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PIECEWISE LINEAR luz (...) AND tar (...) PROJECTIONS. PART 2 – APPLICATION IN MODELLING OF DYNAMIC SYSTEMS WITH FREEPLAY AND FRICTION

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The paper presents the idea and examples of application of a new method to the modelling of mechanical systems with freeplay and friction. This method bases on the piecewise linear luz (...) and tar (...) projections and their original mathematical apparatus. It is very useful for synthesis of simulation models and description of the stick-slip phenomenon in multi-body systems.

Key words: modelling, MBS, freeplay, friction, stick-slip, luz (...) and tar (...) projection

1. Introduction

Basic models of spring elements with freeplay (backlash, clereance) as well as dissipative elements with dry friction are based on piecewise linear characteristics (see Grzesikiewicz, 1990 and state-of-the-art papers: Armstrong-Helouvry *et al.*, 1994; Brogliatto *et al.*, 2002; Ibrahim, 1994; Nordin and Gutman, 2002). Such characteristics cohere (Fig. 1) with the luz (...) and tar (...) projections introduced by the author.

A simple but very efficient mathematical apparatus has been elaborated for the luz (...) and tar (...) projections in the first part of the paper (Żardecki, 2006). Therefore, the proposal of a new method for modelling piecewise linear systems using the luz (...) and tar (...) projections was a natural consequence.

This paper presents detailed rules of use of the luz (...) and tar (...) projections in synthesis of mathematical models of systems with freeplay or friction. The method of modelling is described on a representative example

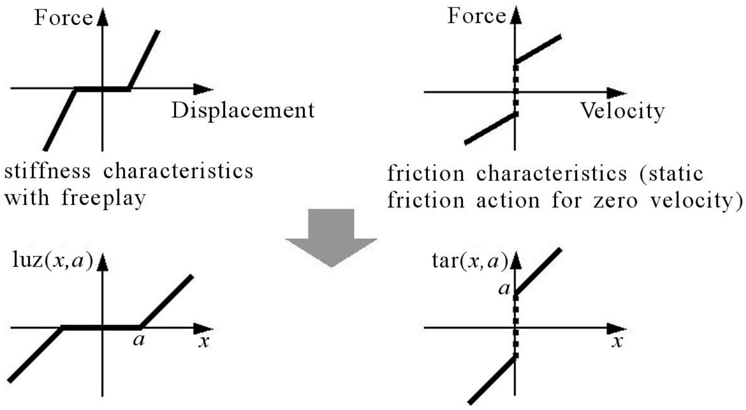

Fig. 1. Idea of description of characteristics by luz (...) and tar (...) projections

of a multi-element gear system (Fig. 10) installed by bearings in a fixed stiff casing.

In the beginning, necessary mathematical models of elementary subsystems with single freeplay or friction will be presented. Then, an original method of synthesis of models of complex systems will be described and applied to the modelling of some exemplary multi-element system.

2. Modelling of systems with single freeplay

2.1. Elementary model of elasticity with freeplay

Notation:

- z_1, z_2 – dispalcements of elements
- Δz_0 – freeplay parameter (0.5 of total freeplay), $\Delta z_0 = (z_1 - z_2)_0$
- F_{S12}, F_{S21} – spring force affecting elements 1 and 2, respectively
- k_{12} – stiffness coefficient

An elementary model of elasticity with freeplay concerns the relationship between the strain force and relative displacement of two coactive toothed elements (Fig. 2).

The relation between the interacting force F_S and relative displacement Δz of the elements expresses piecewise linear characteristics with the "dead zone" (Fig. 3).

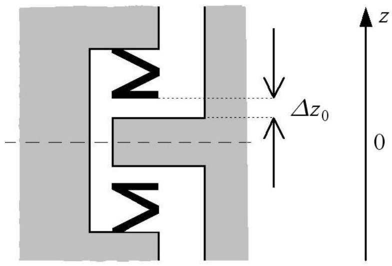

Fig. 2. Elastic toothed elements with freeplay

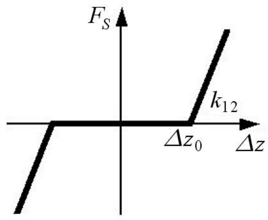

Fig. 3. Strain characteristics with freeplay

Analytical expressions are

$$F_{S12} = k_{12}\,\mathrm{luz}\,(z_1 - z_2, \Delta_0) \qquad F_{S21} = -k_{12}\,\mathrm{luz}\,(z_1 - z_2, \Delta_0)$$

The elementary model of elasticity with freeplay refers to all discrete systems in which a toothed mechanism of rigid solids is given by weightless springs with freeplay. They can be simple sliding or rotation elements (rack and pinion elements, bars, shafts, gears, etc.).

Analytical descriptions of rotation systems with freeplay concern angular characteristics (strain torque versus angular displacement).

2.2. Model of angular elasticity with freeplay for gear elements

Notation:

δ, γ	–	angular displacements of gear wheels (the applied sign convention facilitates the modelling of dynamic systems with gears); $\delta = \alpha_1$, $\gamma = -\alpha_2$
$M_{S\delta\gamma}, M_{S\gamma\delta}$	–	spring torques acting on wheel 1 and 2
$F_{\delta\gamma}, F_{\gamma\delta}$	–	action/reaction spring forces $(F_{\delta\gamma} = -F_{\gamma\delta})$
r_δ, r_γ	–	radii of wheel 1 and 2

p	–	gear ratio (when without freeplay $\delta r_\delta = \gamma r_\gamma$ so $\delta = p\gamma$); $p = r_\gamma / r_\delta$
l	–	perimetric translocation of wheels; $l = r_\delta \delta - r_\gamma \gamma$
l_0	–	perimetric freeplay (0.5 of total freeplay between teeth)
$(\delta - p\gamma)_0$	–	angular freeplay parameter (0.5 of total freeplay in wheel 1); $(\delta - p\gamma)_0 = l_0/r_\delta$
$((\delta/p) - \gamma)_0$	–	angular freeplay parameter (0.5 of total freeplay in wheel 2); $(\delta/p) - \gamma)_0 = l_0/r_\gamma$
K	–	stiffness coefficient of a pair of teeth
$k_{\delta\gamma}, k_{\gamma\delta}$	–	angular stiffness coefficient of a pair of teeth measured from wheel 1 and 2, respectively; $k_{\delta\gamma} = r_\delta^2 K$, $k_{\gamma\delta} = r_\gamma^2 K$

We discuss a simplified model of the gear that consists of two well-coacting toothed wheels characterised by effective radii of wheels (radii determine the gear ratio), the perimetric freeplay and the stiffness coefficient between their teeth. This model concerns rather small disturbances.

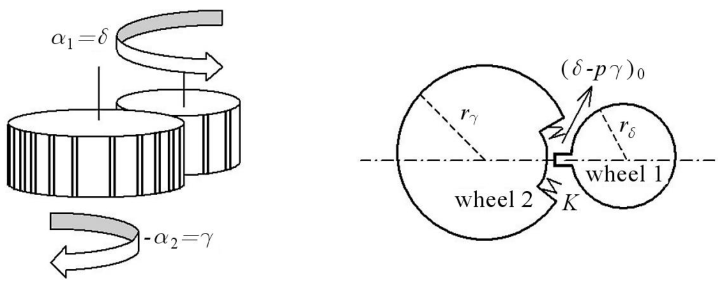

Fig. 4. Gear elements with tooth freeplay

Because of freeplay, the spring force can be described by the following formula

$$F_{\delta\gamma} = K\,\mathrm{luz}\,(l, l_0) = K\,\mathrm{luz}\,(r_\delta \delta - r_\gamma \gamma, l_0) = r_\delta K\,\mathrm{luz}\Big(\delta - \frac{r_\gamma}{r_\delta}\gamma, \frac{l_0}{r_\delta}\Big)$$

$$F_{\gamma\delta} = -K\,\mathrm{luz}\,(l, l_0) = -K\,\mathrm{luz}\,(r_\delta \delta - r_\gamma \gamma, l_0) = -r_\gamma K\,\mathrm{luz}\Big(\frac{r_\delta}{r_\gamma}\delta - \gamma, \frac{l_0}{r_\gamma}\Big)$$

The spring torques of wheels are $M_{S\delta\gamma} = r_\delta F_{\delta\gamma}$, $M_{S\gamma\delta} = r_\gamma F_{\gamma\delta}$, hence

$$M_{S\delta\gamma} = r_\delta^2 K\,\mathrm{luz}\Big(\delta - \frac{r_\gamma}{r_\delta}\gamma, \frac{l_0}{r_\delta}\Big)$$

$$M_{S\gamma\delta} = -r_\gamma^2 K\,\mathrm{luz}\Big(\frac{r_\delta}{r_\gamma}\delta - \gamma, \frac{l_0}{r_\gamma}\Big)$$

The relations between torques and relative angular displacements are expressed by characteristics of the same type as given in Fig. 3, and described by formulas

$$M_{S\delta\gamma} = k_{\delta\gamma} \operatorname{luz}(\delta - p\gamma, (\delta - p\gamma)_0) = \frac{k_{\gamma\delta}}{p} \operatorname{luz}\Big(\frac{\delta}{p} - \gamma, \Big(\frac{\delta}{p} - \gamma\Big)_0\Big)$$

$$M_{S\gamma\delta} = -pk_{\delta\gamma} \operatorname{luz}(\delta - p\gamma, (\delta - p\gamma)_0) = -k_{\gamma\delta} \operatorname{luz}\Big(\frac{\delta}{p} - \gamma, \Big(\frac{\delta}{p} - \gamma\Big)_0\Big) = -pM_{\gamma\delta}$$

2.3. Model of angular elasticity with freeplay for elements twisted by elastic shaft with freeplay in its mounting

Notation:

ψ, δ	–	angular dispalcements of elements; $\psi = \alpha_1$, $\delta = \alpha_2$
$M_{S\psi\delta}, M_{S\delta\psi}$	–	spring torque acting on elements 1 and 2, respectively
$(\psi - \delta)_0$	–	angular freeplay parameter (0.5 of total freeplay)
$k_{\psi\delta}$	–	angular stiffness

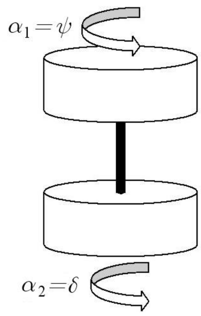

Fig. 5. Elastic shaft with freeplay in its mounting

The relation between torques and relative angular displacements is expressed by the same characteristics as those given in Fig. 3. It can be written as

$$M_{S\psi\delta} = k_{\psi\delta} \operatorname{luz}(\psi - \delta, (\psi - \delta)_0)$$
$$M_{S\delta\psi} = -k_{\psi\delta} \operatorname{luz}(\psi - \delta, (\psi - \delta)_0)$$

3. Modelling of systems with single friction

An elementary dissipation model with dry friction can be formulated for two cases: the first one – when friction exists between a moving element and a fixed base, the second case – when the friction force acts between two moving bodies. We discuss here the first model. It will be applied to the description of the model of a bearing element. The second one (more complicated and not indispensable for modelling of the exemplary system) will be presented in a next special publication.

3.1. Elementary friction model for "moving element – fixed base" system

Notation:

F_T – friction force
$\Delta\dot{z}$ – slip velocity (here $\Delta\dot{z} = \dot{z}$)
F – external force
C – damping coefficient
F_{T0} – maximum value of dry friction
M – mass of block

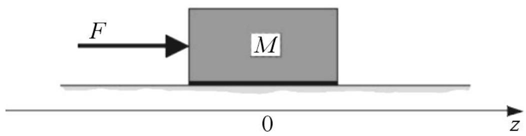

Fig. 6. Moving element – fixed base system

The friction force can be expressed by modified Coulomb's characteristics (which take into consideration the possibility of action of static dry friction at zero velocity).

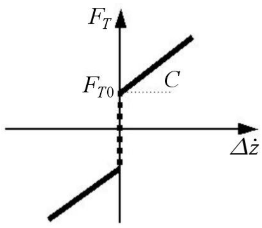

Fig. 7. Modified Coulomb's friction characteristics

The analytical description is given by the formula

$$F_T = C \operatorname{tar}\Big(\Delta\dot{z}, \frac{F_{T0}}{C}\Big)$$

This formula needs only some linear damping and the same maximum absolute values of kinetic as well as static dry friction. Such conditions are usually accomplished, especially when the so called Stribeck effect does not appear. Although its conciseness, this formula contains an ample friction description. It express the friction force as a sum of viscous and dry (kinetic and static) friction for every velocity states

$$F_T = C\,\mathrm{tar}\Big(\dot z, \frac{F_{T0}}{C}\Big) = \begin{cases} C\dot z - F_{T0} & \text{if } \ \dot z < 0 \\ F_{T0}s^* & \text{if } \ \dot z = 0 \\ C\dot z + F_{T0} & \text{if } \ \dot z > 0 \end{cases}$$

where $s^* \in [-1, 1]$ so

$$F_T = \underbrace{C\dot z}_{\text{Viscous friction (damping)}} + \underbrace{\underbrace{F_{T0}\,\mathrm{sgn}\,(\dot z)}_{\text{Kinetic dry friction}} + \underbrace{F_{T0}s^*}_{\text{Static dry friction}}}_{\text{Dry friction}}$$

$$F_T = \underbrace{C\dot z + F_{T0}\,\mathrm{sgn}\,(\dot z)}_{\text{Kinetic friction (Coulomb's)}} + \underbrace{F_{T0}s^*}_{\text{Static friction (also stiction)}}$$

At zero velocity state, the friction force is not calculated on the basis of the modified Coulomb's characteristics. The calculation of $F_{T0}s^*$ needs discussion of the dynamic model.

The model of motion dynamics is determined by the differential inclusion

$$M\ddot z(t) \in F(t) - C\,\mathrm{tar}\Big(\dot z(t), \frac{F_{T0}}{C}\Big) \qquad \text{where} \qquad s^*(t) \in [-1, 1]$$

Note, that for $\dot z(t) = 0$, $M\ddot z(t) \in F(t) - F_{T0}s^*(t)$. The replacement of ambiguity inclusion by an explicit relation demands calculation of $s^*(t)$. This will be shown, firstly – using a heuristic rule, then – using general physic principles.

As it is well known, the description of dry friction can be done by the following heuristic rule: when the slide velocity goes to zero, the static friction force $F_{TS}(t)$ starts, and the stiction state (when $\dot z(t) = 0$, $\ddot z(t) = 0$) may exist until $F_{TS}(t) \in [-F_{T0}, F_{T0}]$. If $\dot z(t) = 0$, while the condition $F_{TS}(t) \in [-F_{T0}, F_{T0}]$, $\ddot z(t) = 0$ is impossible, this means only a temporary static friction without stiction. In such a state, $\ddot z(t) \neq 0$ and $F_{TS}(t) = \pm F_{T0}$, where the sign of the friction force asserts its opposite action. Thus

$$F_{TS}(t) = F_{T0}s^*(t) = \begin{cases} F_{T0} & \text{if } \ F(t) \geqslant F_{T0} & (\text{then } \ddot z(t) \neq 0) \\ F(t) & \text{if } \ -F_{T0} < F(t) < F_{T0} & (\text{then } \ddot z(t) = 0) \\ -F_{T0} & \text{if } \ F(t) \leqslant -F_{T0} & (\text{then } \ddot z(t) \neq 0) \end{cases}$$

in other words

$$F_{TS}(t) = F_{T0}s^*(t) = F(t) - \mathrm{luz}\,(F(t), F_{T0})$$

The heuristic description of dry friction with stiction corresponds to the S-S mathematical procedure (see its definition in Żardecki (2006)). Marking by $s^{**}(t)$ the singularity variable, which balances the equation of motion when $\ddot{z}(t) = 0$ (stiction), we have $0 = F(t) - F_{T0}s^{**}(t)$. Hence, on the basis of the S-S procedure, we obtain

$$F_{T0}s^{*}(t) = F(t) - \operatorname{luz}(F(t), F_{T0})$$

The mathematical description of dry friction action ensuing from the heuristic rule (or from the S-S procedure) is equivalent to application of some general variation principle (Jourdain's or Gauss' principle).

On the basis of the Jourdain principle (with extensions) $\delta\dot{z}$ variation has to be minimized in relation to the s^* singularity in continuity of the $\dot{z} = 0$ state. This means minimization of $|\ddot{z}|$. The task is following

$$s^* : \quad \min_{s^*} |\ddot{z}| \ \wedge \ s^* \in [-1, 1]$$

or

$$s^* : \quad \min_{s^*} \Big|\frac{M\ddot{z}}{F_{T0}}\Big| = \min_{s^*} \Big|\frac{F}{F_{T0}} - s^*\Big| \ \wedge \ s^* \in [-1, 1]$$

The solution is

$$s^* = \begin{cases} 1 & \text{if} \quad s^{**} > 1 \\ s^{**} & \text{if} \quad -1 \leqslant s^{**} \leqslant 1 \\ -1 & \text{if} \quad s^{**} < -1 \end{cases} \qquad \text{where} \qquad s^{**} = \frac{F}{F_{T0}}$$

so

$$s^* = \frac{F}{F_{T0}} - \operatorname{luz}\Big(\frac{F}{F_{T0}}, 1\Big)$$

When we apply the Gauss principle, the so called „acceleration energy" is minimized

$$s^* : \quad \min_{s^*}(M\ddot{z}^2) \ \wedge \ s^* \in [-1, 1]$$

so

$$\begin{aligned} s^* &: \quad \frac{\partial}{\partial s^*}(M\ddot{z}^2) = 0 \ \wedge \ s^* \in [-1, 1] \\ s^* &: \quad \frac{\partial}{\partial s^*}\Big(\frac{(F - F_{T0}s^*)^2}{M}\Big) = 0 \ \wedge \ s^* \in [-1, 1] \\ s^* &: \quad F - F_{T0}s^* = 0 \ \wedge \ s^* \in [-1, 1] \end{aligned}$$

so

$$s^* = \frac{F}{F_{T0}} - \operatorname{luz}\Big(\frac{F}{F_{T0}}, 1\Big)$$

Therefore, applying the Jourdain or Gauss rules

$$F_{T0}s^*(t) = F(t) - \operatorname{luz}(F(t), F_{T0})$$

As we see, all methods have given the same result. They mean that the static dry friction $F_{TS} = F_{T0}s^*$ can be here described by the characteristics shown in Fig. 8.

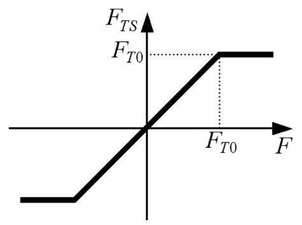

Fig. 8. Characteristics of static dry friction

Applying the $F_{T0}s^*(t)$ formula, for $\dot{z}(t) = 0$, the inclusion description can be replaced by the differential equation

$$M\ddot{z}(t) = \operatorname{luz}(F(t), F_{T0})$$

which perfectly expresses the essence of the "stick-slip" phenomenon. Note, that when $\dot{z}(t) = 0$ and $-F_{T0} < F(t) < F_{T0}$, then also $\ddot{z}(t) = 0$ (stiction state). When for $\dot{z}(t) = 0$, $F(t) < -F_{T0}$ or $F(t) > F_{T0}$, then $\ddot{z}(t) \neq 0$ (no stiction state). In such a case, the state of $\dot{z}(t) = 0$ is only a temporary crossing. When the block is sticky and the excitation $F(t) < -F_{T0}$ or $F(t) > F_{T0}$, then the state of slip begins.

Summing up, for every $\dot{z}(t)$, the dynamic model of the block can be written by a differential motion equation with the singularity $s^*(t)$

$$M\ddot{z}(t) = F(t) - C \operatorname{tar}\Big(\dot{z}(t), \frac{F_{T0}}{C}\Big)$$

where $F_{T0}s^*(t) = F(t) - \operatorname{luz}(F(t), F_{T0})$.

This model may be expressed without the singularity using a variable-structural form

$$M\ddot{z}(t) = \begin{cases} F(t) - C \operatorname{tar}\Big(\dot{z}(t), \dfrac{F_{T0}}{C}\Big) & \text{if} \quad \dot{z}(t) \neq 0 \\ \operatorname{luz}(F(t), F_{T0}) & \text{if} \quad \dot{z}(t) = 0 \end{cases}$$

Both forms of the model are equivalent and both can be used in simulation.

The presented elementary model of the stick-slip phenomenon is equivalent with other models (eg. the well known Karnop model (Karnopp, 1985)). Those models have rather more complicate forms, difficult for analytical operations. A compact, well coherent to parametric operations form of our model is its important feature. Thanks to the luz(...) and tar(...) mathematical apparatus, formal parametric simplification of the friction model is possible and efficient. This is very important for automation of models synthesis and, generally, for a more efficient, the so-called, MBS software.

The easiness of parametric operation on the friction model is shown below. Consider a problem of simplification of the model when the mass of the block is assumed to be negligible. In this case, when $M = 0$, the inclusion model is degenerated to

$$0 \in -C \operatorname{tar}\Big(\dot{z}(t), \frac{F_{T0}}{C}\Big) + F(t)$$

On the basis of tar(...) projection (Theorem 5.3 in the first part (Żardecki, 2006)) such an inclusion passes to the equation form

$$\dot{z}(t) = \frac{1}{C} \operatorname{luz}(F(t), F_{T0})$$

This equation has no $s^*(t)$ and is well determined also for $\dot{z}(t) = 0$. Because of the luz(...) description, the stick-slip problem is solved "automatically":

– when $-F_{T0} < F(t) < F_{T0}$, $\operatorname{luz}(F(t), F_{T0}) = 0$ and $\dot{z}(t) = 0$ is continued,

– when $F(t) < -F_{T0}$ or $F(t) > F_{T0}$, the stiction state is terminated and $\dot{z}(t) \neq 0$.

Analytical descriptions of rotation systems with friction concern angular characteristics of torque and constitutive equations. Here we present a model of a bearing element.

3.2. Friction model of bearing element

Notation:

$\dot{\alpha}$	–	angular velocity
M_α	–	external torque
M_T	–	friction torque
μ	–	damping coefficient
M_{T0}	–	maximum friction torque
I	–	moment of inertia

A stiff solid (inertial or non-inertial) element having a bearing in a fixed base is analysed here.

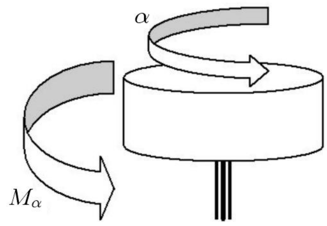

Fig. 9. One-mass element with friction in bearing

In this case, the Coulomb characteristics describing friction in the bearing relates the friction torque with the angular velocity. Assuming that the element may be inertial or non-inertial, one presents two variants of the mathematical model.

- The model of inertial element with friction in the bearing (two variants):
  – equation with singularity

$$I_\alpha \ddot{\alpha}_i(t) = M_\alpha(t) - \mu_\alpha \operatorname{tar}\Big(\dot{\alpha}(t), \frac{M_{T0\alpha}}{\mu_\alpha}\Big)$$

  where for $\dot{\alpha}(t) = 0$

$$M_{T0\alpha} s^*_\alpha(t) = M_\alpha(t) - \operatorname{luz}\Big(M_\alpha(t), M_{T0\alpha}\Big)$$

  – variable-structure equation (without singularity)

$$I_\alpha \ddot{\alpha}(t) = \begin{cases} M_\alpha(t) - \mu_\alpha \operatorname{tar}\Big(\dot{\alpha}(t), \frac{M_{T0\alpha}}{\mu}\Big) & \text{if} \quad \dot{\alpha}(t) \neq 0 \\ \operatorname{luz}(M_\alpha(t), M_{T0\alpha}) & \text{if} \quad \dot{\alpha}(t) = 0 \end{cases}$$

- The model of non-inertial element with friction in the bearing

$$\mu_\alpha \dot{\alpha}(t) = \operatorname{luz}(M_\alpha(t), M_{T0\alpha})$$

The model of the bearing element will be useful for synthesis of the model of the considered multi-body system.

4. Modelling of multi-body systems with freeplay and friction

An efficient method of modelling some class of MBS systems with freeplay and friction has been elaborated on the base of simple piecewise linear models of elementary subsystems and the $\mathrm{luz}(\ldots)$ and $\mathrm{tar}(\ldots)$ projections. This class of systems concerns especially mechanisms which can be treated as rotational systems with fixed axles of rotation. They have invariable mechanical structures, but because of the stick-slip phenomenon their mathematical description have variable structural forms. The method is following:

- Firstly, a complementary discrete physical model is created. It can be built with:

  – stiff solid (inertial or non-inertial) elements

  – spring elements with freeplay

  – dissipative elements with dry friction.

- In the primary stage of modelling, all friction sub-systems are treated as sub-systems having non-zero viscous friction, and all freeplay connections are treated as sub-systems having non-zero elasticity. Solid elements are treated as inertial bodies as well. Therefore, the primary physical model has a redundant form.
- Then equations of motion for the primary model are built. The Lagrange or other well known method can be used. All equations are created by balancing the inertial forces or torques with external excitation, dissipation as well as elasticity ones which are described by the $\mathrm{luz}(\ldots)$ and $\mathrm{tar}(\ldots)$ projections. As a result, we obtain a redundant mathematical model. Coordinates of bodies and their derivatives are the model variables. The mass and geometric parameters of the solid elements and parameters of piecewise linear characteristics are the parameters of the model.
- The applicable model is obtained from the redundant model. This operation is done by formal parametric and assymptotic reduction. This means that we must determine analytical forms of limitations – eg. very small masses or moments of inertia tend to zero, very large stiffness – to infinity. Calculations are supported by the mathematical apparatus of $\mathrm{luz}(\ldots)$ and $\mathrm{tar}(\ldots)$ projections. When the model is provided for simulation investigations, its form should not contain the so-called stiff differential equations as well as any equations of constraints.
- As a result of successive reductions, we obtain successive aproximations of the primary model.

The main advantage of the method is simplicity of the primary model and mathematical formalism of the model reduction. Simplifications of the reduced model ensue from mathematical formulas of the $\mathrm{luz}(\ldots)$ and $\mathrm{tar}(\ldots)$

projections. Oftentimes, reduction of equations seems to be very complicated or even impossible be realised, while application of the luz(...) and tar(...) projections makes them surprisingly simple.

A representative example of application of this method is shown below.

4.1. Model of exemplary multi-body system with freeplay and friction

Notation:

Symbols are the same as in Section 2 and Section 3.

A multi-body rotation system consists of two inertial solids, two shafts and two gear wheels (Fig. 10). We assume that the gear wheels are weightless. There are three freeplays: one between gears teeth and two – in sockets of shafts. The rotation elements have four bearings with friction. This system is driven by two external torques $M_\psi(t)$, $M_\varphi(t)$.

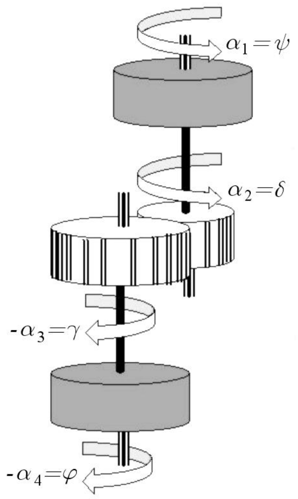

Fig. 10. An example of a multi-body rotation system

The primary redundant model is given by Newton's equations of motion

$$
\begin{aligned}
&I_\psi \ddot{\psi}(t) + \mu_\psi \operatorname{tar}\Big(\dot{\psi}(t), \frac{M_{T0\psi}}{\mu_\psi}\Big) + k_{\psi\delta} \operatorname{luz}\big(\psi(t) - \delta(t), (\psi - \delta)_0\big) = M_\psi(t) \\
&I_\delta \ddot{\delta}(t) + \mu_\delta \operatorname{tar}\Big(\dot{\delta}(t), \frac{M_{T0\delta}}{\mu_\delta}\Big) - k_{\psi\delta} \operatorname{luz}\big(\psi(t) - \delta(t), (\psi - \delta)_0\big) + \\
&\qquad + k_{\delta\gamma} \operatorname{luz}\big(\delta(t) - p\gamma(t), (\delta - p\gamma)_0\big) = 0
\end{aligned}
$$

$$
\begin{aligned}
&I_\delta\dot{\gamma}(t) + \mu_\gamma \operatorname{tar}\Big(\dot{\gamma}(t), \frac{M_{T0\gamma}}{\mu_\gamma}\Big) - pk_{\delta\gamma} \operatorname{luz}\big(\delta(t) - p\gamma(t), (\delta - p\gamma)_0\big) + \\
&\qquad + k_{\gamma\varphi} \operatorname{luz}\big(\gamma(t) - \varphi(t), (\gamma - \varphi)_0\big) = 0 \\
&I_\varphi\dot{\varphi}(t) + \mu_\varphi \operatorname{tar}\Big(\dot{\varphi}(t), \frac{M_{T0\varphi}}{\mu_{\varphi\gamma}}\Big) - k_{\gamma\varphi} \operatorname{luz}\big(\gamma(t) - \varphi(t), (\gamma - \varphi)_0\big) = M_\varphi(t)
\end{aligned}
$$

These equations will be simplified, therefore at this moment we need not to determine their variable-structural forms or equations describing the singularities s^*_φ, s^*_δ, s^*_γ, s^*_ψ, which are necessary for zero velocities.

Simplification 1: when $I_\delta = I_\gamma = 0$ (weightless gear)

The second and third equation assumes a degenerated form

$$
\begin{aligned}
&\mu_\delta \operatorname{tar}\Big(\dot{\delta}(t), \frac{M_{T0\delta}}{\mu_\delta}\Big) - k_{\psi\delta} \operatorname{luz}\big(\psi(t) - \delta(t), (\psi - \delta)_0\big) + \\
&\qquad + k_{\delta\gamma} \operatorname{luz}\big(\delta(t) - p\gamma(t), (\delta - p\gamma)_0\big) = 0 \\
&\mu_\gamma \operatorname{tar}\Big(\dot{\gamma}(t), \frac{M_{T0\gamma}}{\mu_\gamma}\Big) - pk_{\delta\gamma} \operatorname{luz}\big(\delta(t) - p\gamma(t), (\delta - p\gamma)_0\big) + \\
&\qquad + k_{\gamma\varphi} \operatorname{luz}\big(\gamma(t) - \varphi(t), (\gamma - \varphi)_0\big) = 0
\end{aligned}
$$

Applying Theorems 2.1, 3.2, see Żardecki (2006), we obtain these equations in disentangled form. The applicable model of the system has a form

$$
\begin{aligned}
&I_\psi\ddot{\psi}(t) + \mu_\psi \operatorname{tar}\Big(\dot{\psi}(t), \frac{M_{T0\psi}}{\mu_\psi}\Big) + k_{\psi\delta} \operatorname{luz}\big(\psi(t) - \delta(t), (\psi - \delta)_0\big) = M_\psi(t) \\
&\qquad \text{where for } \dot{\psi}(t) = 0 \\
&\qquad\qquad M_{T0\psi} s^*_\psi(t) = M_{\psi\psi}(t) - \operatorname{luz}\big(M_{\psi\psi}(t), M_{T0\psi}\big) \qquad \text{and} \\
&\qquad\qquad M_{\psi\psi}(t) = M_\psi(t) - k_{\psi\delta} \operatorname{luz}\big(\psi(t) - \delta(t), (\psi - \delta)_0\big) \\
&\mu_\delta\dot{\delta}(t) + \operatorname{luz}\Big(k_{\psi\delta} \operatorname{luz}\big(\psi(t) - \delta(t), (\psi - \delta)_0\big) + \\
&\qquad\qquad - k_{\delta\gamma} \operatorname{luz}\big(\delta(t) - p\gamma(t), (\delta - p\gamma)_0\big), M_{T0\delta}\Big) = 0 \\
&\mu_\gamma\dot{\gamma}(t) + \operatorname{luz}\Big(-pk_{\delta\gamma} \operatorname{luz}\big(\delta(t) - p\gamma(t), (\delta - p\gamma)_0\big) + \\
&\qquad\qquad + k_{\gamma\varphi} \operatorname{luz}\big(\gamma(t) - \varphi(t), (\gamma - \varphi)_0\big), M_{T0\gamma}\Big) = 0 \\
&I_\varphi\dot{\varphi}(t) + \mu_\varphi \operatorname{tar}\Big(\dot{\varphi}(t), \frac{M_{T0\varphi}}{\mu_{\varphi\gamma}}\Big) - k_{\gamma\varphi} \operatorname{luz}\big(\gamma(t) - \varphi(t), (\gamma - \varphi)_0\big) = M_\varphi(t) \\
&\qquad \text{where for } \dot{\varphi}(t) = 0 \\
&\qquad\qquad M_{T0\varphi} s^*_\varphi(t) = M_{\varphi\varphi}(t) - \operatorname{luz}\big(M_{\varphi\varphi}(t), M_{T0\varphi}\big) \qquad \text{and} \\
&\qquad\qquad M_{\varphi\varphi}(t) = M_\varphi(t) + k_{\gamma\varphi} \operatorname{luz}\big(\gamma(t) - \varphi(t), (\gamma - \varphi)_0\big)
\end{aligned}
$$

Simplification 2: when $I_\delta = I_\gamma = 0$ and $\mu_\delta = \mu_\gamma = 0$, $M_{T0\delta} = M_{T0\gamma} = 0$ (weightless gear with perfect bearing, but with teeth feeplay).

The second and third equations assume an involved form

$$\begin{aligned}
&k_{\psi\delta}\,\mathrm{luz}\,(\psi(t)-\delta(t),(\psi-\delta)_0) - k_{\delta\gamma}\,\mathrm{luz}\,(\delta(t)-p\gamma(t),(\delta-p\gamma)_0) = 0\\
&-pk_{\delta\gamma}\,\mathrm{luz}\,(\delta(t)-p\gamma(t),(\delta-p\gamma)_0) + k_{\gamma\varphi}\,\mathrm{luz}\,(\gamma(t)-\varphi(t),(\gamma-\varphi)_0) = 0
\end{aligned}$$

These equations are entangled constraints for the first and forth equation. To reduce the variables $\delta(t)$ and $\gamma(t)$, the second and third equations are transformed to the form

$$\begin{aligned}
\mathrm{luz}\,(\psi(t)-\delta(t),(\psi-\delta)_0) &= \frac{k_{\delta\gamma}}{k_{\psi\delta}}\,\mathrm{luz}\,(\delta(t)-p\gamma(t),(\delta-p\gamma)_0)\\
\mathrm{luz}\,(p\gamma(t)-p\varphi(t),p(\gamma-\varphi)_0) &= \frac{p^2k_{\delta\gamma}}{k_{\gamma\varphi}}\,\mathrm{luz}\,(\delta(t)-p\gamma(t),(\delta-p\gamma)_0)
\end{aligned}$$

Applying properties of the luz (...), on the basis of Theorem 4.3 (Żardecki, 2006), we find

$$\begin{aligned}
\mathrm{luz}\,(\psi(t)-\delta(t),(\psi-\delta)_0) &= \frac{\frac{k_{\delta\gamma}}{k_{\psi\delta}}}{\frac{p^2k_{\delta\gamma}}{k_{\gamma\varphi}}+\frac{k_{\delta\gamma}}{k_{\psi\delta}}+1}\cdot\\
&\quad\cdot\mathrm{luz}\,(\psi(t)-p\varphi(t),(\psi-\delta)_0+(\delta-p\gamma)_0+p(\gamma-\varphi)_0)\\
\mathrm{luz}\,(p\gamma(t)-p\varphi(t),p(\gamma-\varphi)_0) &= \frac{\frac{p^2k_{\delta\gamma}}{k_{\gamma\varphi}}}{\frac{p^2k_{\delta\gamma}}{k_{\gamma\varphi}}+\frac{k_{\delta\gamma}}{k_{\psi\delta}}+1}\,\mathrm{luz}\,(\psi(t)-p\varphi(t),(\delta-p\gamma)_0)
\end{aligned}$$

As the final result, we obtain the model without algebraic constraints (!)

$$\begin{aligned}
&I_\psi\ddot{\psi}(t) + \mu_\psi\,\mathrm{tar}\Big(\dot{\psi}(t),\frac{M_{T0\psi}}{\mu_\psi}\Big) + k_{\psi\varphi}\,\mathrm{luz}\,(\psi(t)-p\varphi(t),(\psi-p\varphi)_0) = M_\psi(t)\\
&\qquad\text{where for } \dot{\psi}(t)=0\\
&\qquad\qquad M_{T0\psi}s_\psi^*(t) = M_{\psi\psi}(t) - \mathrm{luz}\,(M_{\psi\psi}(t),M_{T0\psi}) \qquad \text{and}\\
&\qquad\qquad M_{\psi\psi}(t) = M_\psi(t) - k_{\psi\varphi}\,\mathrm{luz}\,(\psi(t)-p\varphi(t),(\psi-p\varphi)_0)\\
&I_\varphi\dot{\varphi}(t) + \mu_\varphi\,\mathrm{tar}\Big(\dot{\varphi}(t),\frac{M_{T0\varphi}}{\mu_{\varphi\gamma}}\Big) - pk_{\psi\varphi}\,\mathrm{luz}\,(\psi(t)-p\varphi(t),(\psi-p\varphi)_0) = M_\varphi(t)\\
&\qquad\text{where for } \dot{\varphi}(t)=0\\
&\qquad\qquad M_{T0\varphi}s_\varphi^*(t) = M_{\varphi\varphi}(t) - \mathrm{luz}\,(M_{\varphi\varphi}(t),M_{T0\varphi}) \qquad \text{and}\\
&\qquad\qquad M_{\varphi\varphi}(t) = M_\varphi(t) + pk_{\psi\varphi}\,\mathrm{luz}\,(\psi(t)-p\varphi(t),(\psi-p\varphi)_0)
\end{aligned}$$

where $k_{\psi\varphi}$ is the reduced stiffness coefficient

$$k_{\psi\varphi} = \Big(\frac{1}{k_{\psi\delta}} + \frac{1}{k_{\delta\gamma}} + \frac{p^2}{k_{\gamma\varphi}}\Big)^{-1}$$

and $(\psi - p\varphi)_0$ is the reduced freeplay parameter

$$(\psi - p\varphi)_0 = (\psi - \delta)_0 + (\delta - p\gamma)_0 + p(\gamma - \varphi)_0$$

Note, when the gear stiffness $k_{\delta\gamma} \to \infty$ (practically $k_{\delta\gamma} \gg k_{\psi\delta}, k_{\gamma\varphi}$), then

$$k_{\psi\varphi} = \Big(\frac{1}{k_{\psi\delta}} + \frac{1}{k_{\delta\gamma}} + \frac{p^2}{k_{\gamma\varphi}}\Big)^{-1} \xrightarrow{k_{\delta\gamma}} \Big(\frac{1}{k_{\psi\delta}} + \frac{p^2}{k_{\gamma\varphi}}\Big)^{-1}$$

This result confirms the possibility of operation with the piecewise linear model by its reduced parameters. Their theoretical as well as well known in practice mathematical forms are compatible.

Simplification 3: where $I_\delta = I_\gamma = 0$, $\mu_\delta = \mu_\gamma = 0$, $M_{T0\delta} = M_{T0\gamma} = 0$, and $k_{\delta\gamma} \gg k_{\psi\delta}, k_{\gamma\varphi}$ $(k_{\delta\gamma} \to \infty)$, $(\psi - \delta)_0 = (\gamma - \varphi)_0 = (\delta - p\gamma) = 0$.
(Ideal kinematic gear with stiff teeth in the system without freeplay and dry friction).

The reduced linear model is

$$\begin{aligned} &I_\psi\ddot{\psi}(t) + \mu_\psi\dot{\psi}(t) + k_{\psi\varphi}(\psi(t) - p\varphi(t)) = M_\psi(t) \\ &I_\varphi\ddot{\varphi}(t) + \mu_\varphi\dot{\varphi}(t) - pk_{\psi\varphi}(\psi(t) - p\varphi(t)) = M_\varphi(t) \end{aligned}$$

As we can see, all these simplifications could be strictly formal.

5. Final remarks

In this paper, the idea and examples of application of a new method to the modelling of mechanical systems with freeplay and friction have been presented. The method is based on the piecewise linear $\text{luz}(\dots)$ and $\text{tar}(\dots)$ projections and their original mathematical apparatus. It is very useful for description of stick-slip processes in multi-body systems which can be described by piecewise – linear equations.

The presented method has been already applied to synthesis of simulation models of steering systems with freeplays and dry friction (see for example Lozia and Żardecki, 2002, 2005; Żardecki, 1998, 2005a,b).

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Przedziałami liniowe odwzorowania luz (...) i tar (...).
Część 2 – Zastosowanie w modelowaniu układów dynamicznych z luzem i tarciem

Streszczenie

Artykuł przedstawia ideę i przykłady zastosowania nowej metody modelowania układów mechanicznych z luzem i tarciem. Metoda bazuje na przedziałami liniowych odwzorowaniach luz (...) i tar (...) oraz ich oryginalnym aparacie matematycznym. Metoda jest bardzo użyteczna dla syntezy modeli symulacyjnych i opisu zjawiska *stick-slip* w układach wielomasowych.

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