49.467 (−0.529%)	56.328 (−0.510%)
			$n = 14$		
16	15.209 (−0.497%)	20.644 (−0.140%)	39.831 (0.141%)	49.545 (−0.372%)	56.514 (−0.182%)
20	15.197 (−0.576%)	20.604 (−0.334%)	39.771 (−0.010%)	49.477 (−0.509%)	56.353 (−0.466%)
24	15.193 (−0.602%)	20.591 (−0.397%)	39.750 (−0.063%)	49.458 (−0.547%)	56.304 (−0.553%)
			PDQM		
16	15.193 (−0.602%)	20.602 (−0.343%)	39.776 (0.003%)	49.545 (−0.372%)	56.515 (−0.180%)
20	15.190 (−0.622%)	20.585 (−0.426%)	39.748 (−0.068%)	49.484 (−0.495%)	56.372 (−0.433%)
24	15.191 (−0.615%)	20.582 (−0.440%)	39.738 (−0.093%)	49.464 (−0.535%)	56.320 (−0.525%)

Tabela 4. C-F-SS-F plate – uniform grid

N	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
			$n = 8$		
16	16.114 (5.424%)	25.226 (22.024%)	52.439 (31.839%)	55.282 (11.164%)	65.627 (15.914%)
20	15.901 (4.030%)	24.082 (16.490%)	50.942 (28.075%)	51.994 (4.553%)	63.494 (12.147%)
24	15.773 (3.193%)	23.395 (13.167%)	48.721 (22.492%)	51.416 (3.390%)	62.185 (9.834%)
			$n = 11$		
16	15.408 (0.805%)	21.223 (2.660%)	40.706 (2.341%)	50.204 (0.953%)	58.501 (3.328%)
20	15.333 (0.314%)	21.002 (1.591%)	40.359 (1.468%)	50.011 (0.565%)	57.777 (2.049%)
24	15.290 (0.033%)	20.877 (0.987%)	40.171 (0.996%)	49.862 (0.265%)	57.337 (1.271%)
			$n = 14$		
16	15.359 (0.484%)	21.023 (1.693%)	40.252 (1.199%)	50.221 (0.987%)	57.951 (2.356%)
20	15.295 (0.065%)	20.865 (0.929%)	40.087 (0.784%)	49.905 (0.352%)	57.290 (1.189%)
24	15.260 (−0.164%)	20.778 (0.508%)	39.987 (0.533%)	49.749 (0.038%)	56.953 (0.593%)
			PDQM		
10	15.492 (1.354%)	21.204 (2.569%)	41.353 (3.967%)	50.585 (1.719%)	58.559 (3.430%)
12	0	0	15.406	21.051	27.976
16	0	0	0	0	0

4. Concluding remarks

The problem of free vibration analysis of plates has been undertaken by the author in order to examine the computational stability of the proposed method on various grid distributions.

The presented results show that the convergence rate of the SDQM is high when a high degree spline is used to approximate the solution. The accuracy can be improved by increasing the number of grid points without concern for losing stability of the method. Unlike the PDQM, the method is not limited to

Tabela 5. C-C-C-C plate – mesh of the Chebyshev-Gauss-Lobatto type

N	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
			$n = 8$		
12	36.021 (0.081%)	73.498 (0.116%)	73.498 (0.116%)	108.549 (0.258%)	131.895 (0.194%)
15	35.995 (0.008%)	73.422 (0.012%)	73.422 (0.012%)	108.308 (0.035%)	131.653 (0.001%)
20	35.987 (−0.014%)	73.399 (−0.019%)	73.399 (−0.019%)	108.233 (−0.034%)	131.593 (−0.036%)
25	35.986 (−0.017%)	73.395 (−0.025%)	73.395 (−0.025%)	108.221 (−0.045%)	131.584 (−0.043%)
30	35.985 (−0.019%)	73.394 (−0.026%)	73.394 (−0.026%)	108.218 (−0.048%)	131.582 (−0.044%)
			$n = 11$		
12	35.990 (−0.006%)	73.417 (0.005%)	73.417 (0.005%)	108.288 (0.017%)	131.659 (0.014%)
15	35.986 (−0.017%)	73.397 (−0.022%)	73.397 (−0.022%)	108.228 (−0.039%)	131.595 (−0.034%)
20	35.985 (−0.019%)	73.394 (−0.026%)	73.394 (−0.026%)	108.217 (−0.049%)	131.582 (−0.044%)
25	35.985 (−0.019%)	73.394 (−0.026%)	73.394 (−0.026%)	108.217 (−0.049%)	131.581 (−0.045%)
30	35.985 (−0.019%)	73.394 (−0.026%)	73.394 (−0.026%)	108.217 (−0.049%)	131.581 (−0.045%)
			PDQM		
12	35.986 (−0.017%)	73.399 (−0.019%)	73.399 (−0.019%)	108.230 (−0.037%)	131.418 (−0.169%)
15	35.985 (−0.019%)	73.394 (−0.026%)	73.394 (−0.026%)	108.217 (−0.049%)	131.580 (−0.046%)
20	35.985 (−0.019%)	73.394 (−0.026%)	73.394 (−0.026%)	108.217 (−0.049%)	131.581 (−0.045%)
25	35.985 (−0.019%)	73.394 (−0.026%)	73.394 (−0.026%)	108.216 (−0.050%)	131.581 (−0.045%)
30	35.985 (−0.019%)	73.394 (−0.026%)	73.394 (−0.026%)	108.217 (−0.049%)	131.581 (−0.045%)

Tabela 6. C-C-C-C plate – uniform grid

N	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
			$n = 8$		
12	36.577 (1.625%)	75.061 (2.245%)	75.061 (2.245%)	112.497 (3.904%)	136.144 (3.421%)
15	36.308 (0.878%)	74.271 (1.169%)	74.271 (1.169%)	110.668 (2.215%)	133.710 (1.572%)
20	36.132 (0.389%)	73.788 (0.511%)	73.788 (0.511%)	109.407 (1.050%)	132.445 (0.612%)
25	36.065 (0.203%)	73.608 (0.266%)	73.608 (0.266%)	108.888 (0.571%)	132.029 (0.296%)
30	36.033 (0.114%)	73.523 (0.150%)	73.523 (0.150%)	108.633 (0.335%)	131.847 (0.157%)
			$n = 11$		
12	36.109 (0.325%)	73.680 (0.364%)	73.680 (0.364%)	109.419 (1.061%)	128.917 (−2.069%)
15	36.037 (0.125%)	73.602 (0.257%)	73.602 (0.257%)	108.887 (0.570%)	131.528 (−0.085%)
20	36.002 (0.028%)	73.475 (0.084%)	73.475 (0.084%)	108.466 (0.181%)	131.795 (0.118%)
25	35.992 (0%)	73.428 (0.020%)	73.428 (0.020%)	108.323 (0.049%)	131.699 (0.045%)
30	35.988 (−0.011%)	73.410 (−0.004%)	73.410 (−0.004%)	108.267 (−0.003%)	131.642 (0.002%)
			PDQM		
10	36.021 (0.081%)	72.996 (−0.568%)	72.996 (−0.568%)	108.348 (0.072%)	130.891 (−0.569%)
12	35.992 (0%)	73.474 (0.083%)	73.474 (0.083%)	78.331 (−27.651%)	78.331 (−40.496%)
15	0	0	0	0	0

special types of grids. Various types of grid point distributions can be applied in the SDQM giving differences only in the convergence rate of the method. These make the SDQM more versatile than the PDQM and in author's opinion it is a good starting point to applying the method to more challenging engineering problems, even nonlinear ones. In the latter, the high rate of convergence and stability of the SDQM should be a great advantage.

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Metody oparte na kwadraturach różniczkowych w zastosowaniu do drgań płyt

Streszczenie

Praca dotyczy metod opartych na kwadraturach różniczkowych i ich aplikacji do zagadnienia drgań własnych płyt. W pracy, jako alternatywę do znanych metod kwadratur różniczkowych, opartych na wielomianie interpolacyjnym (PDQM), przedstawiono metodę bazującą na funkcjach sklejanych (SDQM). W SDQM poszukiwane rozwiązanie przybliżane jest funkcją wielomianową, przedziałami zmienną. W pracy przedstawiono sposób wyznaczenia takiej funkcji interpolacyjnej, jak również sposób obliczenia współczynników wagowych, używanych w metodzie kwadratur różniczkowych. Następnie SDQM użyto do wyznaczenia częstości drgań własnych płyt, gdzie analizowano wpływ stopnia wielomianu, liczby węzłów i ich rozmieszczenia na zbieżność, dokładność i stabilność metody. Otrzymane rezultaty porównano z wynikami uzyskanymi przy pomocy konwencjonalnej metody kwadratur różniczkowych (PDQM).

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