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EXTENSION OF THE HAMILTONIAN APPROACH WITH GENERAL INITIAL CONDITIONS

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In this paper, the Hamiltonian approach is extended for solving vibrations of nonlinear conservative oscillators with general initial conditions. Based on the assumption that the derivative of Hamiltonian is zero, the frequency as a function of the amplitude of vibration and initial velocity is determined. A method for error estimation is developed and the accuracy of the approximate solution is treated. The procedure is based on the ratio between the average residual function and the total energy of the system. Two computational algorithms are described for determining the frequency and the average relative error. The extended Hamiltonian approach presented in this paper is applied for two types of examples: Duffing equation and a pure nonlinear conservative oscillator.

Keywords: nonlinear dynamics, Hamiltonian approach, error estimation

1. Introduction

In 2010, the method denominated "Hamiltonian approach" was introduced by He (2010) for solving strong nonlinear oscillatory systems. Since that time, a significant number of papers, where the suggested method is extended and applied, have been published. In the papers of Akbarzade and Kargar (2011a) and Akbarzade and Kargar (2011), the Hamiltonian method is applied for obtaining accurate analytical solutions to nonlinear oscillators. Using this method, He *et al.* (2010) and Bayat *et al.* (2014) obtained the solution for the Duffing-harmonic equation, and Cveticanin *et al.* (2010a, 2012) derived solutions for the generalized nonlinear oscillator with a fractional power. Belendez *et al.* (2011) pointed out that if a first-order trigonometric approximation function was used then there was an equivalence between the Hamiltonian approach and the first-order harmonic balance method. Yildirim *et al.* (2011b) and Belendez *et al.* (2011) specified the method for oscillators with rational and irrational elastic terms. Xu and He (2010) used the Hamiltonian approach to determine the limit cycle motion for strongly nonlinear oscillators. The method was applied for solving some practical problems, such as nonlinear oscillations of a punctual charge in the electric field of a charged ring (Yildirim *et al.*, 2011a), nonlinear vibrations of micro electro mechanical systems (Sadeghzadeh and Kabiri, 2016), nonlinear oscillations in an engineering structure (Akbarzade and Khan, 2012), among others. The Hamiltonian approach has also been applied to improve the accuracy of the solution for nonlinear oscillators, such as higher order approximations (Durmaz *et al.*, 2010; Yildirim *et al.*, 2011c), multiple coupled nonlinear oscillators (Durmaz *et al.*, 2012) and multiple-parameter ones (Khan *et al.*, 2011), among others.

In this study, the accuracy of the Hamiltonian approach is presented by comparing an analytically obtained solution with the numerical solution. Navarro and Cveticanin (2016) used the

method in oscillators with integer or non-integer terms. The authors proposed an error estimation method based on the ratio between the square root of the averaged residual function and the initial constant of energy.

The aim of this paper is to extend the Hamiltonian approach for solving a conservative nonlinear oscillator

$$\ddot{x} + f(x) = 0 \tag{1.1}$$

with the generalized initial conditions

$$x(0) = A \qquad \dot{x}(0) = v \tag{1.2}$$

where $f(x)$ is a conservative force. The authors also propose an error procedure to measure the quantitative difference between approximate and numerical solutions. In the present work, the error treatment described by Navarro and Cveticanin (2016) is extended for general initial conditions.

This paper is divided in five Sections. After the Introduction, an extended Hamiltonian approach is considered, in Section 2. In that Section, Algorithm 1, called FREQUENCY, is developed. In Section 3, a method for error estimation is introduced. The error is estimated as the integral of the square of the residual function over the period of vibration. Special attention is given to the computation of relative errors. The error calculation is performed using Algorithm 2, called DELTA. In Section 4, a comparison between approximate and numerical solutions is shown. The suggested method of the Hamiltonian approach and error estimation are applied for a Duffing equation and for a pure nonlinear conservative oscillator with various orders of nonlinearity. Finally, conclusions of the numerical experiments are presented.

2. The approximate solution

Equation (1.1) describes a conservative oscillator with kinetic energy

$$K = \frac{\dot{x}^2}{2} \tag{2.1}$$

and potential energy, $F(x)$, is given by

$$\frac{dF(x)}{dx} = f(x) \tag{2.2}$$

The total mechanical energy of the oscillator corresponds to its Hamiltonian

$$H = \frac{1}{2}\dot{x}^2 + F(x) \tag{2.3}$$

For the conservative oscillator, the total mechanical energy keeps unchanged during motion and, consequently, the Hamiltonian of the system is constant, i.e., $H = H_0 = \text{const}$, and due to initial conditions (1.2)

$$H_0 = \frac{1}{2}v^2 + F(A) \tag{2.4}$$

For nonlinear oscillators, it is generally impossible to obtain a closed form analytical solution to (2.3). Consequently, to obtain an approximate solution, we assumed a trigonometric trial solution in the form

$$x(t) = C\cos(\omega t - \phi) \tag{2.5}$$

where ω is the unknown frequency of vibration, C is the amplitude of the oscillator, and ϕ is the phase of the oscillator. Using Eq. (2.5) and initial conditions (1.2), we find the maximum value of $x(t)$ and the phase given, respectively, by

$$C = \sqrt{A^2 + \left(\frac{v}{\omega}\right)^2} \tag{2.6}$$

and

$$\phi = \tan^{-1}\left(\frac{v}{\omega A}\right) \tag{2.7}$$

Substituting (2.5) into (2.3), we have

$$\widetilde{H} = \frac{1}{2}C^2\omega^2 \sin^2(\omega t - \phi) + F(C\cos(\omega t - \phi)) \tag{2.8}$$

Usually, the frequency ω is determined from the derivative of Eq. (2.8)

$$\frac{\partial \widetilde{H}}{\partial C} = 0 \tag{2.9}$$

Unfortunately, the obtained result is far from being accurate. To overcome this problem, He (2010) developed the so called Hamiltonian approach. He introduced a new function, $\overline{H}$ that has a similar form of

$$\overline{H} = \int_{t_1}^{t_1+T} \widetilde{H}\, dt \tag{2.10}$$

where T is the period of vibration and t_1 is an arbitrary initial integration time. After integration, it is $\overline{H} = \widetilde{H}T$, i.e.

$$\widetilde{H} = \frac{\partial \overline{H}}{\partial T} \tag{2.11}$$

According to (2.9) and $T = 2\pi/\omega$, it is

$$\frac{\partial}{\partial C}\left(\frac{\partial \overline{H}}{\partial T}\right) = \frac{\partial}{\partial C}\left(\frac{\partial \overline{H}}{\partial \omega^{-1}}\right) = \frac{\partial}{\partial C}\left(\frac{\partial \overline{H}}{\partial \omega}\right) = 0 \tag{2.12}$$

Solving algebraic equation (2.12), the approximate frequency of the conservative nonlinear oscillator is obtained in function of the amplitude C (see He, 2010). In most cases, by using relations (2.6) and (2.7), we can expressed an explicit relationship for approximate frequency in function of A, v, and other parameters of the system. Thus, for

$$\overline{H}(x, \dot{x}) = \int_{t_1}^{t_1+T} \left(\frac{1}{2}\dot{x}^2 + F(x)\right) dt \tag{2.13}$$

and trial function (2.5), we can express $\overline{H}$ as

$$\overline{H}(C, \omega) = \int_{t_1}^{t_1+T} \left[\frac{1}{2}C^2\omega^2 \sin^2(\omega t - \phi) + F(C\cos(\omega t - \phi))\right] dt \tag{2.14}$$

Substituting in Eq. (2.14) the variable $\psi = \omega t$ and the period of vibration $T = 2\pi/\omega$, and defining $\psi^* = t_1/T$, we have

$$\overline{H}(C,\omega) = \int_{2\pi\psi^*}^{2\pi(1+\psi^*)} \Big[\frac{1}{2}C^2\omega\sin^2(\psi-\phi) + \omega^{-1}F(C\cos(\psi-\phi))\Big]\, d\psi \tag{2.15}$$

where ψ^* is an arbitrary value that can be defined as $\psi^* = n/m$, with $n \in \mathbb{N}$ and $m \in \mathbb{N}^*$, to avoid numerical errors. The trivial choice for ψ^* is $\psi^* = 0$ ($t_1 = 0$) so that Eq. (2.15) becomes

$$\overline{H}(C,\omega) = \int_{0}^{2\pi} \Big[\frac{1}{2}C^2\omega\sin^2(\psi-\phi) + \omega^{-1}F(C\cos(\psi-\phi))\Big]\, d\psi \tag{2.16}$$

Equations (2.12) and (2.16) can be used to determine a relationship for approximate frequency in function of C.

We can see that Eq. (2.5) reaches its maximum value when $\omega t - \phi = 2\pi n$, where $n = 0, \pm1, \pm2, \ldots$. The maximum value for $n = 0$ occurs when $t = \phi/\omega$. Adopting the initial time $t_1 = \phi/\omega$ and substituting the variable $\tau = t - t_1$ into Eq. (2.14), we obtain

$$\overline{H}(C,\omega) = \int_{0}^{T} \Big[\frac{1}{2}C^2\omega^2\sin^2(\omega\tau) + F(C\cos(\omega\tau))\Big]\, d\tau \tag{2.17}$$

As before, substituting into Eq. (2.17) the variables $\psi = \omega\tau$ and $T = 2\pi/\omega$, we have

$$\overline{H}(C,\omega) = \int_{0}^{2\pi} \Big[\frac{1}{2}C^2\omega\sin^2(\psi) + \omega^{-1}F(C\cos(\psi))\Big]\, d\psi \tag{2.18}$$

We can note that Eq. (2.18) is the common expression used for determining the frequency-amplitude relationship with particular initial conditions $x(0) = A$, $\dot{x}(0) = 0$ in works developed by several authors (e.g., Akbarzade and Kargar, 2011a,b; He, 2010; Akbarzade and Khan, 2012; Askari, 2013; Belendez *et al.*, 2011; Bayat *et al.*, 2014; Cveticanin *et al.*, 2010a, 2012; Durmaz *et al.*, 2010, 2012; He *et al.*, 2010; Khan *et al.*, 2011; Navarro and Cveticanin, 2016; Sadeghzadeh and Kabiri, 2016; Xu and He, 2010; Yildirim *et al.*, 2011a,b,c, 2012), i.e, when $C = A$ and $\phi = 0$. Comparing Eq. (2.16) with Eq. (2.18), it is observed that both can be used with Eq. (2.12) to determine a relationship for the approximate frequency in function of C and of other parameters of the system. Therefore, it is recommended to use Eq. (2.18) because of the following reasons: (a) it is simpler; (b) it is easy to determine the variation of the signal of the function potential F with the angle ψ, i.e., depending on the signal of F, the limits of integration can vary, for example, from 0 to π or 0 to $\pi/2$.

In some cases, an explicit relationship for the approximate frequency (that is a function of the initial amplitude A, the initial velocity v and parameters of the system) can be derived by Eq. (2.18) (frequency in function of C and parameters of the system) and Eq. (2.6). When it is not possible to derive an explicit relation, the values of the frequency, for specific values of A and v, can be obtained numerically by Algorithm 1, where *cs* are generic parameters of the system. After computation of the variables ω and C by Algorithm 1, the phase ϕ is determined by Eq. (2.7).

The approximate solution, Eq. (2.5), can also be represented by

$$x(t) = C\sin(\omega t + \phi_1) \tag{2.19}$$

where the phase is given by $\phi_1 = \pi/2 - \phi$, i.e,

$$\phi_1 = \tan^{-1}\Big(\frac{\omega A}{v}\Big) \tag{2.20}$$

Algorithm 1 Frequency

```
procedure FREQUENCY(A, v, cs)
    C2 ← sqrt(A^2 + v^2)
    repeat
        ω1 ← ω(C2, cs)                          ▷ Using Eqs. (2.12) and (2.18)
        C1 ← sqrt(A^2 + (v/ω1)^2)               ▷ Using Eq. (2.6)
        ω2 ← ω(C1, cs)                          ▷ Using Eqs. (2.12) and (2.18)
        C2 ← sqrt(A^2 + (v/ω2)^2)               ▷ Using Eq. (2.6)
    until |(ω2 − ω1)/ω2| < Tolerance
    return ω2, C2
```

3. Error estimation

The Hamiltonian calculated for approximate solution (2.5) differs from the total mechanical energy H_0. The difference between $\widetilde{H}$ and H_0 gives us the instantaneous residual

$$R(t) = \widetilde{H} - H_0 = \frac{1}{2}C^2\omega^2 \sin^2(\omega t - \phi) + F(C\cos(\omega t - \phi)) - \Big(\frac{1}{2}\dot{v}^2 + F(A)\Big) \tag{3.1}$$

As the residual varies in time, to determine the average value, the error is estimated as the integral of the square of the residual function over the period of vibration. The following functional is introduced for error estimation

$$\Delta_2 = \frac{1}{T}\int_{t_1}^{t_1+T} R(t)^2\, dt = \frac{1}{T}\int_{t_1}^{t_1+T} (\widetilde{H} - H_0)^2\, dt \tag{3.2}$$

Substituting (3.1) into (3.2), we have

$$\Delta_2 = \frac{1}{T}\int_{t_1}^{t_1+T} \Big[\frac{1}{2}C^2\omega^2 \sin^2(\omega t - \phi) + F(C\cos(\omega t - \phi)) - \Big(\frac{1}{2}\dot{v}^2 + F(A)\Big)\Big]^2\, dt \tag{3.3}$$

using $t_1 = \phi/\omega$ and $\tau = t - t_1$

$$\Delta_2 = \frac{1}{T}\int_{0}^{T} \Big[\frac{1}{2}C^2\omega^2 \sin^2(\omega\tau) + F(C\cos(\omega\tau)) - \Big(\frac{1}{2}\dot{v}^2 + F(A)\Big)\Big]^2\, d\tau \tag{3.4}$$

and for $\psi = \omega\tau$, $T = 2\pi/\omega$, and Eq.(2.6)

$$\Delta_2 = \frac{1}{2\pi}\int_{0}^{2\pi} \Big[\frac{1}{2}(A^2\omega^2 + v^2)\sin^2(\psi) + F\Big(\frac{1}{\omega}\sqrt{A^2\omega^2 + v^2}\cos(\psi)\Big) - \Big(\frac{1}{2}\dot{v}^2 + F(A)\Big)\Big]^2\, d\psi \tag{3.5}$$

The obtained solution is a function of the initial amplitude, initial velocity and coefficients of the system: $\Delta_2 = \Delta_2(A, v, c_k)$. The relative error is calculated as the ratio between the square root of the average residual function and the initial constant energy function H_0

$$\Delta = \frac{\sqrt{\Delta_2}}{H_0} \tag{3.6}$$

The relative error is suitable to be presented in the percent form: $\Delta_{\%} = (\sqrt{\Delta_2}/H_0)100\%$. When the explicit relation for the frequency is not available, the average relative error Δ can be determined numerically by Algorithm 2, where *cs* are generic parameters of the system. Algorithm 2, DELTA, calls Algorithm 1 and uses the integral given by Eq. (3.4).

Next, we apply the procedure proposed in Sections 2 and 3 in two examples: Duffing equation and a pure nonlinear conservative oscillator.

Algorithm 2 Delta

procedure DELTA(A, v, cs)
  $\omega = f(A, v, cs)$ ▷ Analytically or using Algorithm 1
  Compute C ▷ Eq. (2.6)
  Create vector: $\tau_i \leftarrow 0 : \Delta\tau : T$
  $y_i \leftarrow C\cos(\omega\tau_i)$
  $\dot{y}_i \leftarrow C\omega\sin(\omega\tau_i)$
  Compute H_0 ▷ Eq. (2.4)
  $\widetilde{H}_i \leftarrow \frac{1}{2}\dot{y}_i^2 + F(y_i)$ ▷ F is obtained using Eq. (2.2)
  $R_i \leftarrow \widetilde{H}_i - H_0$
  $\Delta_2 \leftarrow \frac{1}{n}\sum_{i=1}^{n} {R_i}^2$ ▷ Eq. (3.4)
  $\Delta \leftarrow \frac{\sqrt{\Delta_2}}{H_0}$
  return Δ

4. Comparison between approximate and numerical solutions

Consider an oscillator represent by the Duffing equation

$$\ddot{x} + c_1^2 x + c_2^2 x^3 = 0 \tag{4.1}$$

with initial conditions (1.2). The origin of the name of this equation is shown by Cveticanin (2013). Translation of sections from Duffing's original book Duffing (1918) is found in the work of Kovacic and Brennan (2011). Considering Eq. (4.1), the potential energy is written as

$$F(x) = \frac{1}{2}c_1^2 x^2 + \frac{1}{4}c_2^2 x^4 \tag{4.2}$$

and the Hamiltonian of the oscillator is expressed by

$$H = \frac{1}{2}\dot{x}^2 + \frac{1}{2}c_1^2 x^2 + \frac{1}{4}c_2^2 x^4 \tag{4.3}$$

For an approximate solution (2.5), function (2.18) becomes

$$\begin{aligned}\overline{H}(C,\omega) &= 4\int_0^{\pi/2}\Big[\frac{1}{2}C^2\omega\sin^2(\psi) + \omega^{-1}\Big(\frac{1}{2}c_1^2C^2\cos^2(\psi) + \frac{1}{4}c_2^2C^4\cos^4(\psi)\Big)\Big]\,d\psi \\ &= \frac{C^2\pi(8c_1^2 + 3C^2c_2^2 + 8\omega^2)}{16\omega}\end{aligned} \tag{4.4}$$

setting Eq. (2.12)

$$\frac{\partial}{\partial C}\Big(\frac{\partial\overline{H}}{\partial\omega}\Big) = 2\pi C - \frac{3\pi c_2^2C^3}{8\omega^2} - \frac{\pi C(8c_1^2 + 3c_2^2C^2 + 8\omega^2)}{8\omega^2} = 0 \tag{4.5}$$

we obtain an approximate frequency relationship with the maximum amplitude

$$\omega = \sqrt{c_1^2 + \frac{3}{4}c_2^2C^2} \tag{4.6}$$

In this case, we can write an explicit relationship for the approximate frequency in function of the initial amplitude A, initial velocity v, and parameters c_1 and c_2. Thus, using Eqs. (2.6) and (4.6), the approximate frequency is given by

$$\omega = \sqrt{\frac{c_1^2}{2} + \frac{3}{8}c_2^2A^2 + \sqrt{\Big(\frac{c_1^2}{2} + \frac{3}{8}c_2^2A^2\Big)^2 + \frac{3}{4}c_2^2v^2}} \tag{4.7}$$

and according Eq. (2.7), the phase is

$$\phi = \tan^{-1}\left(\frac{v}{A\sqrt{\frac{c_1^2}{2}+\frac{3}{8}c_2^2A^2+\sqrt{\left(\frac{c_1^2}{2}+\frac{3}{8}c_2^2A^2\right)^2+\frac{3}{4}c_2^2v^2}}}\right) \tag{4.8}$$

If $v = 0$, the approximate frequency and phase become, respectively,

$$\omega = \sqrt{c_1^2 + \frac{3}{4}c_2^2A^2} \qquad \phi = 0 \tag{4.9}$$

and for $A = 0$

$$\omega = \frac{\sqrt{2}}{2}\sqrt{c_1^2+\sqrt{c_1^4+3c_2^2v^2}} \qquad \phi = \frac{\pi}{2} \tag{4.10}$$

The approximate period is computed by

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{c_1^2}{2}+\frac{3}{8}c_2^2A^2+\sqrt{\left(\frac{c_1^2}{2}+\frac{3}{8}c_2^2A^2\right)^2+\frac{3}{4}c_2^2v^2}}} \tag{4.11}$$

Now, we compare approximate period (4.11) with the numerically obtained one. The 'exact' numerical period of vibration T_n is calculated using the numerical solution of Eq. (4.1) with initial conditions (1.2) by the Runge-Kutta method. The vibration period T_n is computed for the time in the interval $0 \leqslant t \leqslant T$ with a time-step of Δt, where T is the approximate period, see Eq. (4.11). The relative error in percentage is given by

$$\Delta_T = \left|\frac{T_n - T}{T_n}\right| \cdot 100\% \tag{4.12}$$

Figure 1 illustrates the variation of the relative error in percentage versus the initial amplitude A for three values of the initial velocity, i.e., $v = 0$ (solid line), $v = 10$ (dashed line) and $v = 100$ (dotted line). In Fig. 1, the parameters of the system are $c_1^2 = c_2^2 = 1$ and the amplitude A varies in the interval $0 \leqslant A \leqslant 50$ with an increment $\Delta A = 0.5$. The time-step is $\Delta t = 10^{-5}T$. We can note that for $v = 0$ the relative error has a monotonic growth, while for $v = 10$ and $v = 100$ the relative error starts with a higher value and then oscillates converging to $A = 50$.

Table 1 shows the relative errors in percentage, Eq. (4.12), for the parameters $c_1^2 = c_2^2 = 1$ and for several values of initial conditions, amplitude A and velocity v. In Table 1 the time-step used is $\Delta t = 10^{-7}T$. The numbers in Table 1 are rounded to three decimal places. The same behavior obtained in Figure 1 is shown in Table 1. We can also see a monotonic growth for $v = 0$. For values of $v > 0$, the relative error starts in higher values for $A = 0$ and then, increasing A, the relative error decreases up to a minimum value. After that, the relative error starts to increase converging to 2.172% when $A \to \infty$. We can also observe that for high values of v and $A = 0$, the error is larger and this error converges to 8.264% ($A = 0$, $v \to \infty$).

Now, an estimative for the average error between the approximate and exact solutions is computed. For the initial conditions $x(0) = A$ and $\dot{x}(0) = v$, the Hamiltonian of the system is

$$H_0 = \frac{1}{2}v^2 + \frac{1}{2}c_1^2A^2 + \frac{1}{4}c_2^2A^4 \tag{4.13}$$

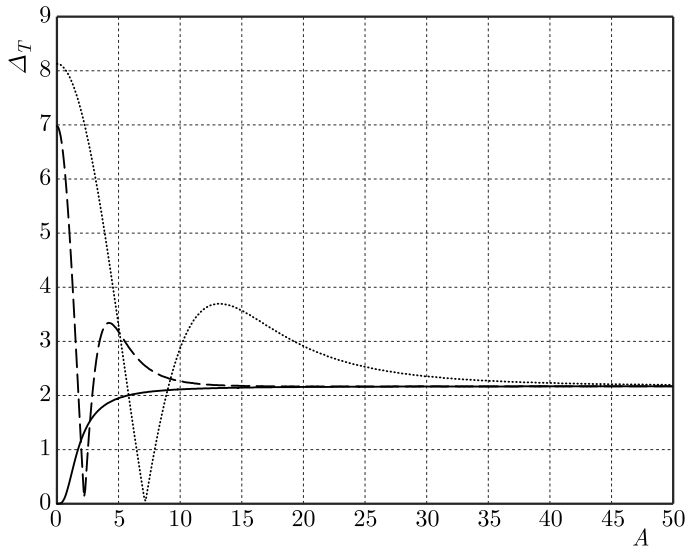

Fig. 1. Relative errors Δ_T, Eq. (4.12), versus the amplitude A for $v = 0$ (solid line), $v = 10$ (dashed line) and $v = 100$ (dotted line)

Table 1. Relative errors Δ_T, Eq. (4.12), for $c_1^2 = c_2^2 = 1$ and various initial conditions A, v

v \ A	0	1	5	10	50	100	500	1000
0	—	0.385	1.949	2.113	2.170	2.172	2.172	2.172
1	1.354	1.010	1.972	2.115	2.170	2.172	2.172	2.172
5	5.861	2.807	2.431	2.152	2.170	2.172	2.172	2.172
10	6.977	5.070	3.179	2.263	2.170	2.172	2.172	2.172
50	7.993	7.542	0.106	3.615	2.176	2.172	2.172	2.172
100	8.127	7.898	3.294	2.873	2.196	2.173	2.172	2.172
500	8.237	8.190	7.108	4.180	2.707	2.212	2.172	2.172
1000	8.250	8.227	7.675	6.067	3.494	2.327	2.172	2.172

Substituting approximate solution (2.5) into (4.3) and using relations (3.2), (4.13), the error function, according to (3.5), is

$$\Delta_2 = \frac{2}{\pi}\int_0^{\pi/2}\Big[\frac{1}{2}C^2\omega^2\sin^2(\psi) + \frac{1}{2}c_1^2C^2\cos^2(\psi) + \frac{1}{4}c_2^2C^4\cos^4(\psi) - \Big(\frac{1}{2}v^2 + \frac{1}{2}c_1^2A^2 + \frac{1}{4}c_2^2A^4\Big)\Big]^2\,d\psi \tag{4.14}$$

Using Eqs. (2.6) and (4.7) and after some calculation, Δ_2 is written as

$$\Delta_2 = \frac{A^4c_1^4}{6} + \frac{23c_1^8}{1296c_2^4} + \frac{A^2c_1^6}{12c_2^2} + \frac{1}{8}A^6c_1^2c_2^2 + \frac{135A^8c_2^4}{4096} + \frac{85}{576}A^2c_1^2v^2 + \frac{23c_1^4v^2}{432c_2^2} + \frac{107}{768}A^4c_2^2v^2 + \frac{23v^4}{1152} - \frac{263A^4c_1^2R}{9216} - \frac{23c_1^6R}{5184c_2^4} - \frac{121A^2c_1^4R}{6912c_2^2} - \frac{121A^6c_2^2R}{12288} - \frac{49A^2v^2R}{4608} - \frac{23c_1^2v^2R}{3456c_2^2} \tag{4.15}$$

where

$$R = \sqrt{16c_1^4 + 24A^2c_1^2c_2^2 + 9A^4c_2^4 + 48c_2^2v^2} \tag{4.16}$$

and according to Eqs. (3.6) and (4.13), the error Δ is given by

$$\Delta = \frac{\sqrt{\Delta_2}}{\frac{1}{2}v^2 + \frac{1}{2}c_1^2A^2 + \frac{1}{4}c_2^2A^4} \tag{4.17}$$

As before, Fig. 2 illustrates the relative errors Δ, Eq. (4.17) versus the initial amplitude $0 \leqslant A \leqslant 50$ ($\Delta A = 0.1$) for three values of the initial velocity, i.e., $v = 0$ (solid line), $v = 10$ (dashed line) and $v = 100$ (dotted line), with $c_1^2 = c_2^2 = 1$. The same pattern shown in Fig. 1 is also obtained for the average relative error Δ.

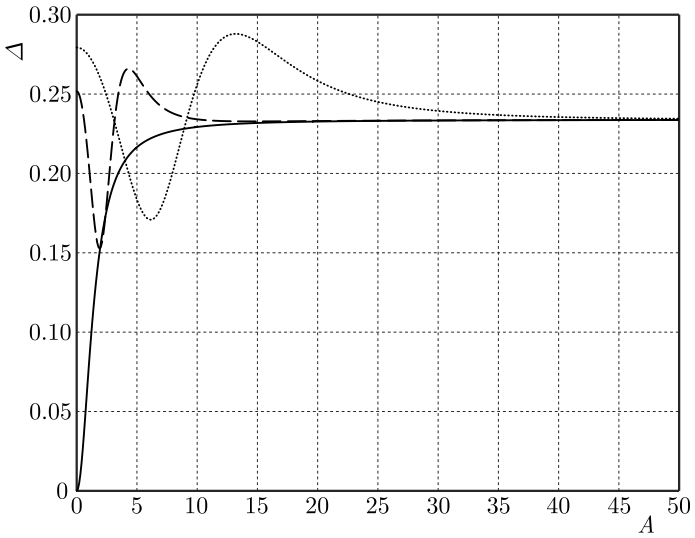

Fig. 2. Relative errors Δ, Eq. (4.17), versus the amplitude A, for $v = 0$ (solid line), $v = 10$ (dashed line) and $v = 100$ (dotted line)

4.1. Pure nonlinear conservative oscillator

Considering the following equation for a pure nonlinear conservative oscillator

$$\ddot{x} + c_1^2 x|x|^{\alpha-1} = 0 \tag{4.18}$$

where α is the order of nonlinearity, integer or non-integer. There are several studies of oscillators where the nonlinearity has an order which is any rational number (integer or non-integer) (see, for example, Cveticanin *et al.*, 2010a,b, 2012; Herişanu and Marinca, 2010; Kovacic *et al.*, 2010; Cveticanin, 2014; Cveticanin and Pogany, 2012). According to (2.5), the approximate solution is

$$x(t) = C\cos(\omega t - \phi) \tag{4.19}$$

and using Eqs. (2.12) and (2.18), the frequency is

$$\omega^2 = c_1^2|C|^{\alpha-1}K_\alpha \tag{4.20}$$

with

$$K_\alpha = \frac{2}{\pi}\mathrm{B}\Big(\frac{1}{2}, \frac{\alpha+2}{2}\Big) \tag{4.21}$$

where B is the Euler beta function. If $v = 0$, the approximate frequency and phase become, respectively,

$$\omega = \sqrt{c_1^2|A|^{\alpha-1}K_\alpha} \qquad \phi = 0 \tag{4.22}$$

for $A = 0$

$$\omega = \left(c_1^2 |v|^{\alpha-1} K_\alpha\right)^{\frac{1}{\alpha+1}} \qquad \phi = \frac{\pi}{2} \tag{4.23}$$

and for general values of A and v, the frequency is calculated using Eq. (4.20) and Algorithm 1. Then, the amplitude C and phase ϕ are determined by Eqs. (2.6) and (2.7), respectively. The approximate period is given by $T = 2\pi/\omega$, where the frequency is determined by Algorithm 1. The approximate period T is compared with T_n obtained numerically solving Eq. (4.18) with initial conditions (1.2) by the Runge-Kutta method in the time interval $0 \leqslant t \leqslant T$ with a time-step $\Delta t = 10^{-7}T$. The relative error Δ_T is computed by Eq.(4.12). The average relative error Δ is calculated by Algorithm 2. In Table 2, the value of relative errors Δ_T and Δ for various values of α and for $c_1^2 = 1$ are shown. When the initial velocity is zero, the errors Δ_T and Δ are independent of the initial amplitude A and of the constant c_1^2. Table 2 shows that for $\alpha < 1$ the error decreases with an increase in α, and for $\alpha > 1$ occurs the reverse, i.e., the error increases with α. For the linear case, $\alpha = 1$, the approximate solution corresponds to the exact solution, $x(t) = C\cos(|c_1|t - \phi)$ and, apart from this value ($\alpha = 0$ or $\alpha = 3$), the relative errors increase due to nonlinear effects. Navarro and Cveticanin (2016) studied the solutions when $v = 0$ and showed that for high values of α the relative errors diverge. We can also note that $\Delta_T(A, 0) = \Delta_T(A \to \infty, v)$ and $\Delta(A, 0) = \Delta(A \to \infty, v)$ for a specific value of α. These equalities can be observed in Table 2 for $v = 10$.

Table 2. Values of relative errors $\Delta_T(A, v)$ and $\Delta(A, v)$ for various values of the power α

α	0	1/2	1	3/2	2	5/2	3
$\Delta_T(A, 0)$	1.5649	0.3103	0	0.2117	0.7203	1.3987	2.1723
$\Delta(A, 0)$	0.11096	0.05957	0	0.06079	0.12047	0.17823	0.23385
$\Delta_T(0, 10)$	23.370	2.7389	0	1.1411	3.3583	5.8275	8.2642
$\Delta(0, 10)$	0.52475	0.17855	0	0.11022	0.18556	0.24056	0.28260
$\Delta_T(1, 10)$	22.164	2.5547	0	0.9867	2.7627	4.5399	6.0783
$\Delta(1, 10)$	0.49605	0.16669	0	0.09817	0.16116	0.20409	0.2379
$\Delta_T(5, 10)$	17.866	1.3809	0	0.1548	1.1174	2.4310	3.4975
$\Delta(5, 10)$	0.39674	0.09721	0	0.05788	0.13734	0.21773	0.28097
$\Delta_T(10, 10)$	13.516	0.3133	0	0.3714	1.0361	1.6631	2.3273
$\Delta(10, 10)$	0.30175	0.05655	0	0.07108	0.13361	0.18736	0.23881
$\Delta_T(20, 10)$	7.4054	0.4490	0	0.2667	0.7731	1.4251	2.1824
$\Delta(20, 10)$	0.18253	0.06318	0	0.06402	0.12251	0.17910	0.23417
$\Delta_T(50, 10)$	0.1732	0.5300	0	0.2181	0.7238	1.3998	2.1725
$\Delta(50, 10)$	0.10583	0.06565	0	0.06115	0.12060	0.17826	0.23386
$\Delta_T(A \to \infty, 10)$	1.5649	0.3103	0	0.2117	0.7203	1.3987	2.1723
$\Delta(A \to \infty, 10)$	0.11096	0.05957	0	0.06079	0.12047	0.17823	0.23385

Figures 3 and 4 show, respectively, the period relative errors Δ_T and the average relative errors Δ (Algorithm 2) versus the initial amplitude $0 \leqslant A \leqslant 50$ ($\Delta A = 0.1$) for the initial velocity $v = 10$, parameter $c_1^2 = 1$, and three values of the power, i.e., $\alpha = 1/2$ (solid line), $\alpha = 3/2$ (dashed line) and $\alpha = 3$ (dotted line). We can see that in both figures the errors start in a higher value and then oscillate converging when $A = 50$. The minimum values of the relative errors occur for lower values of A with the increase of the power α.

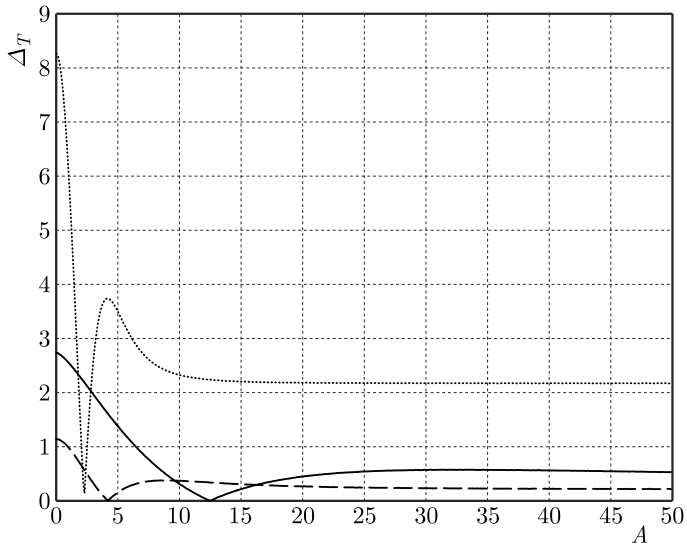

Fig. 3. Relative errors Δ_T, Eq. (4.12) versus the amplitude A for $v = 10$, and $\alpha = 1/2$ (solid line), $\alpha = 3/2$ (dashed line) and $\alpha = 3$ (dotted line)

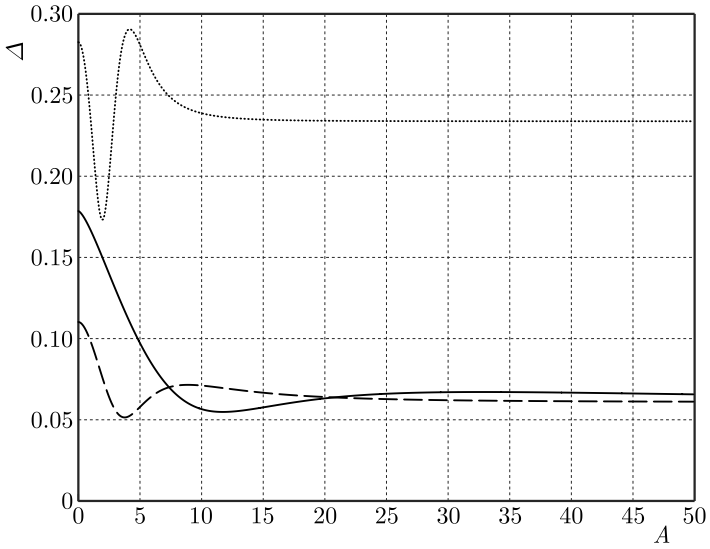

Fig. 4. Relative errors Δ (Algorithm 2) versus the amplitude A for $v = 10$, and $\alpha = 1/2$ (solid line), $\alpha = 3/2$ (dashed line) and $\alpha = 3$ (dotted line)

5. Conclusions

The Hamiltonian approach is extended for nonlinear conservative oscillators with general initial conditions. A method is proposed for obtaining a relationship for the frequency as a function of the initial amplitude, initial velocity and parameters of the system. An error estimation procedure is investigated. Computational algorithms are proposed as an alternative procedure to determine the frequency, amplitude, phase and relative errors in the adopted approximate solution. Considering the obtained results, we have concluded the following:

- In the case of oscillators governed by the Duffing equation, the relative error for the vibration period Δ_T converges to 2.172% for $(A \to \infty, v)$ and to 8.264% for $(A = 0, v \to \infty)$. The average relative error Δ converges to 0.233854 for $(A \to \infty, v)$ and to 0.282597 for $(A = 0, v \to \infty)$.
- In pure nonlinear conservative oscillators, the relative errors for the vibration period Δ_T, and for the average solution Δ decrease with the increasing α up to $\alpha = 1$, where the

errors are zero. Beyond this point $\alpha > 1$, the relative errors start to increase. According to Table 2, when the velocity $v = 0$, there is a limit or maximum values for the errors, i.e., $\Delta_T(A, 0) = \Delta_T(A \to \infty, v)$ and $\Delta(A, 0) = \Delta(A \to \infty, v)$. The minimum values of the relative errors occur for lower values of A with an increase in the power α.

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