ON THE MODELLING OF HEAT CONDUCTION IN A NON-PERIODICALLY LAMINATED LAYER

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Heat conduction in a thick layer made of non-periodically distributed micro-laminas of two rigid conductors is considered. Macroscopic properties of the layer are continuously graded across its thickness (FGL layer). The aim of the paper is to propose an averaged model of non-stationary heat conduction in the thick layer with functionally graded macroscopic properties. Model equations are derived in the framework of the modified tolerance averaging technique, cf.Woźniak and Wierzbicki (2000). Moreover, the model is applied to show the microstructural effect on heat conduction in the FGL layer.

Key words: heat conduction, functionally graded material, non-periodic layer

1. Introduction

The main object of considerations is a thick layer made of two rigid conductors, non-periodically distributed in the form of micro-laminas along the thickness of the layer. It is assumed that this laminated composite has macroscopic properties continuously varying across its thickness. Materials of this kind are called Functionally Graded Materials (FGM), cf. Suresh and Mortensen (1998).

The geometry of microstructure of those composites cannot be described exactly. Thus, thermomechanical behaviour of such composites can be analysed only within micromechanical models with idealised geometries. To describe FGM-type media, methods proposed to investigate macroscopically homogeneous composites, e.g. periodic composites (laminates), can be applied. Despite of the fact that FGM-type media are not macroscopically homogeneous, those methods are modified and adapted to analyse their overall behaviour. Some methods were discussed by Reiter *et al.* (1997) and in a monograph by

Suresh and Mortensen (1998). Between these methods, we have to mention methods based on the asymptotic *homogenization* proposed and developed for periodic composites and structures, cf. Bakhvalov and Panasenko (1984), Jikov *et al.* (1994), and applied to heat conduction problems e.g. by Gałka *et al.* (1994). Heat transfer problems were also analysed in the framework of models with *microlocal parameters*, cf. Matysiak (1991). However, models based on the asymptotic homogenization as well as microlocal models neglect usually *the effect of microstructure size* (called *the length-scale effect* or *the effect of period lengths*) on the overall behaviour of laminates with microperiodic structure. Homogenization procedures were applied to investigate problems of heat conduction and/or thermal stresses in functionally graded materials in many papers, see e.g. Itoh *et al.* (1996), where also the aforementioned effect was neglected.

In order to circumvent the above drawback, one can apply *the tolerance averaging technique*, proposed and discussed for periodic composites in a book by Woźniak and Wierzbicki (2000). This method was used to analyse special problems of different periodic structures in a series of papers, e.g. for thin plates in Jędrysiak (2001), for wavy plates in Michalak (2001), for laminates in Szymczyk and Woźniak (2006). Heat conduction problems of periodic composites were investigated in the framework of this technique in some papers by e.g. Woźniak *et al.* (1996), Ignaczak and Baczyński (1997), Woźniak and Wierzbicki (2000), Woźniak M. *et al.* (2001), Łaciński (2005). This approach leads from equations with functional, periodic, highly-oscillating and, in general, non-continuous coefficients to a system of differential equations with constant coefficients, describing *the effect of microstructure size* on the overall behaviour of a periodic composite.

In the last time, the tolerance averaging technique was modified and adopted to investigate mechanical problems of FGM-type structures, e.g. for FGL plates by Jędrysiak *et al.* (2005), shells by Woźniak *et al.* (2005), for laminates with microdefects by Rychlewska *et al.* (2006).

The aim of this paper is twofold. Firstly, to propose a certain *averaged microstructural model of heat conduction* in a non-periodically laminated layer, which has macroscopic properties functionally varying along its thickness. Secondly, to apply the new model to show some *microstructural effects* in heat conduction of that layer.

2. Preliminaries

Subscripts $i, j, \ldots$ run over $1, 2, 3$ and are related to the coordinate system $Ox_1x_2x_3$; subscripts $\alpha, \beta, \ldots$ run over $2, 3$ and are related to the system

Ox_2x_3; summation convention holds. Denote by $\boldsymbol{x} \equiv (x_2, x_3)$, $x \equiv x_1$ and introduce t as the time coordinate. It is assumed that the non-periodically laminated layer under consideration occupies the region $(0, L_1) \times (0, L_2) \times (0, L_3)$ in the physical space. Denote by $H = L_1$ the thickness of the layer along the x-axis. The layer is made of two materials distributed in m laminas having the same thickness l; hence $H = ml$. It is assumed that the number m of laminas is large $(m^{-1} \ll 1)$. Thus, the condition $l \ll H$ is satisfied and the thickness l will be called *the microstructure parameter*. The nth lamina is defined as the region $I_n \times \Pi$, $n = 1, \ldots, m$, where $I_n \equiv ((n-1)l, nl)$, $\Pi \equiv (0, L_2) \times (0, L_3)$. Moreover, every lamina consists of two homogeneous sub-laminas with thicknesses l'_n, l''_n. Properties of the sub-laminas are described by specific heats c', c'' and heat conduction tensors k'_{ij}, k''_{ij}, $i, j = 1, 2, 3$. Material volume fractions in the nth lamina are denoted by $\nu'_n \equiv l'_n/l$, $\nu''_n \equiv l''_n/l$. Assuming the sequence $\{\nu'_n\}$, $n = 1, \ldots, m$, to be monotone and for every $n = 1, \ldots, m-1$ to satisfy condition $|\nu'_{n+1} - \nu'_n| \ll 1$, the thick layer can be treated as made of a FGM and will be called *the functionally graded laminated layer* (*the FGL layer*). Similar conditions are satisfied by the sequence $\{\nu''_n\}$ because of $\nu'_n + \nu''_n = 1$. Under the above requirements, the sequences $\{\nu'_n\}$, $\{\nu''_n\}$, $n = 1, \ldots, m$, can be approximated by continuous functions $\nu'(\cdot)$, $\nu''(\cdot)$, which describe the gradation of material properties across the layer thickness. The choice of these functions is very important in design and optimization problems for FGL composites. The functions $\nu'(\cdot)$, $\nu''(\cdot)$ will be called *the fraction ratios of materials*. Hence, *the non-homogeneity ratio* ν can be introduced as a function defined by the formula $\nu(\cdot) \equiv \sqrt{\nu'(\cdot)\nu''(\cdot)}$. The aforementioned functions of fraction ratios are assumed to be slowly varying (this concept is defined in the book by Woźniak and Wierzbicki (2000) and will be recalled in the subsequent section).

A fragment of the macrostructure of the FGL layer (as the layer made of a *functionally graded material*) is shown in Fig. 1; however, a fragment of the FGL layer on the micro level is presented in Fig. 2.

Assuming small oscillations of an unknown temperature field θ and introducing the thermal load q (the intensity of heat sources), the heat conduction in the FGL layer can be considered in the framework of the Fourier model, described by the governing equation

$$-(k_{ij}\theta_{,j})_{,i} + c\dot{\theta} = q \tag{2.1}$$

It has to be emphasized that the above equation has coefficients $k_{ij} = k_{ij}(x)$, $c = c(x)$, being highly-oscillating, non-continuous functions in x. In order to replace differential equation (2.1) by a system of differential equations with continuous, slowly varying functional coefficients, *the tolerance averaging method* will be adopted. This method was proposed for periodic structures by Woźniak and Wierzbicki (2000).
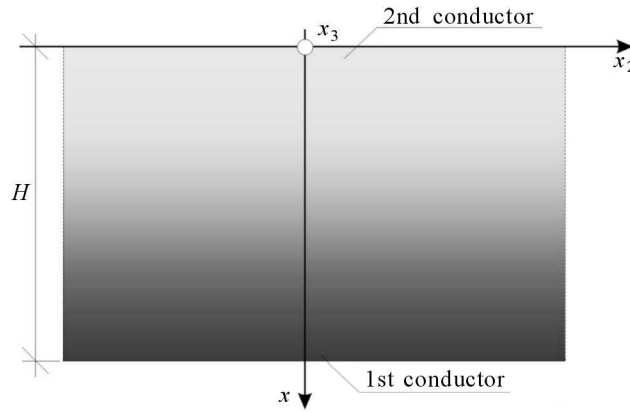
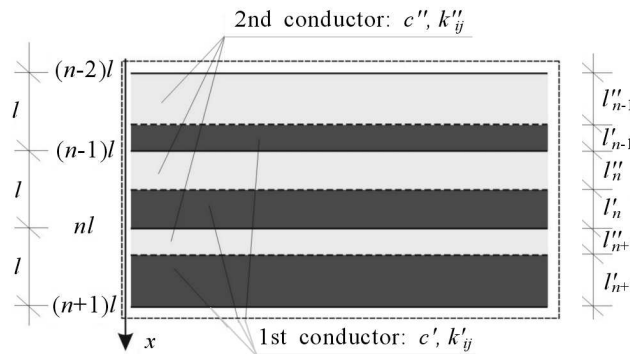Fig. 1. A fragment of the macrostructure of the non-periodically laminated layer

Fig. 2. A fragment of the microstructure of the non-periodically laminated layer

3.   Tolerance averaging technique

3.1.   Foundations

Denote by f an arbitrary integrable function defined in $[0, H]$, which can also depend on $\boldsymbol{x}$ and t as parameters. *The averaging operator* will be defined by

$$\langle f \rangle(\overline{x}) = l^{-1} \int\limits_{\overline{x}-\frac{l}{2}}^{\overline{x}+\frac{l}{2}} f(x)\, dx \qquad \overline{x} \in \left[\frac{l}{2}, H - \frac{l}{2} \right] \qquad (3.1)$$

where l is the constant thickness of every lamina.

Let Ψ be a differentiable function, $\Psi \in C^1([0, H])$, which can depend on $\boldsymbol{x}$ and t as parameters. The function Ψ will be called *a slowly varying function* (for a certain tolerance $\varepsilon \ll 1$ and with respect to the microstructure parameter l) and denoted $\Psi \in SV$, if functions $l\partial\Psi$ (where $\partial(\cdot)$ is differen-

tiation operation with respect to x) and $O(\varepsilon\Psi)$ are of the same order, i.e. $l\partial\Psi \in O(\varepsilon\Psi)$, $0 < \varepsilon \ll 1$. We also introduce the concept of *the fluctuation shape function* $g(x)$, $x \in [0, H]$, defined by the following formula

$$g(x) = \begin{cases} -l\sqrt{3}\dfrac{\nu(\overline{x})}{\nu''(\overline{x})}\left[2\dfrac{x}{l} + \nu'(\overline{x})\right] & \text{for} \quad x \in \left(-\dfrac{l}{2}, -\dfrac{l}{2} + l\nu''(\overline{x})\right) \\ l\sqrt{3}\dfrac{\nu(\overline{x})}{\nu'(\overline{x})}\left[2\dfrac{x}{l} - \nu''(\overline{x})\right] & \text{for} \quad x \in \left(\dfrac{l}{2} - l\nu'(\overline{x}), \dfrac{l}{2}\right) \end{cases}$$

where $\overline{x}$ is the centre of the interval $(-l/2, l/2)$. This function is assumed to be continuous, linear across every sub-lamina thickness and of an order $O(l)$. Moreover, it has values $l\sqrt{3}\nu(\overline{x})$ at the interfaces between laminas, and it takes values $-l\sqrt{3}\nu(\overline{x})$ at the interfaces between the adjacent sub-laminas within the lamina. Because the non-homogeneity ratio $\nu(\cdot)$ is a slowly varying function, it can be shown that the mean value of the function $g(\cdot)$ in every lamina is equal to zero. An example of the fluctuation shape function is shown in Fig. 3.
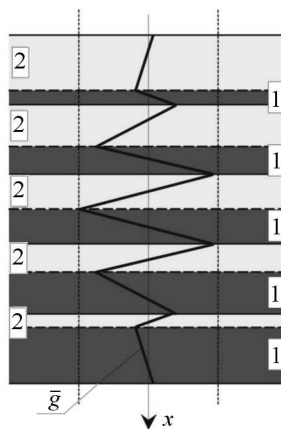

Fig. 3. An example of the fluctuation shape function $g(x)$ (1 – the 1st conductor, 2 – the 2nd conductor)

A detailed discussion of the above concepts is shown in Woźniak and Wierzbicki (2000).

3.2. Modelling assumptions

In the framework of the tolerance averaging technique, additional assumptions are formulated, cf. Woźniak and Wierzbicki (2000).

The first modelling assumption is *the thermal assumption*, which states that the temperature field $\theta = \theta(x, \boldsymbol{x}, t)$, $x \in [0, H]$, $\boldsymbol{x} \in \overline{\Pi}$, is restricted by the following formula

$$\theta(x, \boldsymbol{x}, t) = \Theta(x, \boldsymbol{x}, t) + g(x)\vartheta(x, \boldsymbol{x}, t) \tag{3.2}$$

where $\Theta(\cdot, \boldsymbol{x}, t) \in SV$ is called *the averaged temperature*, $\vartheta(\cdot, \boldsymbol{x}, t) \in SV$ is called *the fluctuation variable* of temperature. The functions $\Theta(\cdot)$, $\vartheta(\cdot)$ are basic unknowns describing temperature in the FGL layer.

The second assumption, called *the tolerance approximation*, states that for an arbitrary slowly varying function Ψ, $\Psi \in SV$, the approximation $\Psi + O(\varepsilon\Psi) \cong \Psi$ will be employed, which neglects terms of the order $O(\varepsilon)$ as small when compared to 1.

3.3.   The outline of the modelling procedure

The modelling procedure for FGM-type composites is similar to the procedure of the tolerance averaging, cf. Woźniak and Wierzbicki (2000), and can be divided into three steps:

1) to average equation (2.1), use formula (3.1),

2) to formulate a variational condition (cf. the aforementioned book), multiply equation (2.1) by the test function which has characteristic features as the fluctuation shape function, and then to average the resulting equation use formula (3.1),

3) substitute formula (3.2) into the aforementioned averaged equations, and after some calculations, differential equations for the averaged temperature Θ and the fluctuation variable ϑ are obtained.

4.   Governing equations of the microstructural model

Applying the aforementioned modelling procedure and denoting

$$
\begin{aligned}
&K_{ij} \equiv \langle k_{ij} \rangle && \widetilde{K}_{ij} \equiv \langle k_{ij} g_{,j} \rangle && \breve{K}_{ij} \equiv \langle k_{ij} g_{,i} g_{,j} \rangle \\
&\widehat{K}_{ij} \equiv l^{-1} \langle k_{ij} g \rangle && \overline{K}_{ij} \equiv l^{-2} \langle k_{ij} gg \rangle && C \equiv \langle c \rangle \\
&\widehat{C} \equiv l^{-1} \langle cg \rangle && \overline{C} \equiv l^{-2} \langle cgg \rangle && Q \equiv \langle q \rangle \\
&\widehat{Q} \equiv l^{-1} \langle qg \rangle
\end{aligned} \tag{4.1}
$$

we arrive at the following equations of *the microstructural model of the heat conduction in the thick FGL layer* (the thick layer with an internal non-periodically laminated structure)

$$
\begin{aligned}
&K_{\alpha j}(x)\Theta_{,j\alpha} + [K_{1j}(x)\Theta_{,j}]_{,1} - C(x)\dot{\Theta} + [\widetilde{K}_{11}(x)\vartheta]_{,1} + \widetilde{K}_{\alpha 1}(x)\vartheta_{,\alpha} + \\
&\quad + l\{[\widehat{K}_{1\alpha}(x)\vartheta_{,\alpha}]_{,1} + \widehat{K}_{\alpha i}(x)\vartheta_{,\alpha i} - \widehat{C}(x)\dot{\vartheta}\} = -Q \\[4pt]
&\widetilde{K}_{1j}(x)\Theta_{,j} + l[\widehat{C}(x)\dot{\Theta} - \widehat{K}_{\alpha j}(x)\Theta_{,j\alpha}] + \\
&\quad + \breve{K}_{11}(x)\vartheta + l^2[\overline{C}(x)\dot{\vartheta} - \overline{K}_{\alpha\beta}(x)\vartheta_{,\alpha\beta}] = -l\widehat{Q}
\end{aligned} \tag{4.2}
$$

with terms dependent on the microstructure parameter l.

It has to be emphasized that model equations (4.2) have three characteristic features:

1) coefficients of these equations are slowly varying functions in x, because the functions $\nu'(\cdot)$, $\nu''(\cdot)$, $\nu(\cdot)$ are slowly varying,

2) since equations (4.2) depend on the microstructure parameter l, the proposed model describes certain microstructural phenomena,

3) boundary conditions for the fluctuation variable ϑ have to be formulated only on that part of the boundary, which intersects lamina interfaces, because the equation for ϑ is independent of the derivative of ϑ with respect to x.

Summarizing, *the microstructural heat conduction model* is determined by:

- equations (4.2) for the unknowns $\Theta(\cdot, \boldsymbol{x}, t)$, $\vartheta(\cdot, \boldsymbol{x}, t)$,

- conditions of applicability of the model, i.e. equations (4.2) have physical sense for the unknowns $\Theta(\cdot, \boldsymbol{x}, t)$, $\vartheta(\cdot, \boldsymbol{x}, t)$ being slowly varying functions in x for every $\boldsymbol{x}$ and t which can be treated as parameters,

- the temperature field of the non-periodically laminated layer can be approximated by means of formula (3.2).

At the end of this section, let us observe that the formal passage $l \to 0$ in equations (4.2) makes it possible to eliminate the fluctuation variable ϑ from (4.2), which is given by the formula

$$\vartheta = -\frac{\widetilde{K}_{1j}(x)}{\breve{K}_{11}(x)}\Theta_{,j} \tag{4.3}$$

After substituting (4.3) into equation $(4.2)_1$, a simplified equation representing a certain asymptotic model, called *the macrostructural heat conduction model of the FGL layer*, is derived

$$K_{\alpha j}(x)\Theta_{,j\alpha} + [K_{1j}(x)\Theta_{,j}]_{,1} - C(x)\dot{\Theta} - \Big[\frac{\widetilde{K}_{11}(x)\widetilde{K}_{1j}(x)}{\breve{K}_{11}(x)}\Theta_{,j}\Big]_{,1} +$$
$$-\frac{\widetilde{K}_{\alpha 1}(x)\widetilde{K}_{1j}(x)}{\breve{K}_{11}(x)}\Theta_{,j\alpha} = -Q \tag{4.4}$$

where the effect of the microstructure parameter l is not taken into account.

It can be observed that for constant values of the parameters ν', ν'', $\nu' + \nu'' = 1$, equations (4.2) or (4.4) represent certain heat conduction models of a periodically laminated layer. It is necessary to emphasize that the governing equations of the proposed microstructural model can be employed to periodically laminated layers.

5.   Applications

5.1.   Heat conduction in a thick FGL layer

Let us consider a thick FGL layer subjected to a thermal gradient in the direction parallel to laminas. Hence, a plane problem of heat conduction in this layer bounded by the planes $x = 0$, $x = H$ and $x_2 = 0$, $x_2 = L$ is investigated.

Sub-laminas of the layer are made of two different isotropic materials. Properties of the sub-laminas are described by specific heats c', c'' and heat conduction tensors with elements $k'_{11} = k'_{22} = k'_{33} = k'$, $k''_{11} = k''_{22} = k''_{33} = k''$, where k', k'' are heat conduction coefficients. Assume that heat sources are zero, $Q = \widehat{Q} = 0$ and denote $K_1 \equiv K_{11}$, $K_2 \equiv K_{22}$, $\check{K} \equiv \check{K}_{11}$, $\overline{K} \equiv \overline{K}_{22}$, $\widetilde{K} \equiv \widetilde{K}_{11}$ and $z \equiv x_2$. All unknowns in equations (4.2) are functions of x, z, t, i.e. $\Theta = \Theta(x, z, t)$, $\vartheta = \vartheta(x, z, t)$.

Hence, *the microstructural model equations* take the form of two differential equations

$$[K_1(x)\Theta_{,1}]_{,1} + K_2(x)\Theta_{,22} - C(x)\dot{\Theta} = -[\widetilde{K}(x)\vartheta]_{,1}$$

$$\check{K}(x)\vartheta + l^2[\overline{C}(x)\dot{\vartheta} - \overline{K}(x)\vartheta_{,22}] = -\widetilde{K}(x)\Theta_{,1} \tag{5.1}$$

In the framework of *the macrostructural model* we obtain from (4.4)

$$\left\{\left(K_1(x) - \frac{[\widetilde{K}(x)]^2}{\check{K}(x)}\right)\Theta_{,1}\right\}_{,1} + K_2(x)\Theta_{,22} - C(x)\dot{\Theta} = 0 \tag{5.2}$$

which describes only the macrotemperature Θ.

Let us introduce a new unknown Ψ instead of ϑ given by the formula

$$\Psi \equiv \vartheta + \frac{\widetilde{K}(x)}{\check{K}(x)}\Theta_{,1} \tag{5.3}$$

It can be observed that in the framework of the macrostructural model the function Ψ is equal to zero. It can be called *the microstructural variable*. Because the macrotemperature $\Theta(\cdot, z, t)$ and the fluctuation variable $\vartheta(\cdot, z, t)$ are slowly varying functions in x for every z and t, the new unknown – the microstructural variable $\Psi(\cdot, z, t)$ is also slowly varying in x.

Substituting (5.3) into (5.1), we obtain modified equations

$$C(x)\dot{\Theta} - K_2(x)\Theta_{,22} - \left\{\left(K_1(x) - \frac{[\widetilde{K}(x)]^2}{\check{K}(x)}\right)\Theta_{,1}\right\}_{,1} = [\widetilde{K}(x)\Psi]_{,1}$$

$$\check{K}(x)\Psi + l^2[\overline{C}(x)\dot{\Psi} - \overline{K}(x)\Psi_{,22}] = l^2[\overline{C}(x)\dot{\Theta}_{,1} - \overline{K}(x)\Theta_{,122}]\frac{\widetilde{K}(x)}{\check{K}(x)} \tag{5.4}$$

The right hand side of $(5.4)_2$ can be rewritten as

$$l^2[\overline{C}(x)\dot{\Theta}_{,1} - \overline{K}(x)\Theta_{,122}]\frac{\widetilde{K}(x)}{\check{K}(x)} =$$

$$= l^2[\nu(x)]^2[C(x)\dot{\Theta}_{,1} - K_2(x)\Theta_{,122}]\frac{\widetilde{K}(x)}{\check{K}(x)}$$

Using equation $(5.4)_1$ and taking into account that all coefficients are slowly varying functions in x, we can write the above equation in the form

$$l^2[\overline{C}(x)\dot{\Theta}_{,1} - \overline{K}(x)\Theta_{,122}]\frac{\widetilde{K}(x)}{\check{K}(x)} =$$

$$= l^2[\nu(x)]^2\frac{\widetilde{K}(x)}{\check{K}(x)}\left\{\left[\left(K_1(x) - \frac{[\widetilde{K}(x)]^2}{\check{K}(x)}\right)\Theta_{,1} + \widetilde{K}(x)\Psi\right]_{,11}\right\}$$

Hence, the right hand side of $(5.4)_2$ is small comparing to the left hand side of this equation, bearing in mind that the macrotemperature $\Theta(\cdot, z, t)$ and the microstructural variable $\Psi(\cdot, z, t)$ are slowly varying functions in x. The right hand side of $(5.4)_2$ can be neglected in the first approximation of this model, and equations (5.4) can be written in the form

$$\left\{\left(K_1(x) - \frac{[\widetilde{K}(x)]^2}{\check{K}(x)}\right)\Theta_{,1}\right\}_{,1} + K_2(x)\Theta_{,22} - C(x)\dot{\Theta} = -[\widetilde{K}(x)\Psi]_{,1}$$

$$\check{K}(x)\Psi + l^2[\overline{C}(x)\dot{\Psi} - \overline{K}(x)\Psi_{,22}] = 0$$

(5.5)

Hence, for the microstructural variable, we obtain independent equation $(5.5)_2$. Similar formulas for vibrations were derived: for FGL plates by Jędrysiak *et al.* (2005), for FGL shells by Woźniak *et al.* (2005). Equations (5.5) can be treated as the first approximation of equations (5.1). They describe a certain *approximated microstructural model*. At the same time, formula for the temperature (3.2) takes the following form

$$\theta = \Theta - g(x)\widetilde{K}(x)[\check{K}(x)]^{-1}\Theta_{,1} + g(x)\Psi$$

Below, our considerations are restricted to the problem of changeability of the microstructural variable Ψ, described by equation $(5.5)_2$. Hence, *the effect of the microstructure size* on the microtemperature in the FGL layer is analysed.

5.2. The effect of microstructure size on the microstructural variable Ψ

The effect of microstructure size on the microtemperature, described by equation $(5.5)_2$, is investigated. Let us assume that the solution to this equation has to satisfy the following initial-boundary conditions:
— the initial condition

$$\Psi(x, z, 0) = f(x, z) \tag{5.6}$$

— the boundary conditions (only for the boundaries $z = 0$, $z = L$)

$$\Psi(x, 0, t) = f(x, 0) \exp(-\omega t) = f_0(x) \exp(-\omega t) \qquad \Psi(x, L, t) = 0 \quad (5.7)$$

It can be observed that the boundary conditions for the function Ψ can be formulated only for the z-axis. Hence, the solution to equation $(5.5)_2$ has the form

$$\Psi(x, z, t) = f(x, z) \exp(-\omega t) \tag{5.8}$$

Denoting

$$\chi^2 \equiv \frac{\overline{K}(x)}{\overline{C}(x)} \qquad\qquad \eta^2 \equiv \frac{\check{K}(x)}{l^2 \overline{C}(x)}$$

and substituting solution (5.8) into equation $(5.5)_2$, we obtain a differential equation for the function f

$$f_{,22} - f(\eta^2 - \omega)\chi^{-2} = 0 \tag{5.9}$$

It can be observed that the argument x can be treated as a parameter. Following (5.7), the boundary conditions for the function f have the form

$$f(x, 0) = f_0(x) \qquad\qquad f(x, L) = 0 \tag{5.10}$$

Introduce non-dimensional denotations ζ and ϕ such that

$$\zeta \equiv \frac{z}{L} \qquad\qquad f(x, z) = f_0(x)\phi(x, \zeta)$$

Hence, boundary conditions (5.10) take the form

$$\phi(0) = 1 \qquad\qquad \phi(1) = 0$$

Denoting

$$\partial(\cdot) \equiv \frac{\partial(\cdot)}{\partial\zeta} \qquad\qquad \lambda \equiv \frac{l}{L} \qquad\qquad \check{\rho}^2 \equiv \frac{\check{K}}{\overline{K}} \qquad\qquad \Omega \equiv \frac{\omega}{\eta^2}$$

differential equation (5.9) can be written as

$$\partial^2\phi - \phi(1 - \Omega)\frac{\check{\rho}^2}{\lambda^2} = 0 \tag{5.11}$$

After some manipulations, we arrive at the following cases of solutions to equation (5.11):

1. if $\Omega = 0$ and setting $\kappa^2 \equiv \breve{\rho}^2 \lambda^{-2}$ then

$$\phi(\zeta) = \exp(-\kappa\zeta)\frac{1 - \exp[2\kappa(\zeta - 1)]}{1 - \exp(-2\kappa)} \tag{5.12}$$

2. if $0 < \Omega < 1$ and setting $\kappa_\Omega^2 \equiv (1 - \Omega)\breve{\rho}^2\lambda^{-2}$ then

$$\phi(\zeta) = \frac{\exp(-\kappa_\Omega\zeta)}{1 - \exp(-2\kappa_\Omega)} + \frac{\exp(\kappa_\Omega\zeta)}{1 - \exp(2\kappa_\Omega)} \tag{5.13}$$

3. if $\Omega = 1$ then

$$\phi(\zeta) = 1 - \zeta \tag{5.14}$$

4. if $\Omega > 1$ and $\gamma^2 \equiv (\Omega - 1)\breve{\rho}^2\lambda^{-2} \neq n^2\pi^2$ $(n = 1, 2, \ldots)$ then

$$\phi(\zeta) = \frac{\sin[\gamma(1 - \zeta)]}{\sin\gamma} \tag{5.15}$$

5. if $\Omega > 1$ and $\gamma^2 = n^2\pi^2$ $(n = 1, 2, \ldots)$ then

$$\Omega_n = n^2\pi^2\lambda^2\breve{\rho}^{-2} + 1 \tag{5.16}$$

Hence, using formulas (5.12)-(5.16), the microstructural variables Ψ of temperature, i.e. solutions to equation $(5.5)_2$, can be obtained:

1. if $\Omega = 0$ and setting $\kappa^2 \equiv \breve{\rho}^2\lambda^{-2}$ then

$$\Psi(x, z) = f_0(x)\exp\left(-\frac{\kappa z}{L}\right)\frac{1 - \exp\left[2\kappa\left(\frac{z}{L} - 1\right)\right]}{1 - \exp(-2\kappa)}$$

2. if $0 < \omega < \eta^2$ and setting $\kappa_\Omega^2 \equiv (1 - \Omega)\breve{\rho}^2\lambda^{-2}$ then

$$\Psi(x, z) = f_0(x)\left[\frac{\exp\left(-\frac{\kappa_\Omega z}{L}\right)}{1 - \exp(-2\kappa_\Omega)} + \frac{\exp\left(\frac{\kappa_\Omega z}{L}\right)}{1 - \exp(2\kappa_\Omega)}\right]\exp(-\omega t)$$

3. if $\omega = \eta^2$ then

$$\Psi(x, z) = f_0(x)\left(1 - \frac{z}{L}\right)\exp(-\omega t)$$

4. if $\omega > \eta^2$ and $\gamma^2 \equiv (\Omega - 1)\breve{\rho}^2\lambda^{-2} \neq n^2\pi^2$ $(n = 1, 2, \ldots)$ then

$$\Psi(x, z) = f_0(x)\frac{\sin\left[\gamma\left(1 - \frac{z}{L}\right)\right]}{\sin\gamma}\exp(-\omega t)$$

5. if $\omega > \eta^2$ and $\gamma^2 = n^2\pi^2$ $(n = 1, 2, \ldots)$ then

$$\omega_n = n^2\pi^2\chi^2 L^{-2} + \eta^2$$

It can be observed that the microstructural variable Ψ related to the microstructural effect depends on the frequency ω. The microstructural variable decays exponentially for cases 1-2 and linearly for case 3. Certain values of the frequency cause a non-decayed form of the microstructural variable Ψ (case 4). Moreover, the solution to equation $(5.5)_2$ does not exist (the case 5) for certain frequencies.

It has to be emphasized that the microstructural variable has to be a slowly varying function in x. Hence, some solutions do not satisfy the aforementioned modelling condition, because they can have points of singularity, e.g. case 4.

The above microstructural effects cannot be analysed in the framework of the macrostructural heat conduction model, described by equation (4.4) or (5.2).

Some calculational results illustrating the above formulas are shown in the subsequent section.

6. Calculational results

Let us assume two different cases of the material distribution across the layer thickness:

1. the first case of the fraction ratios of materials – linear functions

$$\nu'(x) = \frac{x}{H} \qquad \nu''(x) = 1 - \nu'(x) \tag{6.1}$$

2. the second case of the fraction ratios of materials – exponential functions

$$\nu'(x) = \frac{1 - \exp\left(\frac{2x}{H}\right)}{1 - \exp(2)} \qquad \nu''(x) = 1 - \nu'(x) \tag{6.2}$$

These functions and proper non-homogeneity ratios $\nu = \sqrt{\nu'\nu''}$ are shown in Fig. 4.

Some calculational results are shown in Figs. 5-8. In Figs. 5a,b, plots of solutions to equation (5.11), i.e. non-dimensional parts ϕ of microstructural variables Ψ versus the non-dimensional coordinate $\zeta \in [0, 1]$ are presented. These curves are found for the following parameters: $l/L = 0.1$, $k''/k' = 0.5$, $x/H = 0.5$. Decaying solutions ($\Omega \leqslant 1$: $\Omega = 0$, $\Omega = 0.9$, $\Omega = 1$) are shown in Fig. 5a, and oscillating solutions ($\Omega > 1$: $\Omega = 1.1$) in Fig. 5b.
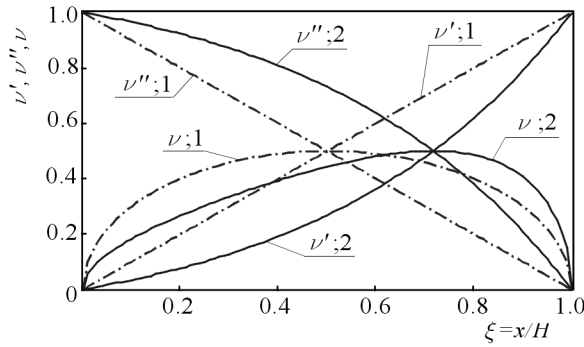
Fig. 4. The fraction ratios of materials ν', ν'' and the non-homogeneity ratio ν versus the non-dimensionless coordinate $\xi = x/H$; (1) – for formulas (6.1), (2) – for formulas (6.2)
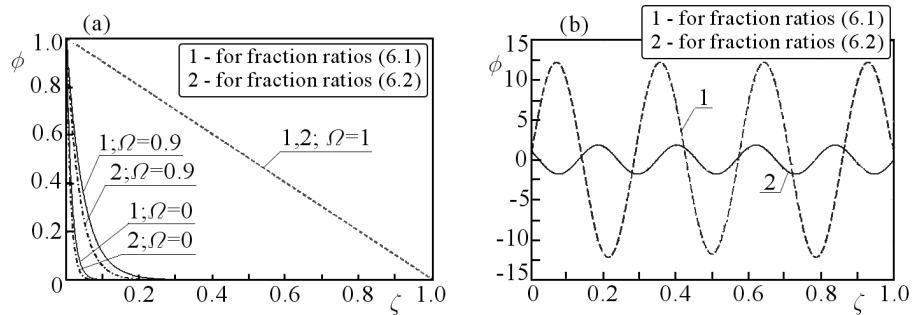

Fig. 5. Diagrams of non-dimensional parts ϕ of microstructural variables Ψ versus the non-dimensional coordinate ζ; (a) decaying solutions ($\Omega \leqslant 1$: $\Omega = 0$, 0.9, 1), (b) oscillating solutions ($\Omega > 1$: $\Omega = 1.1$) (for parameters: $l/L = 0.1$, $k''/k' = 0.5$, $x/H = 0.5$)

Plots of non-dimensional functions ϕ versus the ratio $x/H \in [0, 1]$ (the non-dimensional parameter ξ) are presented in Figs. 6a,b. These diagrams are determined for the following parameters: $l/L = 0.1$, $k''/k' = 0.5$, $\zeta = z/L = 0.05$. Plots for the exponentially decaying solutions ($\Omega < 1$: $\Omega = 0$, $\Omega = 0.9$) are shown in Fig. 6a, and for the oscillating solutions ($\Omega > 1$: $\Omega = 1.1$) in Fig. 6b.

It can be observed that the oscillating solutions (for $\Omega > 1$: $\Omega = 1.1$) are not slowly varying functions. Hence, they cannot be treated as solutions in the framework of the microstructural model, because they do not satisfy the modelling conditions of the tolerance averaging. Below, only decaying solutions will be considered.

In Fig. 7, we have diagrams of non-dimensional parts ϕ versus the ratio $k''/k' \in [0, 1]$. These curves are calculated for the following parameters: $l/L = 0.1$, $x/H = 0.5$, $\zeta = z/L = 0.05$. The presented plots are given only for the exponentially decaying solutions ($\Omega < 1$: $\Omega = 0$, $\Omega = 0.9$).
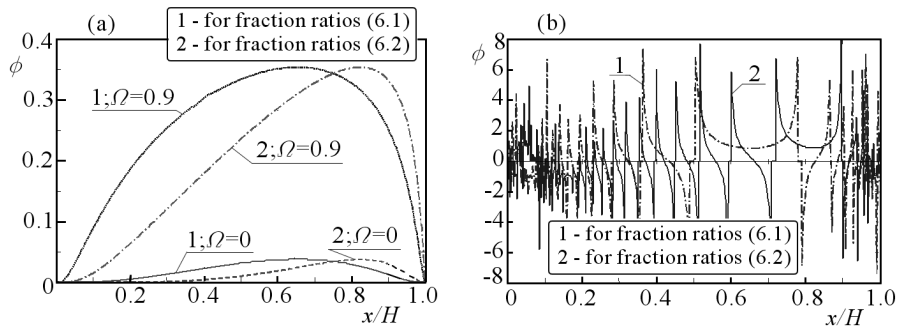
Fig. 6. Diagrams of non-dimensional parts ϕ of microstructural variables Ψ versus the ratio x/H; (a) for exponentially decaying solutions ($\Omega < 1$: $\Omega = 0$, $\Omega = 0.9$), (b) for oscillating solutions ($\Omega > 1$: $\Omega = 1.1$) (for parameters: $l/L = 0.1$, $k''/k' = 0.5$, $z/L = 0.05$)
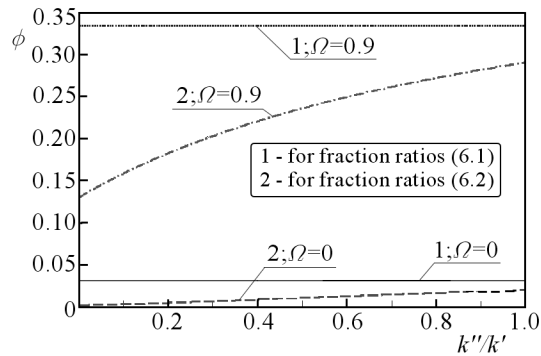

Fig. 7. Diagrams of non-dimensional parts ϕ of microstructural variables Ψ versus the ratio k''/k' for exponentially decaying solutions, i.e. $\Omega < 1$: $\Omega = 0$, $\Omega = 0.9$ (for parameters: $l/L = 0.1$, $x/H = 0.5$, $z/L = 0.05$)

Curves of non-dimensional functions ϕ, being the exponentially decaying solutions ($\Omega < 1$: $\Omega = 0$, $\Omega = 0.9$), versus the ratio $l/L \in [0,1]$ are shown in Fig. 8. These plots correspond to the following parameters: $x/H = 0.5$, $\zeta = z/L = 0.05$, $k''/k' = 0.5$.

Analysing the obtained results, some remarks can be formulated:

1. the shape of fluctuation variables depends on the non-dimensional parameter Ω, describing fluctuation frequencies, see Fig. 5, i.e.:

   (a) for $0 \leqslant \Omega < 1$ the fluctuations decay exponentially,

   (b) for $\Omega = 1$ they decay linearly,

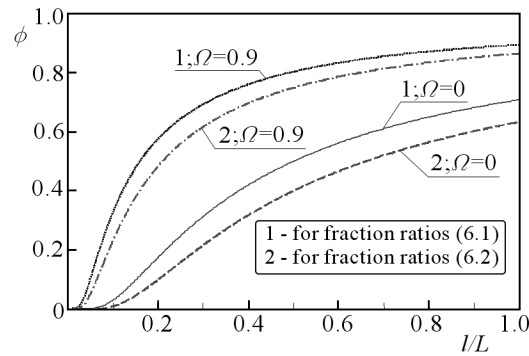   (c) for $\Omega > 1$ they oscillate;

Fig. 8. Diagrams of non-dimensional parts ϕ of microstructural variables Ψ versus the ratio l/L for exponentially decaying solutions, i.e. $\Omega < 1$: $\Omega = 0$, $\Omega = 0.9$ (for parameters: $k''/k' = 0.5$, $x = 0.5$, $z/L = 0.05$)

2. it can be observed for parameters which are not taken account in Fig. 5, that:

   (a) for small values of the parameter Ω from the interval $[0, 1)$, the microstructural variables strongly decay (Fig. 5a and Fig. 6a),

   (b) for small values of the ratios l/L, k''/k' and for $0 \leqslant \Omega < 1$, the microstructural variables strongly decay,

   (c) for fraction ratios (6.2), the solutions more decay than for fraction ratios (6.1), see Fig. 5a,

   (d) with increasing values of the parameter Ω (for $\Omega > 1$), the microstructural variables strongly oscillate;

3. from the results shown in Fig. 6, it can be observed that:

   (a) the maximum values of the decaying microstructural variables for the exponential functions of the fraction ratios of materials (6.2) are obtained for greater values of the ratio x/H than for the linear functions of fraction ratios (6.1),

   (b) for fixed values of the ratios: l/L, k''/k', z/L, the microstructural variables do not exist for certain values of the ratio x/H, determined by formula (5.16), (see points of singularity in Fig. 6b),

   (c) for parameters $\Omega > 1$ (e.g. $\Omega = 1.1$), we obtain the oscillating solutions which have points of singularity (Fig. 6b); hence, these solutions are not slowly varying functions in x and they do not hold the modelling conditions;

4. analysing the results presented in Fig. 7, we can observe that decaying microstructural variables for the exponential functions of the fraction

ratios of materials (6.2) increase with the increasing of the ratio k''/k', in contrast to the variables for linear fraction ratios (6.1), which are constant;

5. from the results shown in Fig. 8, it follows that decaying microstructural variables for linear (6.1) and exponential (6.2) functions of the fraction ratios of materials increase with the increasing ratio l/L.

7.   Final remarks

Applying the modified tolerance averaging method proposed for periodic composites by Woźniak and Wierzbicki (2000) to the equation of heat conduction in *thick non-periodically laminated layers (FGL layers)*, governing equations of *the non-asymptotic model* of such layers have been derived.

Summarizing our considerations, the following general remarks regarding some results concerning the heat conduction in FGL layers can be formulated.

- Derived non-asymptotic heat conduction model of FGL layers takes into account the effect of microstructure size (the lamina thickness) in contrast to the asymptotic model which describes only the overall behaviour of the layers. Hence, the proposed model is called *the microstructural model* and the asymptotic one – *the macrostructural.*

- Both models are governed by equations with functional, continuous coefficients, being slowly varying functions of the argument describing the layer along the direction perpendicular to laminas.

- The microstructural model makes it possible to formulate initial and boundary conditions not only for the averaged temperature but also for fluctuations of the temperature.

Analysing the application of the proposed model, some remarks can be formulated.

- On exemplary applications of the microstructural model, the effect of microstructure size on the problem of heat conduction in a thick FGL layer has been analysed. It has been shown that changes of the microstructural variable (hence, the fluctuation of the temperature) can be investigated independently of the changes of macrotemperature (the averaged temperature). Hence, this effect is called *the intrinsic microstructural effect.* However, changes of the averaged temperature cannot be analysed independently of the changes of temperature fluctuation, but this problem will be analysed in forthcoming papers.

- The microstructural variables are different for different values of fluctuation frequencies, i.e. there exists a special value of the fluctuation frequency, such that:

  - the microstructural variables decay exponentially for frequencies smaller than this value,

  - the variables decay linearly for the frequency equal to the special value,

  - the variables oscillate for frequencies greater than this value (5.15) and they have points of singularity,

  - the variables do not exist for the frequencies determined from formula (5.16).

- It can be observed that the microstructural variables are not slowly varying functions for frequencies greater than this special value of the fluctuation frequency, and these variables are not solutions in the framework of the model assumptions.

- The changeability of the microstructural variables depends on:

  - different functions of the fraction ratios of materials, e.g. exponential or linear functions,

  - differences between the heat conduction coefficients k', k''.

Other problems of heat conduction in FGL layers in the framework of the microstructural model will be considered in forthcoming papers, e.g. the effect of changes of the fluctuation of temperature on changes of the averaged temperature.

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O modelowaniu przewodnictwa ciepła w nieperiodycznie laminowanej warstwie

Streszczenie

W pracy rozpatrywane jest przewodnictwo ciepła w grubej warstwie zbudowanej z nieperiodycznie rozmieszczonych mikrolamin, wykonanych z dwóch przewodników. Własności makroskopowe warstwy zmieniają się w sposób ciągły wzdłuż jej grubości (tzw. warstwa typu FGL – ang. *functionally graded laminate*), por. rys. 1. Celem pracy jest przedstawienie uśrednionego modelu niestacjonarnego przewodnictwa ciepła w grubej warstwie o funkcyjnie zmieniających się własnościach makroskopowych. Równania modelu otrzymano w ramach zmodyfikowanej techniki tolerancyjnego uśredniania, por. Woźniak i Wierzbicki (2000). Ponadto, zaproponowany model będzie wykorzystany w przykładzie dla warstwy typu FGL, który pozwoli pokazać wpływ mikrostruktury.