JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
44, 1, pp. 107-125, Warsaw 2006

ALGEBRA OF SYSTEMS OF FORCES APPLIED TO THE FLAT MATERIAL LINE

Stefan Piechnik

Cracow University of Technology
e-mail: sp@limba.wil.pk.edu.pl

The research presents certain theorems concerning the algebra of vector systems of forces applied to a flat material line, rigid only in its plane. The considerations are based on a modification of one of two axiomatic equivalences assumed in the algebra of systems of forces applied to a rigid body. The problem is of particular importance for the theory of thin-walled beams.

Key words: system of forces, equivalent systems, system reduction

1. Introduction

This work continues, in a sense, the paper entitled *Kinematyczna równoważność układów sił* [Kinematic equivalence of force systems] (Piechnik, 1978), that gave rise to a heated discussion in MTiS, not so much due to its content, but the use of the adjective "kinematic". That contribution proposed a certain definition of the equivalence of systems of forces applied to the central line of a thin-walled bar with an open profile. This study assumes a different definition, based on two obvious axioms of the equilibrium of forces, and shows a proof of the former definition as a theorem. This approach illustrates much better the idea behind and the need for such a theorem that may probably find an application in other domains of mechanics as well. The kinematic equivalence theorem in the theory of thin-walled bars has the same significance as the static equivalence theorem in the solid bar theory.

2. System of forces

If forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots, \boldsymbol{P}_n$ anchored at points $Q_1, Q_2, \ldots, Q_n$, respectively, are applied to a material line in the form of an open (no loops) broken line with rigid sections, rigidly connected the plane of the line, then we say that the system of forces (Fig. 1), described as

$$\mathsf{A} = \begin{bmatrix} \boldsymbol{P}_1 & \boldsymbol{P}_2 & \ldots & \boldsymbol{P}_n \\ Q_1 & Q_2 & \ldots & Q_n \end{bmatrix}$$

is applied to the line.

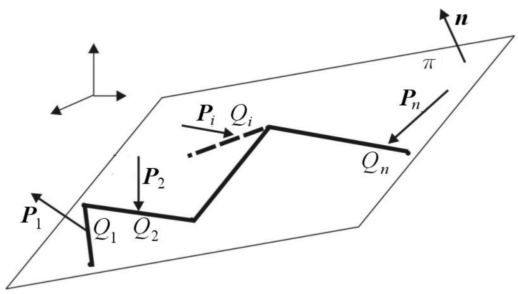

Fig. 1.

The force anchoring point may also be located off the line. Then, it can be imagined to be connected to the line with a rigid bar without a mass; the line remains open. Among possible systems, it is possible to differentiate special cases that shall receive separate names.

2.1. Special cases of force systems

Zero system

The zero system $\{\mathbf{0}\}$ is a system where all force vectors are equal to $\boldsymbol{0}$.

Single-element system

The single-element system is a system consisting of only one force $\boldsymbol{P}$, anchored at any arbitrary point Q

$$\mathsf{A} = \begin{bmatrix} \boldsymbol{P} \\ Q \end{bmatrix}$$

Couple of forces

The couple of forces, or the couple, for short, is a set of two non-zero vectors of opposite forces that are not located in a single line and are applied at the points that belong to the same section of the broken line (Fig. 2)

$$\mathsf{A} = \begin{bmatrix} \boldsymbol{P} & -\boldsymbol{P} \\ Q_1 & Q_2 \end{bmatrix}$$

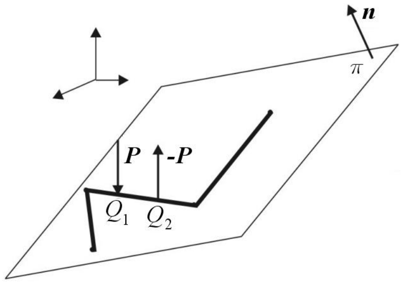

Fig. 2.

Bicouple of forces

The bicouple of forces, or the bicouple, for short, is a couple of couples located in parallel planes containing line sections or rigid sections without a mass where particular couples are applied (Fig. 3)

$$\mathsf{A} = \begin{bmatrix} \boldsymbol{P} & -\boldsymbol{P} & -\boldsymbol{P} & \boldsymbol{P} \\ Q_1 & Q_2 & R_1 & R_2 \end{bmatrix}$$

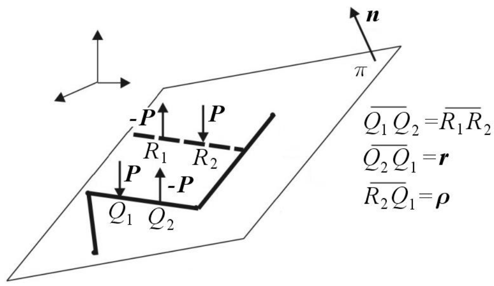

Fig. 3.

2.2. The question of reduction of force systems

Theoretical mechanics deals with systems of forces applied to a rigid body and allows one to formulate notions of the *static* and *dynamic operation* of the system. This is formulated in the following manner in Królikiewicz (1959):

- If a rigid body is motionless and keeps motionless after the application of a force system **A** and the same effect is produced under the impact of a system **B**, it is said that the static operation of both systems **A** and **B** is identical.
- If a rigid body is in motion under the impact of the force system **A** and the same motion is produced by the impact of the system **B**, it is said that the dynamic operation of both systems **A** and **B** is identical.

In the considered case of a material line that is not a rigid object, motion can be decomposed – according to the principles of kinematics – into translation and rotation of the rigid body and its deformation. In order to take advantage of the notions and the algorithm of operation provided by theoretical mechanics and to assume the above-mentioned kinematic method of the description of motion, the notion of *kinematic operation*[1] shall be introduced.

> If a deformable material line gets deformed under the impact of the force system **A** and the same deformation is produced by the impact of the system **B**, it is said that the kinematic operation of both systems **A** and **B** is identical.

The question of the reduction of a system of forces operating on a material line shall be reduced to the replacing of a given system **A** with another system of forces **R**, identical with the system **A** wit respect to the static, dynamic and kinematic operation. The present research shall be limited to the reduction with respect to a pole.

First, however, one should ask, whether there can exist two such systems of forces applied to a material line that produce the same static and dynamic effect and the same deformation, and whether such systems may be determined deductively?

[1]In the case of a rigid body, the notion of "dynamic operation" would certainly be sufficient as dynamics is the division of mechanics that deals with motion and equilibrium of material bodies. Similarly, the notion of dynamic operation would be sufficient in the analysis of a material line, because dynamics deals with both rigid bodies and bodies under deformation. The introduction of the notions of static or kinematic operation is aimed at making clear distinctions in the research.

The answer is obviously ”no”. Mathematics is insufficient, as there is a need for experiments and appropriate measurements.

While considering systems of forces applied to a rigid body in the framework of theoretical mechanics (Paluch, 2001), we reduce the analysis of a similar problem, amitting deformation, to purely geometrical operations by introducing intuitive axioms of the equivalence of systems of forces that allowed us to define the notion of elementary transformation, understood as follows:

α) removing two opposite vectors located on one straight line from the system or adding them to it,

β) removing or adding several vectors with the identical origin and the sum equal to $\boldsymbol{0}$.

The two definitions allowed us to produce the basic definition in the following form:

> Two systems $\mathbf{X}$ and $\mathbf{Y}$ are statically equivalent if and only if a finite number of elementary transformations α and β applied to the system $\mathbf{X}$ produces the system $\mathbf{Y}$ (Królikiewicz, 1959).

This definition provided the ground for proving the basic theorem on the equivalence of systems of forces:

> For two systems to be statically equivalent, it is necessary and sufficient for them to have equal sums and moments with respect to one pole.

Thus, the question of both system equivalence and reduction to a simpler system with respect to any pole can be considered in the sense of the basic definition, exclusively in deductive terms, without violating the physical sense.

The above definition of equivalence is not sufficient in the case in question, where a deformation of the material line can occur. Therefore, if we want to consider the problem of ”equivalent operation” of two or more applied systems of forces in purely deductive terms, we must assume a different basic definition of the equivalence and thus modify the problem of the reduction of forces.

3. Kinematic equivalent systems

Elementary transformations

Let us first consider a flat material line in the form of a broken line consisting of rigid sections connected in a rigid manner only in the line plane. It can be easily guessed that the analysis of any curve can be obtained through the limit transition.

In the case in question, intuition prompts us to assume a slightly modified version of the above- mentioned notions of elementary transformations, and of the transformation α in particular.

Further on, we shall assume the following definitions of elementary transformations:

> Adding to or removing from the already applied set of forces of:
>
> $\widetilde{\alpha}$) two opposite forces located along the same straight line belonging to the line plane,
>
> β) two or more forces anchored at the same point of the line and with the sum equal to zero
>
> at any point shall not change the kinematic or static and dynamic effects of the material line.

The operations $\widetilde{\alpha}$ and β applied to a given system shall be called elementary transformations, as in theoretical mechanics.

Secondary elementary transformations

The following operations shall be called the secondary elementary transformations:

- a system of forces with a common origin can be replaced with their sum with the same origin,
- each force in a system of forces can be decomposed into a sum of several components, anchored at the same point of origin,
- each force located in the plane of a line can be shifted in the direction of its operation to another point of the line.

It can be easily shown with the elementary transformations that all these transformations produce systems equivalent with the original one.

3.1. Basic definition

> Two systems of forces $\mathbf{A}$ and $\mathbf{B}$ applied to a flat material broken line made of rigid elements joined in a rigid way in its plane and infinitely limp in the direction perpendicular to the plane shall be called *kinematically equivalent* if and only if a finite number of appropriate elementary operations $\widetilde{\alpha}$ and β applied to the system $\mathbf{A}$ produces the system $\mathbf{B}$.

In view of this definition, the following theorems are obviously true:

- $\mathsf{A} \equiv \mathsf{A}$
- if $\mathsf{A} \equiv \mathsf{B}$ then $\mathsf{B} \equiv \mathsf{A}$
- if $\mathsf{A} \equiv \mathsf{B}$ and $\mathsf{B} \equiv \mathsf{C}$ then $\mathsf{A} \equiv \mathsf{C}$

which show that the relation of kinematic equivalence is reflexive, symmetric and transitive. It should be added that the algebra of equivalent systems often uses the notion of a *representative* – usually the simplest system, i.e. the system consisting of the smallest number of forces. The notion "representative" shall also be used in this text.

The sum and difference of systems

If there are two arbitrary sets of vectors given

$$\mathsf{A} = \begin{bmatrix} \boldsymbol{P}_1 & \boldsymbol{P}_2 & \dots & \dots & \boldsymbol{P}_n \\ Q_1 & Q_2 & \dots & \dots & Q_n \end{bmatrix} \qquad \mathsf{B} = \begin{bmatrix} \boldsymbol{S}_1 & \boldsymbol{S}_2 & \dots & \dots & \boldsymbol{S}_m \\ R_1 & R_2 & \dots & \dots & R_m \end{bmatrix}$$

the following system shall constitute their sum

$$\mathsf{C} = \mathsf{A} + \mathsf{B} = \begin{bmatrix} \boldsymbol{P}_1 & \boldsymbol{P}_2 & \dots & \dots & \boldsymbol{P}_n & \boldsymbol{S}_1 & \boldsymbol{S}_2 & \dots & \dots & \boldsymbol{S}_m \\ Q_1 & Q_2 & \dots & \dots & Q_n & R_1 & R_2 & \dots & \dots & R_m \end{bmatrix}$$

and the following system shall constitute their difference

$$\mathsf{D} = \mathsf{A} - \mathsf{B} = \begin{bmatrix} \boldsymbol{P}_1 & \boldsymbol{P}_2 & \dots & \dots & \boldsymbol{P}_n & -\boldsymbol{S}_1 & -\boldsymbol{S}_2 & \dots & \dots & -\boldsymbol{S}_m \\ Q_1 & Q_2 & \dots & \dots & Q_n & R_1 & R_2 & \dots & \dots & R_m \end{bmatrix}$$

In particular, if $m = n$ in the system B and the forces are anchored at the points $Q_1, Q_2, \dots, Q_n$, respectively, then the sum can be described as

$$\mathsf{A} = \begin{bmatrix} \boldsymbol{P}_1 + \boldsymbol{S}_1 & \boldsymbol{P}_2 + \boldsymbol{S}_2 & \dots & \dots & \boldsymbol{P}_n + \boldsymbol{S}_n \\ Q_1 & Q_2 & \dots & \dots & Q_n \end{bmatrix}$$

The same description can be provided for the difference of systems.

The theorem on the sum of equivalent systems

Let us assume that

$$\mathsf{A} + \mathsf{B} \equiv \mathsf{C}$$

and that

$$\mathsf{A} \equiv \mathsf{D} \qquad \text{and} \qquad \mathsf{B} \equiv \mathsf{E}$$

By means of the theorems on reflexivity, symmetry and transitivity of the equivalence relations, we shall easily show that the following obvious theorem is true

$$\mathsf{D} + \mathsf{E} \equiv \mathsf{C}$$

This theorem shall be used to reduce any system applied to a material line.

System decomposition into "flat" and "normal" components

Let us assume a given system A. Let us decompose it into two systems: P and N by means of the latter secondary elementary transformation

$$\mathsf{A} = \mathsf{P} + \mathsf{N}$$

Let the system P be a flat system located within the line plane and the system N be a system of forces perpendicular to the line plane.

Further on, we shall consider only systems N[2]. For this reason, line deformation shall be called warping further on in the text.

The sum of a system

The sum of a random system N of forces

$$\mathsf{N} = \begin{bmatrix} \boldsymbol{P}_1 & \boldsymbol{P}_2 & \dots & \boldsymbol{P}_n \\ Q_1 & Q_2 & \dots & Q_n \end{bmatrix}$$

is understood to be the vector

$$\boldsymbol{S} = \sum_{i-1}^{n} \boldsymbol{P}_i$$

It is obvious that if the transformation $\widetilde{\alpha}$ or β is applied to the system N, the resultant system has the same sum. Thus, it is possible to formulate the following theorem:

> The sum of the system N is an invariant with respect to elementary transformations.

[2]The algebra of P-type systems is a special (flat) case of the algebra of systems of forces applied to a rigid body, considered by theoretical mechanics.

The moment of a system

➢ The moment of a single-element system with respect to a given pole

Let us consider the following system in an oriented space

$$\mathbf{J} = \begin{bmatrix} \boldsymbol{P} \\ Q \end{bmatrix}$$

and an arbitrary point R belonging to the material line or rigidly joined with it. The moment of the system $\mathbf{J}$ with respect to the pole R (Fig. 4) is understood to be the following vector

$$\boldsymbol{M}_R(\mathbf{J}) = \overline{RQ} \times \boldsymbol{P} = \boldsymbol{r} \times \boldsymbol{P}$$

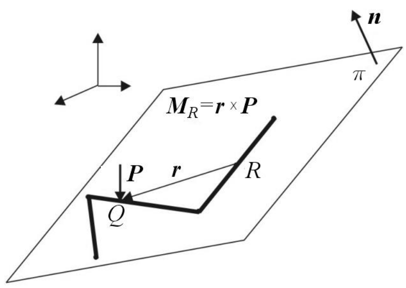

Fig. 4.

It should be noted that the vector of the moment with respect to $\boldsymbol{r}$ belonging to the material line plane π and with respect to $\boldsymbol{P} \perp \pi$ is parallel to π.

➢ The moment of the system $\mathbf{N}$ with respect to a given pole

The moment of a system with respect to a selected pole is understood to be the sum of moments of all forces of the system with respect to the pole

$$\boldsymbol{M}_Q(\mathbf{N}) = \sum_{i-1}^{n} \boldsymbol{r}_i \times \boldsymbol{P}_i \qquad \boldsymbol{r}_i = \overline{R_i Q} \qquad i = 1, 2, \ldots, n$$

It is obvious that if the transformation $\widetilde{\alpha}$ or β is applied to the system $\mathbf{N}$, the resultant system has the same moment. Thus, it is possible to formulate the following theorem:

> The moment of the system $\mathbf{N}$ is an invariant with respect to elementary transformations.

The bimoment of a system

➢ The bimoment of a bicouple

A bicouple (Fig. 3) can be assigned a number called a bimoment that is equal to the triple product

$$B_\omega = [\boldsymbol{P}, \boldsymbol{r}, \boldsymbol{\rho}]$$

where

- $\boldsymbol{P}$ – any vector of the bicouple, e.g. $\boldsymbol{P}$, anchored at Q_1 as shown in Fig. 3,
- $\boldsymbol{r}$ – the radius-vector of the anchoring point of a selected force, the origin of which is located at any point of the straight line of the application of the second force in the bicouple (e.g. $\boldsymbol{r} = \overline{Q_2Q_1}$),
- $\boldsymbol{\rho}$ – the radius-vector of the anchoring point of the above-selected force vector, with the origin at any point of the plane of the second bicouple, e.g. $\boldsymbol{\rho} = \overline{R_2Q_1}$.

➢ The bimoment with respect to a selected pole

If the anchoring point of the vector $\boldsymbol{\rho}$ is distinguished, as the pole is R_2 in Fig. 3, one can speak of a bimoment with respect to this pole.

➢ The bimoment of the system $\mathbf{N}$

A bimoment of the system $\mathbf{N}$ with respect to the pole R is the sum of bimoments of all bicouples if the plane of the "second" couple goes through R

$$B_\omega(\mathbf{N}) = \sum_{i-1}^{n} [\boldsymbol{P}_i, \boldsymbol{r}_i, \boldsymbol{\rho}_i]$$

It is obvious that if the transformation $\widetilde{\alpha}$ or β is applied to the system $\mathbf{N}$, the resultant system has the same bimoment. Thus, it is possible to formulate the following theorem:

> The bimoment of the system $\mathbf{N}$ is an invariant with respect to elementary transformations.

3.2. The basic theorem on kinematic equivalence of systems

For two systems of forces $\mathbf{A}$ and $\mathbf{B}$ applied to a material flat broken line with rigid elements joined in a rigid manner in their plane and infinitely limp in the perpendicular direction

$$\mathbf{A} = \begin{bmatrix} \boldsymbol{P}_1 & \boldsymbol{P}_2 & \dots & \dots & \boldsymbol{P}_n \\ Q_1 & Q_2 & \dots & \dots & Q_n \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} \boldsymbol{S}_1 & \boldsymbol{S}_2 & \dots & \dots & \boldsymbol{S}_m \\ R_1 & R_2 & \dots & \dots & R_m \end{bmatrix}$$

to be kinematically equivalent, it is necessary and sufficient for them to have equal sums, moments with respect to the same pole and bimoments

$$\begin{aligned}&\boldsymbol{S}(\mathbf{A}) = \boldsymbol{S}(\mathbf{B})\\&\boldsymbol{M}_Q(\mathbf{A}) = \boldsymbol{M}_Q(\mathbf{B})\\&B_\omega(\mathbf{A}) = B_\omega(\mathbf{B})\end{aligned}$$

First, the necessary condition of this theorem shall be shown.

If it is assumed that $\mathbf{A} \equiv \mathbf{B}$, then the equivalence of sums, moments and bimoments of both systems follows directly from the fact that the sum and the moment with respect to a selected pole and the bimoment are invariants with respect to elementary transformations.

The sufficient condition shall be shown in two steps.

(i) Firstly, we shall show that for two systems $\mathbf{A}$ and $\mathbf{B}$ to be equivalent, it is necessary and sufficient for their difference to be a system equivalent to a zero system, i.e.

$$\mathbf{A} - \mathbf{B} \equiv \mathbf{0} \Rightarrow \begin{bmatrix} \boldsymbol{P}_1 & \dots & \boldsymbol{P}_n & -\boldsymbol{S}_1 & \dots & -\boldsymbol{S}_m \\ Q_1 & \dots & Q_v & R_1 & \dots & R_m \end{bmatrix} \equiv \mathbf{0}$$

We shall assume first that $\mathbf{A} \equiv \mathbf{B}$. If so, then the application of the elementary transformations transforming $\mathbf{A}$ into $\mathbf{B}$ applied to the above system leads to

$$\begin{aligned}\mathbf{A} - \mathbf{B} &= \begin{bmatrix} \boldsymbol{S}_1 & \dots & \boldsymbol{S}_m & -\boldsymbol{S}_1 & \dots & -\boldsymbol{S}_m \\ R_1 & \dots & R_m & R_1 & \dots & R_m \end{bmatrix} = \\ &= \begin{bmatrix} \boldsymbol{S}_1 - \boldsymbol{S}_1 & \dots & \boldsymbol{S}_m - \boldsymbol{S}_m \\ R_1 & \dots & R_m \end{bmatrix} = \begin{bmatrix} 0 & \dots & 0 \\ R_1 & \dots & R_m \end{bmatrix} = \mathbf{0}\end{aligned}$$

We found out that if two systems are equivalent, then their difference is equivalent to a zero system.

Let us now assume that $\mathbf{A} - \mathbf{B} \equiv \mathbf{0}$ and add $\mathbf{B}$ to both sides of the system. We get

$$[(\mathbf{A} - \mathbf{B} + \mathbf{B}] \equiv [\mathbf{0} + \mathbf{B}] \equiv [\mathbf{A} + \mathbf{0}] \Rightarrow \mathbf{A} \equiv \mathbf{B}$$

(ii) Let us assume the origin of the natural coordinate s to be located at the end of the line O_1 (Fig. 5). Let us reduce the system of forces applied the ith section of the line l_i to the point O_i which constitutes the beginning of the section.

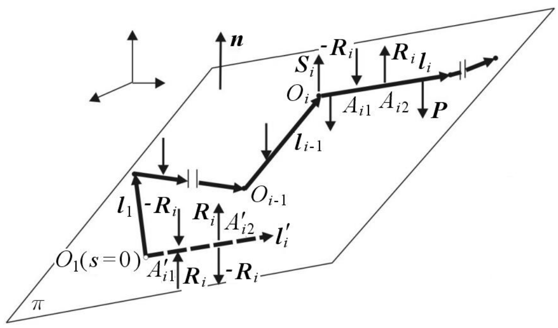

Fig. 5.

We shall use here the theorem on the reduction of a system of forces applied to a rigid bordy. At the point O_i, we anchor the sum $\boldsymbol{S}_i$ of all forces applied to this section and the pair

$$\begin{bmatrix} \boldsymbol{R}_i & -\boldsymbol{R}_i \\ A_{i2} & A_{i1} \end{bmatrix}$$

with the moment $\boldsymbol{M}_i = \boldsymbol{r}_i \times \boldsymbol{R}_i$, where $\boldsymbol{r}_i = \overline{A_{i1}A_{i2}}$, equal to the moment of all forces applied to this section with respect to the point O_i. As we are considering a system of forces normal to the plane of line π, the pair is located in the plane perpendicular to π, where the section l_i is located. Let us ascribe the force $\boldsymbol{S}_i$ to the point at the end of the section l_{i-1} of the spanning tree and reduce the system containing this force and the remaining forces applied to the section down to the point O_{i-1} by anchoring there the sum $\boldsymbol{S}_i + \boldsymbol{S}_{i-1}$ and the couple with the moment $\boldsymbol{M}_{i-1}$ calculated with respect to the pole O_{i-1} of all forces applied to the equivalent section with the sum $\boldsymbol{S}_{i-1}$. By following this procedure up to the first section of the line, we obtain the force $\boldsymbol{S} = \sum_{i-1}^{n} \boldsymbol{S}_n = \boldsymbol{0}$ at the point O_1, according to our assumptions and the relevant couples for each section. Let us join n non-material rigid sections to the material line at the point O_1, so that particular sections are parallel to successive sections of the material line. Let us consider the section parallel to l_i (the dotted line in Fig. 5) and apply the following system of forces to the section

$$\begin{bmatrix} \boldsymbol{R}_i & -\boldsymbol{R}_i & -\boldsymbol{R}_1 & \boldsymbol{R}_1 \\ A'_{i1} & A'_{i1} & A'_{i2} & A'_{i2} \end{bmatrix}$$

where $\overline{A'_{i1}A'_{i2}} = \boldsymbol{r}_i = \overline{A_{i1}A_{i2}}$.

It should be noted that the system

$$\begin{bmatrix} \boldsymbol{R}_i & -\boldsymbol{R}_i & -\boldsymbol{R}_i & \boldsymbol{R}_i \\ A_{i2} & A_{i1} & A'_{i2} & A'_{i1} \end{bmatrix}$$

defines a bicouple with the bimoment

$$B_{\omega i} = [\boldsymbol{R}_i, \boldsymbol{r}_i, \boldsymbol{\rho}_i]$$

where $\boldsymbol{\rho}_i = \overline{A'_{i1}A_{i2}}$, while the system

$$\begin{bmatrix} \boldsymbol{R}_i & -\boldsymbol{R}_i \\ A'_{i2} & A'_{i1} \end{bmatrix}$$

defines a couple in the plane going through the point O_1.

Following the same procedure, i.e. crating a similar transformation on each line l_i, we obtain n bicouples and n couples located in planes going through the point O_1, whose vectors of moments are located in the plane π. The sum of these vectors is, as assumed, equal to

$$\sum_{i-1}^{n} \boldsymbol{r}_i \times \boldsymbol{R}_i = \boldsymbol{0}$$

Thus, the remaining system of forces consists only of bicouples with the sum of bimoments equal to zero, as assumed. Therefore

$$B_\omega = \sum_{i-1}^{n} B_{\omega i} = \sum_{i-1}^{n} [\boldsymbol{R}_i, \boldsymbol{r}_i, \boldsymbol{\rho}_i] = 0$$

As $\boldsymbol{r}_i \neq \boldsymbol{0}$ and $\boldsymbol{\rho}_i \neq \boldsymbol{0}$, $\boldsymbol{R}_i = \boldsymbol{0}$ must be for $i = 1, 2, \ldots, n$. This proves that if a system of forces has a sum, moment and bimoment being equal to zero, it is a system equivalent to the zero system.

4. System reduction

The theorem on transferring a force to another point

The theorem on "transferring" a force vector applied to a point of a rigid body to another point Q of the body states:

> "any vector of a system can be transferred to any point Q, while adding to the system a couple with the moment equal to the moment of the vector with respect to the pole Q",

and it remains valid for the line in question as well. It should be remembered, however, that the assumed notion of the couple defines it as a system of two opposite forces that are not located on the same line and whose application points are located on the same section. In other words, to take advantage of this theorem, we can transfer a force, but only to the points of the same section where the force is applied.

The theorem on transferring a couple onto another parallel plane

The sections containing points Q_1, Q_2, points of application of the pair of forces $\boldsymbol{P}, -\boldsymbol{P}$ belonging to the plane α and points R_1, R_2 belonging to a material or rigid non-material line belonging to the parallel plane β are mutually parallel (Fig. 6). Let us apply the following systems to the points R_1 and R_2, respectively

$$\begin{bmatrix} \boldsymbol{P} & -\boldsymbol{P} \\ R_1 & R_1 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} -\boldsymbol{P} & \boldsymbol{P} \\ R_2 & R_2 \end{bmatrix}$$

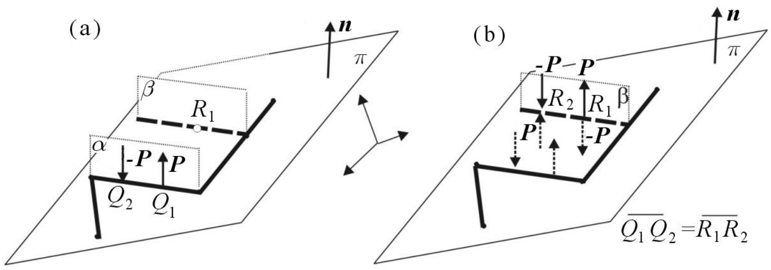

Fig. 6.

It should be noted that the system

$$\begin{bmatrix} \boldsymbol{P} & -\boldsymbol{P} \\ R_1 & R_2 \end{bmatrix}$$

has the same moment vector as that applied to the plane α. In terms of the adopted definition, these are not equivalent couples, although they have the same moment vectors because they are applied to different sections of the line. It can only be said that the couple applied to the section R_1, R_2, located in the plane β is a couple applied to the points Q_1, Q_2, "shifted" to a parallel plane. The remaining components, apart from the shifted couple, constitute a bicouple

$$\begin{bmatrix} \boldsymbol{P} & -\boldsymbol{P} & -\boldsymbol{P} & \boldsymbol{P} \\ Q_1 & Q_2 & R_1 & R_2 \end{bmatrix}$$

Thus, the following theorem has been:

> When transferring a couple from one plane to a parallel plane, one has to add to the transferred couple a bicouple consisting of two opposite couples located in both planes in question.

Reduction of a system with respect to the selected pole

The process shall not lose its general character if the reduction is shown on a material line consisting of three elements located in the plane XY of the global reference system (Fig. 7a). Let us assume first that the system consists of a single force $\boldsymbol{P}_3$ anchored at the edge l_3 at the point Q_3.

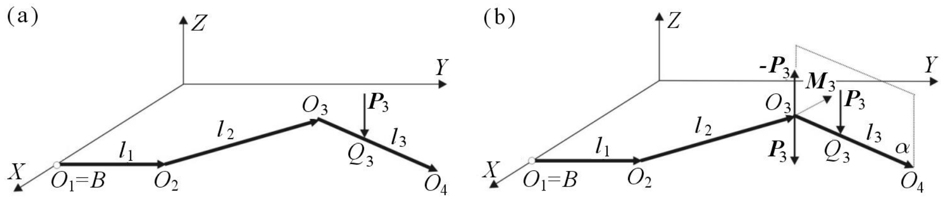

Fig. 7.

The reduction of this system with respect to the pole B shall begin with a transfer of the force $\boldsymbol{P}_3$ to the point O_3 constituting the beginning of the edge where the force is applied. To achieve this, we apply two forces $-\boldsymbol{P}_3$ and $\boldsymbol{P}_3$ to the point O_3 (Fig. 7b). We obtain a new system – the following sum

$$\begin{bmatrix} \boldsymbol{P}_3 \\ O_3 \end{bmatrix} + \begin{bmatrix} -\boldsymbol{P}_3 & \boldsymbol{P}_3 \\ O_3 & Q_3 \end{bmatrix}$$

i.e. the sum of the force $\boldsymbol{P}_3$ anchored at O_3 and a couple applied to the edge l_3, located in the plane α perpendicular to XY, with the moment $\boldsymbol{M}_3 = \overline{O_3Q_3} \times \boldsymbol{P}_3$ located in the plane XY.

The next step of reduction consists in adding the non-material rigid section at the point B, parallel to l_3, and applying two forces $-\boldsymbol{P}_3$ and $\boldsymbol{P}_3$ to the point B and the forces $\boldsymbol{P}_3$ and $-\boldsymbol{P}_3$ to the point Q_3^* ($\overline{BQ_3^*} = \overline{O_3Q_3}$). The system shown in Fig. 8a may be described as a sum of the following three systems

$$\begin{bmatrix} \boldsymbol{P}_3 \\ O_3 \end{bmatrix} + \begin{bmatrix} -\boldsymbol{P}_3 & \boldsymbol{P}_3 \\ B & Q_3^* \end{bmatrix} + \begin{bmatrix} -\boldsymbol{P}_3 & \boldsymbol{P}_3 & \boldsymbol{P}_3 & -\boldsymbol{P}_3 \\ O_3 & Q_3 & B & Q_3^* \end{bmatrix}$$

in which:

- the vector $\boldsymbol{P}_3$ is anchored at O_3,
- the couple applied to the edge l_3^* with the moment $\boldsymbol{M}_3 = \overline{O_3Q_3} \times \boldsymbol{P}_3$,
- and a bicouple with the bimoment

$$B_{\omega 3} = [\boldsymbol{P}_3, \overline{O_3Q_3}, \overline{BQ_3}]$$

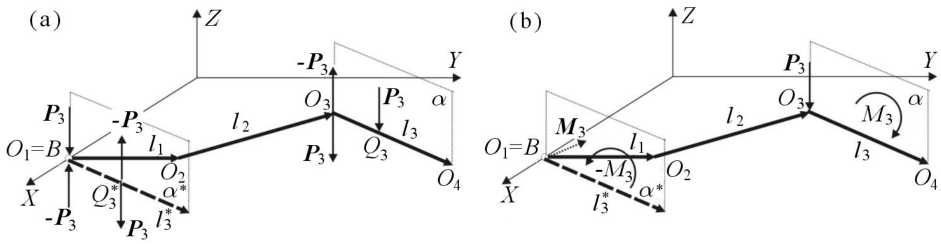

Fig. 8.

Figure 8a shall become more legible if the couple located in the plane α and belonging to the bicouple is replaced by its representative in the form of a directed arc, and the same is done with the second bicouple in the plane α^* (the arc with the opposite direction), while the couple applied to the line l_3^* – that can be treated as transferred from the plane α to α^* – is replaced with the vector $\boldsymbol{M}_3$[3] as it is shown in Fig. 8b. Further on, we shall use such quantities wherever it is not doubtful to do so.

Now, we shall transfer the force $\boldsymbol{P}_3$ in a similar way. It can be ascribed to the edge l_2 at the point O_2, i.e. two forces $-\boldsymbol{P}_3$ and $\boldsymbol{P}_3$ are applied to the point O_2 (Fig. 9a). We obtain a new system that can be described as a set of four systems: the force $\boldsymbol{P}_3$ anchored at O_2 and the couple $-\boldsymbol{P}_3$ and $\boldsymbol{P}_3$ anchored at O_2 and O_3, respectively, with the moment $\boldsymbol{M}_2 = \bar{l}_1 \times \boldsymbol{P}_3$ (the results of the last transformation), and the formerly discussed bicouple and

[3]A couple applied to a rigid element can be transferred along the section according to the algebra of forces applied to a rigid body. Moreover, couples anchored in the same section can be replaced with a single couple with moment equal to the sum of moments of applied couples.

couple $\boldsymbol{M}_3$ applied to the point B. As a result of the reduction, all the four systems are described as

$$\begin{bmatrix}\boldsymbol{P}_3\\O_2\end{bmatrix}+\begin{bmatrix}-\boldsymbol{P}_3 & \boldsymbol{P}_3\\B & Q_3^*\end{bmatrix}+\begin{bmatrix}-\boldsymbol{P}_3 & \boldsymbol{P}_3 & \boldsymbol{P}_3 & -\boldsymbol{P}_3\\O_3 & Q_3 & B & Q_3^*\end{bmatrix}+\begin{bmatrix}-\boldsymbol{P}_3 & \boldsymbol{P}_3\\O_2 & O_3\end{bmatrix}$$

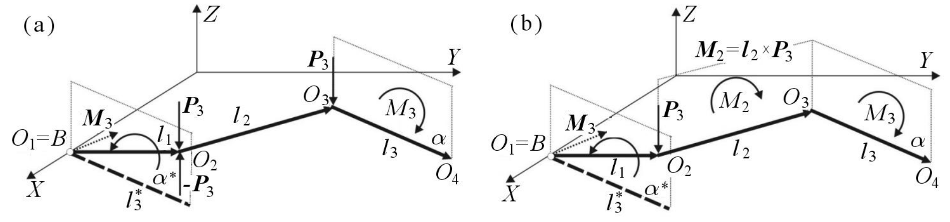

Fig. 9.

Figure 9a can be presented in a more legible manner as shown in Fig. 9b, i.e. the force $\boldsymbol{P}_3$ anchored at O_2, a couple with the moment vector $\boldsymbol{M}_3$, a bicouple located in the planes α and α^* with the bimoment $B_{\omega 3}$ and the representative of the couple located in the plane going through l_2 with the moment $\boldsymbol{M}_2 = \overline{O_2O_3} \times \boldsymbol{P}_3$.

In the next step of reduction, we add to the line the non-material section l_2^* (Fig. 10a) anchored at the point B and parallel to l_2, defining in this way the plane β^*, parallel to β. Then, the couple located in the plane β is transferred to β^* and drawn as a moment vector $\boldsymbol{M}_2$ anchored at the point B, while a bicouple is added and one of its couples, located in the plane β^*, is drawn in the form of a representative with the value $-M_2$ and the bimoment equal to $[\boldsymbol{P}_3, \overline{O_2O_3}, \overline{BO_3}]$.

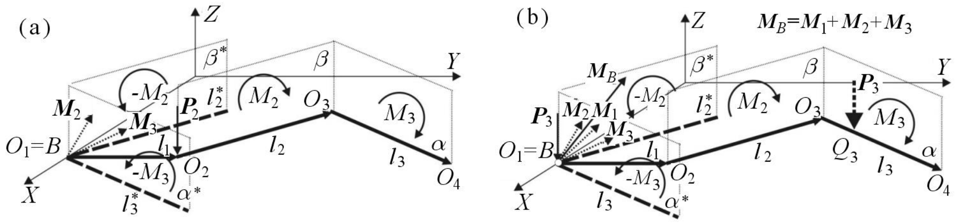

Fig. 10.

The last step of the reduction consists in transferring the force $\boldsymbol{P}_3$ anchored at O_2 to the point B, while adding a couple with the moment $\boldsymbol{M}_1 = \overline{BO_2} \times \boldsymbol{P}_3$, which is presented in Fig. 10b as the vector $\boldsymbol{M}_1$ anchored at B. Let us replace all the moment vectors anchored at B with their sum, described as

$\boldsymbol{M}_B = \boldsymbol{M}_1 + \boldsymbol{M}_2 + \boldsymbol{M}_3$. This sum can be described by a slightly different formula

$$\begin{aligned}\boldsymbol{M}_B &= \overline{BO_2} \times \boldsymbol{P}_3 + \overline{O_2O_3} \times \boldsymbol{P}_3 + \overline{O_3Q_3} \times \boldsymbol{P}_3 = \\ &= (\overline{BO_2} + \overline{O_2O_3} + \overline{O_3Q_3}) \times \boldsymbol{P}_3 = \overline{BQ_3} \times \boldsymbol{P}_3\end{aligned}$$

In conclusion, as a result of the process of reducing the system of the force $\boldsymbol{P}_3$ anchored at Q_3 to the point B, the following components are obtained at that point (Fig. 11):

- vector of the force $\boldsymbol{P}_3$,
- vector of the moment $\boldsymbol{M}_B = \overline{BQ_3} \times \boldsymbol{P}_3$, where $\overline{BQ_3}$ is a radius-vector,
- two bicouples located in the planes: (α, α^*) with the bimoment

$$B_{\omega 3} = [\boldsymbol{P}_3, \overline{O_3Q_3}, \overline{BQ_3}]$$

and (β, β^*) with the bimoment $[\boldsymbol{P}_3, \overline{O_2O_3}, \overline{BO_3}]$.

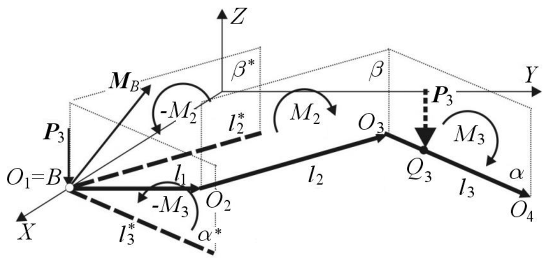

Fig. 11.

Both the initial system and the system reduced to the pole B are kinematically equivalent, as the reduction has been carried out exclusively by means of the elementary transformation with respect to which the sums, moments and bimoments are invariant.

If the initial system consists of numerous forces $\boldsymbol{P}_i$ applied to points Q_i of the material line, an analogous procedure could be used. Thus, we are justified to formulate the following theorem on the reduction of a system with respect to any pole B, belonging to the material line or to a rigid non-material section, joined to the material line without creating a loop:

> **Theorem:** A system is kinematically equivalent to a system consisting of a sum of that system, anchored at any selected pole B, a pair with the moment equal to the moment of this system with

respect to the pole, and bicouples located in parallel planes going through particular material lines and through the pole B, which are ascribed to the bimoment equal to the bimoment of all bicouples.

5. Concluding remark

It has already been stressed that all the theorems can be easily transferred to a flat material line of any shape. For the broken line under consideration it can be treated as an element of a sequence of approximate sums. The introduction of notion of the sectorial coordinate into the analysis considerably simplifies all the theorems.

References

1. Królikiewicz I., Siedmiograj Z., 1959, *Mechanika teoretyczna. Part I – Rachunek wektorowy, układy równoważne wektorów*, Wyd. Techn. PK, **1**
2. Paluch M., 2001, *Mechanika teoretyczna*, Textbook for students of technical universities, PK
3. Piechnik S., 1978, Kinematyczna równoważność układów sił, *Mechanika Teoretyczna i Stosowana*, **16**, 1

Algebra układów wektorów sił przyłożonych do płaskiej linii materialnej

Streszczenie

W pracy przedstawiono pewne twierdzenia dotyczące algebry układów wektorów sił przyłożonych do płaskiej, sztywnej jedynie w swej płaszczyźnie, linii materialnej. U podstaw rozważań leży modyfikacja jednego z dwóch aksjomatów równoważności przyjmowanych w algebrze układów sił przyłożonych do bryły sztywnej. Problem jest szczególnie ważny w teorii belek cienkościennych.

Manuscript received June 10, 2005; accepted for print October 10, 2005