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PIECEWISE LINEAR MODELING OF FRICTION AND STICK-SLIP PHENOMENON IN DISCRETE DYNAMICAL SYSTEMS

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The paper presents an idea and application of a new method of the modeling of mechanical systems with friction. This method is based on the piecewise linear $\mathrm{luz}(\ldots)$ and $\mathrm{tar}(\ldots)$ projections and their original mathematical apparatus. The presented models of systems with friction describe the stick-slip phenomenon in detail.

Key words: piecewise linear projections, mathematical modeling, friction, stick-slip

1. Introduction

The modeling of dynamic systems with friction is one of the most important and difficult problems of mechanical science and engineering. Two main categories of problems are noticed: the first one concerns "microscopic" friction models and is representative for the tribology and general contact theory. The second one concerns "macroscopic" descriptions of friction actions (stick-slip phenomena) in discrete systems and is representative for the MBS (multibody systems) as well as the theory of mechanisms. Even though the macroscopic description of systems has a simplified character (piecewise linear friction force characteristic is preferred), the synthesis and analysis of such MBS models is usually very sophisticated and must be supported by a mathematical theory distinctive for non-smooth systems with constraints (variable-structure differential-algebraic equations and inclusions). Such an approach was discussed eg. by Grzesikiewicz (1990) and Brogliatto *et al.* (2002).

Problems of the modeling of friction systems are presented in many papers. They are discussed in several surveys, eg. by Amstrong-Helouvry *et al.*

(1994), Feeny *et al.* (1998), Ferri (1995), Gaul and Nitche (2001), Ibrachim (1994a,b), Martins *et al.* (1990), Oden and Martins (1985), Tworzydlo *et al.* (1992). Exploring the bibliography, we notice that the most recommended reference concerning the friction and stick-slip "macroscopic" modeling seems to be the article by Karnopp (1985). It contains derivations of variable-structure piecewise linear models for elementary single-mass and two-mass systems. So, Karnopp's models can be treated as the reference models for modeling of other friction system.

In this paper, we focus on the "macroscopic" piecewise linear approach too. The aim of the paper is a detailed presentation of the original method of modeling of discrete dynamical friction systems basing on the piecewise linear $\mathrm{luz}(\ldots)$ and $\mathrm{tar}(\ldots)$ projections and their mathematical apparatus. Using this method, the models have compact analytical forms enabling parametrically-made operations (eg. reductions). In comparison with other methods, just this feature seems to be the main advantage. The single- and two-mass stick-slip models discussed here are strictly compatible with Karnop's models. The derivations include also their degenerate versions when masses go to zero or infinity.

2. Idea of piecewise linear friction and stick-slip description

Even so the friction force characteristics can have different nonlinear forms, the simplest piecewise linear representations shown in Fig. 1 express the essence of kinetic and static friction action.

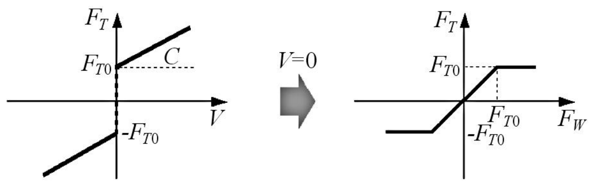

Fig. 1. Typical piecewise linear friction force characteristics: $F_T(V)$ – Coulomb-like kinetic friction characteristic, $F_T(F_W)$ – saturation static friction characteristics. Notation: F_T – friction force, V – relative velocity of abrading elements, F_W – acting force, F_{T0} – maximum dry friction force (here the same for kinetic and static friction), C – damping factor)

Such characteristics can be analytically written using the luz (...) and tar (...) piecewise linear projections (compare Fig. 1 and Fig. 2)

$$\begin{aligned} \operatorname{luz}(x,a) &= x + \frac{|x-a| - |x+a|}{2} \\ \operatorname{tar}(x,a) &= \operatorname{luz}(x,a)^{-1} = x + a \operatorname{sgh}(x) \end{aligned}$$

where

$$\operatorname{sgh}(x) = \begin{cases} -1 & \text{if} \quad x < 0 \\ s^* \in [-1,1] & \text{if} \quad x = 0 \\ 1 & \text{if} \quad x > 0 \end{cases}$$

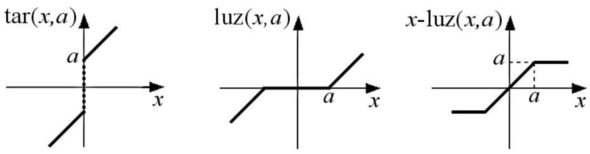

Fig. 2. Geometric interpretations of piecewise linear projections

The luz (...) and tar (...) projections have a surprisingly simple mathematical apparatus which was formulated with proofs by Żardecki (2001, 2006b). Here we will explore only particular formulas. They will be used when necessary.

Applying the tar (...) projection in description of Coulomb's-like characteristics, we can express in a compact formula both friction forces, the kinetic force – for a non-zero relative velocity as well as the static one – for the zero velocity. Such "extended Coulomb characteristics" have form

$$F_T = C \operatorname{tar}\Big(V, \frac{F_{T0}}{C}\Big) = \underbrace{CV + F_{T0} \operatorname{sgn} V}_{\text{Kinetic friction } (V \neq 0)} + \underbrace{F_{T0} s^*}_{\text{Static friction } (V=0)}$$

The kinetic friction force is expressed one-to-one, while the static friction F_{TS} demands an additional description in form of a function of the acting force. Having the form of saturation characteristics, the static friction characteristics can be can be analytically written with using the luz (...) projection

$$F_{TS} = F_{T0} s^* = F_w - \operatorname{luz}(F_w, F_{T0})$$

Obviously, the acting force F_W depends on the system configuration. Only in the simplest single-mass system (Fig. 3), the force F_W is identical with the external input force F. In multibody systems with many frictions sources, where the structure of an object depends on many stick-slip processes, the forces F_W may have a complicate character and must be derived for a concrete but temporary structure of the system.

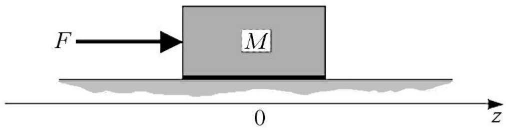

Fig. 3. Single-mass system with friction; M – mass, F – external acting force (notation of friction parameters in accordance with Fig. 1)

Having analytical forms of kinetic and static fiction forces, we can formulate the stick-slip system model. For the single-mass system, the model is (Żardecki, 2006b)

$$M\ddot{z}(t) + C\,\mathrm{tar}\Big(\dot{z}(t), \frac{F_{T0}}{C}\Big) = F(t)$$

where

$$F_{T0}s^*(t) = F(t) - \mathrm{luz}\,(F(t), F_{T0})$$

The model in a compact form can be rewritten as a variable structure equation

$$M\ddot{z}(t) = \begin{cases} F(t) - C\,\mathrm{tar}\Big(\dot{z}(t), \frac{F_{T0}}{C}\Big) & \text{if} \quad \dot{z}(t) \neq 0 \\ \mathrm{luz}\,(F(t), F_{T0}) & \text{if} \quad \dot{z}(t) = 0 \end{cases}$$

This variable-structure formula strictly corresponds to the single-mass Karnopp model (1985) and clearly describes the stick-slip phenomenon in the system. When $\dot{z}(t) = 0$ and $|F(t)| \leqslant F_{T0}$, we obtain $\mathrm{luz}\,(F(t), F_{T0}) = 0$, hence also $\ddot{z}(t) = 0$. It means the stiction. When $|F(t)| > F_{T0}$, we have $\mathrm{luz}\,(F(t), F_{To}) \neq 0$ and $\ddot{z}(t) \neq 0$ – the slip state.

Advantages of using the $\mathrm{luz}\,(\dots)$ and $\mathrm{tar}\,(\dots)$ projections concern not only brief analytical forms of friction characteristics and clear stick-slip description. Using their mathematical apparatus, we can transform the stick-slip models by parametric operations, which is the main benefit. For example:

- We can reduce analytically the order of the model. For the single-mass system, when $M = 0$, the compact model simplifies to the form

$$C\,\mathrm{tar}\Big(\dot{z}(t), \frac{F_{T0}}{C}\Big) = F(t)$$

Using the theorem on degeneration and the formula $k\,\mathrm{tar}\,(x,a) = \mathrm{tar}\,(kx,ka)$ (Żardecki, 2006a), we obtain after inversion the final uncoupled form

$$C\dot{z}(t) = \mathrm{luz}\,(F(t), F_{T0})$$

which effectively explains the stick-slip phenomenon in the mass less system (no motion for $-F_{T0} \leqslant F(t) \leqslant F_{T0}$)

- We can simplify non–homonegous friction models using substitutive parameters. For such a single-mass but multi-friction system, its compact model has form

$$M\ddot{z}(t) + \sum_{i=1}^{i=n} C_i\,\mathrm{tar}\left(\dot{z}(t), \frac{F_{T0i}}{C_i}\right) = F(t)$$

By the formula

$$k_1\,\mathrm{tar}\,(x,a_1) + k_2\,\mathrm{tar}\,(x,a_2) = (k_1+k_2)\,\mathrm{tar}\left(x, \frac{k_1a_1+k_2a_2}{k_1+k_2}\right)$$

(Żardecki, 2006a), the model is reduced to the form with substitutive parameters

$$M\ddot{z}(t) + C\,\mathrm{tar}\left(\dot{z}(t), \frac{F_{T0}}{C}\right) = F(t)$$

where

$$C = \sum_{i}^{n} C_i \qquad\qquad F_{T0} = \sum_{i}^{n} F_{T0i}$$

The $\mathrm{luz}\,(\dots)$ and $\mathrm{tar}\,(\dots)$ projections are especially useful in the synthesis and analysis of non-trivial friction system models. This is presented in the next sections.

3. Modeling of dynamics of a two-mass system with single friction

The simplest example of a two-mass system with friction (Fig. 4) concerns two moving blocks placed one on another. The blocks are subject to two external forces and the friction force between them.

The friction force is described according to the extended Coulomb characteristics. Because the action of static friction is not given as an explicit

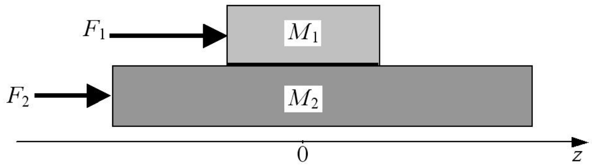

Fig. 4. Two-mass system with single friction; M_1, M_2 – masses of blocks, F_1, F_2 – external forces, F_{T12} – friction force, F_{T012} – maximum dry friction force, C_{12} – damping factor

dependence yet, the primary dynamical system model must be expressed by inclusion forms. Here

$$M_1\ddot{z}_1(t) + C_{12}\,\mathrm{tar}\Big(\dot{z}_1(t) - \dot{z}_2(t), \frac{F_{T012}}{C_{12}}\Big) \in F_1(t)$$

$$M_2\ddot{z}_2(t) - C_{12}\,\mathrm{tar}\Big(\dot{z}_1(t) - \dot{z}_2(t), \frac{F_{T012}}{C_{12}}\Big) \in F_2(t)$$

where $s^*_{12}(t) \in [-1, 1]$.

If $\dot{z}_1(t) - \dot{z}_2(t) = 0$ they are

$$M_1\ddot{z}_1(t) + F_{T012}s^*_{12}(t) \in F_1(t)$$

$$M_2\ddot{z}_2(t) - F_{T012}s^*_{12}(t) \in F_2(t)$$

Formal calculation of $s^*_{12}(t)$ (or static friction force $F_{T012}s^*_{12}(t)$) can be based on the Gauss rule. This requires minimization of the so-called acceleration energy Q. Thus

$$s^*_{12}\ :\quad \min_{s^*_{12}}\Big(Q(s^*_{12}) = \frac{M_1(\ddot{z}_1(s^*_{12}))^2 + M_2(\ddot{z}_2(s^*_{12}))^2}{2}\Big) \quad\wedge\quad s^*_{12} \in [-1, 1]$$

For concrete s^*_{12}

$$M_1(\ddot{z}_1(s^*_{12}))^2 = \frac{(F_1 - F_{T012}s^*_{12})^2}{M_1} \qquad\qquad M_2(\ddot{z}_2(s^*_{12}))^2 = \frac{(F_2 + F_{T012}s^*_{12})^2}{M_2}$$

So

$$s^*_{12}\ :\quad \min_{s^*_{12}}\Big(\frac{(M_1 + M_2)F_{T012}}{2M_1M_2}\Big(s^*_{12} - \frac{M_2F_1 - M_1F_2}{(M_1 + M_2)F_{T012}}\Big)^2 + \frac{(F_1 + F_2)^2}{2M_1M_2}\Big)$$
$$\wedge\quad s^*_{12} \in [-1, 1]$$

For $s^*_{12} \in [-1, 1]$ optimal solution is

$$s^*_{12} = \frac{M_2F_1 - M_1F_2}{(M_1 + M_2)F_{T012}}$$

For arbitrary F_1 and F_2, the solution $s^*_{12}(F_1, F_2)$ must be saturated. Therefore finally

$$
\begin{aligned}
s^*_{12}(t) &= \frac{M_2F_1(t) - M_1F_2(t)}{(M_1+M_2)F_{T012}} - \operatorname{luz}\Big(\frac{M_2F_1(t) - M_1F_2(t)}{(M_1+M_2)F_{T012}}, 1\Big)\\
F_{TS12}(t) &= F_{T012}s^*_{12}(t) =\\
&= \frac{M_2F_1(t) - M_1F_2(t)}{M_1+M_2} - \operatorname{luz}\Big(\frac{M_2F_1(t) - M_1F_2(t)}{M_1+M_2}, F_{T012}\Big)
\end{aligned}
$$

The derivation of these formulas may be done in different ways. Because the s^*_{12} singularity is related to the cross-friction description, when $\Delta\dot{z} = \dot{z}_1 - \dot{z}_2$ is an independent variable, a new inclusion based on $\Delta\dot{z}$ and $\Delta\ddot{z}$ variables is useful. After combination operations, we obtain

$$
\frac{M_1M_2}{M_1+M_2}[\ddot{z}_1(t) - \ddot{z}_2(t)] + C\operatorname{tar}\Big(\dot{z}_1(t) - \dot{z}_2(t), \frac{F_{T012}}{C}\Big) \in \frac{M_2F_1(t) - M_1F_2(t)}{M_1+M_2}
$$

Comparing this inclusion with the compact single-mass system model (Section 2), we notice that they have the same mathematical structure. The substitutive parameters and variables of the model may be interpreted in the same way. So, in the analysis of relative motion, we can directly use the methods that have been applied to the single-mass friction model, also the semi-heuristic S-S or formal variation procedure (Żardecki, (2006a,b). Such calculations give the same results as F_{TS12} obtained on the basis of the Gauss principle.

Taking into account the result concerning the static friction force F_{TS12}, we obtain for $\dot{z}_1(t) = \dot{z}_2(t)$ the following equation of relative motion

$$
\frac{M_1M_2}{M_1+M_2}[\ddot{z}_1(t) - \ddot{z}_2(t)] = \operatorname{luz}(F_{12}(t), F_{T012})
$$

where

$$
F_{12}(t) = \frac{M_2F_1(t) - M_1F_2(t)}{M_1+M_2}
$$

This equation expresses the stick-slip phenomenon in relation to the resultant excitation F_{12} and the limitations F_{T012} on the static friction force F_{TS12}. Particularly, when $-F_{T012} \leqslant F_{12}(t) \leqslant F_{T012}$ also $\ddot{z}_1(t) = \ddot{z}_2(t)$, thus the stiction state starts. In the stiction state $F_{12}(t) = F_{TS12}(t)$ (linear part of saturation characteristics). When $F_{12}(t) \leqslant -F_{T012}$ or $F_{12}(t) \geqslant F_{T012}$ we have $\ddot{z}_1(t) \neq \ddot{z}_2(t)$, and the slip state happens. In such a situation $\dot{z}_1(t) = \dot{z}_2(t)$ entails temporary static friction without stiction.

A new aspect of the stick-slip description is taken up by the analysis mentioned below. When $\dot{z}_1(t) = \dot{z}_2(t)$, the equations for $z_1(t)$, $z_2(t)$ are

$$M_1\ddot{z}_1(t) = \frac{M_1}{M_1+M_2}[F_1(t)+F_2(t)] + \operatorname{luz}\Big(\frac{M_2F_1(t)-M_1F_2(t)}{M_1+M_2}, F_{T012}\Big)$$
$$M_2\ddot{z}_2(t) = \frac{M_2}{M_1+M_2}[F_1(t)+F_2(t)] - \operatorname{luz}\Big(\frac{M_2F_1(t)-M_1F_2(t)}{M_1+M_2}, F_{T012}\Big)$$

Furthermore if

$$-F_{T012} \leqslant F_{12}(t) = \frac{M_2F_1(t)-M_1F_2(t)}{M_1+M_2} \leqslant F_{T012}$$

then

$$(M_1+M_2)\ddot{z}_1(t) = F_1(t)+F_2(t) \qquad \text{or} \qquad (M_1+M_2)\ddot{z}_2(t) = F_1(t)+F_2(t)$$

These equations have identical forms. It means that $\ddot{z}_1(t) = \ddot{z}_2(t)$ (stiction state), and motion is described by one of these equations only.

Summing up, the compact-form model of a two-mass frictional system is

$$M_1\ddot{z}_1(t) + C\operatorname{tar}\Big(\dot{z}_1(t)-\dot{z}_2(t), \frac{F_{T012}}{C}\Big) = F_1(t)$$
$$M_2\ddot{z}_2(t) - C\operatorname{tar}\Big(\dot{z}_1(t)-\dot{z}_2(t), \frac{F_{T012}}{C}\Big) = F_2(t)$$

where

$$F_{T012}s^*_{12}(t) = \frac{M_2F_1(t)-M_1F_2(t)}{M_1+M_2} - \operatorname{luz}\Big(\frac{M_2F_1(t)-M_1F_2(t)}{M_1+M_2}, F_{T012}\Big)$$

This model can be expressed in a variable-structure form without s^*_{12}:
— if $\dot{z}_1(t) \neq \dot{z}_2(t)$

$$M_1\ddot{z}_1(t) = F_1(t) - C_{12}\operatorname{tar}\Big(\dot{z}_1(t)-\dot{z}_2(t), \frac{F_{T012}}{C_{12}}\Big)$$
$$M_2\ddot{z}_2(t) = F_2(t) + C_{12}\operatorname{tar}\Big(\dot{z}_1(t)-\dot{z}_2(t), \frac{F_{T012}}{C_{12}}\Big)$$

— if $\dot{z}_1(t) = \dot{z}_2(t)$

$$M_1\ddot{z}_1(t) = \frac{M_1}{M_1+M_2}[F_1(t)+F_2(t)] + \operatorname{luz}\Big(\frac{M_2F_1(t)-M_1F_2(t)}{M_1+M_2}, F_{T012}\Big)$$
$$M_2\ddot{z}_2(t) = \frac{M_2}{M_1+M_2}[F_1(t)+F_2(t)] - \operatorname{luz}\Big(\frac{M_2F_1(t)-M_1F_2(t)}{M_1+M_2}, F_{T012}\Big)$$

The presented model is strictly compatible with Karnop's model (1985). It is clear when our model is rewritten in a traditional form without the $\operatorname{luz}(\dots)$ and $\operatorname{tar}(\dots)$ projections.

Now, we can analyze particular situations when one of the mass parameters (for example M_2) goes to infinity or goes to zero.

- When $M_2 \to \infty$, the state $\ddot{z}_2(t) = 0$ must be steady. It means permanent blockade of this block. Hence also $\dot{z}_2(t) = 0$. After an asymptotic transformation, this compact-form model can be given by one equation for the moving mass M_1

$$M_1\ddot{z}_1(t) + C_{12}\operatorname{tar}\Big(\dot{z}_1(t), \frac{F_{T012}}{C_{12}}\Big) = F_1(t)$$

where

$$F_{T012}s_{12}^*(t) = F_1(t) - \operatorname{luz}(F_1(t), F_{T012})$$

In such a case, the two-mass system with cross-friction becomes the single-mass elementary system presented in Section 2.

- When $M_1 \to 0$, motion of the massless element has a kinetic character. After inversion, the first inclusion of the primary model transforms to the equation

$$\dot{z}_1(t) - \dot{z}_2(t) = \frac{1}{C}\operatorname{luz}(F_1(t), F_{T012})$$

Finally, the second equation can be written as $M_2\ddot{z}_2(t) = F_1(t) + F_2(t)$. Note: when also $-F_{T012} \leqslant F_1(t) \leqslant F_{T012}$, we have $\dot{z}_1(t) - \dot{z}_2(t) = 0$. It means the stiction state. It is independent of action of the second mass.

4. Modeling of dynamics of a two-mass system with two friction forces

The simplest system with simultaneous action of several various friction forces is the two-mass system with two friction forces (Fig. 5). It is an extension of the two-mass system discussed in Section 3. In this case, the bottom block interacts with friction forces not only with the top block but also with the fixed basis.

The primary mathematical inclusion model is

$$\begin{aligned}
&M_1\ddot{z}_1 + C_{12}\operatorname{tar}\Big(\dot{z}_1 - \dot{z}_2, \frac{F_{T012}}{C_{12}}\Big) \in F_1\\
&M_2\ddot{z}_2 - C_{12}\operatorname{tar}\Big(\dot{z}_1 - \dot{z}_2, \frac{F_{T012}}{C_{12}}\Big) + C_{20}\operatorname{tar}\Big(\dot{z}_2, \frac{F_{T020}}{C_{20}}\Big) \in F_2
\end{aligned}$$

where $s_{12}^* \in [-1, 1]$, $s_{20}^* \in [-1, 1]$.

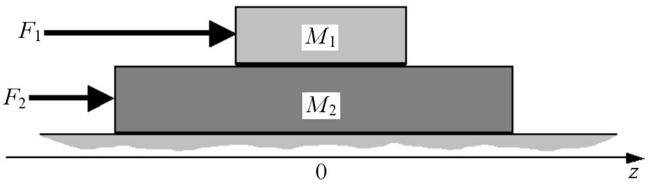

Fig. 5. Two-mass system with two friction forces; M_1, M_2 – masses of blocks, F_1, F_2 – external forces, F_{T12}, F_{T20} – friction forces, F_{T012}, F_{T020} – maximal dry friction forces, C_{12}, C_{20} – damping factors

The singularities of dry friction characteristics concern the velocities $\dot{z}_1-\dot{z}_2$ or $\dot{z}_2$ at zero points. Therefore, we must analyze three cases:

(1) $\dot{z}_1=\dot{z}_2\neq 0$ (then $\dot{z}_1-\dot{z}_2=0$ and $\dot{z}_2\neq 0$ – problem of s^*_{12})

(2) $\dot{z}_1\neq 0$, $\dot{z}_2=0$ (then $\dot{z}_1-\dot{z}_2\neq 0$ and $\dot{z}_2=0$ – problem of s^*_{20})

(3) $\dot{z}_1=\dot{z}_2=0$ (then $\dot{z}_1-\dot{z}_2=0$ and $\dot{z}_2=0$ – problem of s^*_{12} and s^*_{20})

Analysis (the Gauss rule is applied):

— If $\dot{z}_1=\dot{z}_2\neq 0$ the minimization task is

$$s^*_{12}: \qquad \min_{s^*_{12}}\Big(Q(s^*_{12})=\frac{M_1(\ddot{z}_1(s^*_{12}))^2+M_2(\ddot{z}_2(s^*_{12}))^2}{2}\Big) \quad\wedge\quad s^*_{12}\in[-1,1]$$

where

$$\begin{aligned}
M_1(\ddot{z}_1(s^*_{12}))^2 &= \frac{(F_1-F_{T012}s^*_{12})^2}{M_1}\\
M_2(\ddot{z}_2(s^*_{12}))^2 &= \frac{\Big(F_2-C_{20}\,\mathrm{tar}\Big(\dot{z}_2,\frac{F_{T020}}{C_{20}}\Big)+F_{T012}s^*_{12}\Big)^2}{M_2}
\end{aligned}$$

For $s^*_{12}\in[-1,1]$

$$s^*_{12}=\frac{M_2F_1-M_1\Big(F_2-C_{20}\,\mathrm{tar}\Big(\dot{z}_2,\frac{F_{T020}}{C_{20}}\Big)\Big)}{(M_1+M_2)F_{T012}}$$

For arbitrary excitations F_1, F_2, $\dot{z}_2$, the solution $s^*_{12}(F_1,F_2,\dot{z}_2)$ must be saturated. Thus finally

$$s^*_{12}(t)=\frac{F_w(t)}{F_{T012}}-\mathrm{luz}\Big(\frac{F_w(t)}{F_{T012}},1\Big)$$

or

$$F_{T012}s_{12}^*(t) = F_w(t) - \operatorname{luz}(F_w(t), F_{T012})$$

where

$$F_w(t) = \frac{M_2F_1(t) - M_1F_2(t)}{M_1+M_2} + \frac{M_1C_{20}}{M_1+M_2}\operatorname{tar}\Big(\dot{z}_2(t), \frac{F_{T020}}{C_{20}}\Big)$$

— If $\dot{z}_1 \neq 0$, $\dot{z}_2 = 0$ the minimization task has a form

$$s_{20}^* : \qquad \min_{s_{12}^*}\Big(Q(s_{20}^*) = \frac{M_1(\ddot{z}_1(s_{20}^*))^2 + M_2(\ddot{z}_2(s_{20}^*))^2}{2}\Big) \quad \wedge \quad s_{20}^* \in [-1,1]$$

where

$$M_1(\ddot{z}_1(s_{20}^*))^2 = \frac{1}{M_1}\Big[F_1 - C_{12}\operatorname{tar}\Big(\dot{z}_1 - \dot{z}_2), \frac{F_{T012}}{C_{12}}\Big)\Big]^2 = A$$

(independent of s_{20}^*) and

$$M_2(\ddot{z}_2(s_{20}^*))^2 = \frac{1}{M_2}\Big[F_2 + C_{12}\operatorname{tar}\Big(\dot{z}_1 - \dot{z}_2), \frac{F_{T012}}{C_{12}}\Big) - F_{T20}s_{20}^*\Big]^2$$

Thus, for $s_{20}^* \in [-1,1]$ the solution is

$$s_{20}^* = \frac{1}{F_{T020}}\Big[F_2 - C_{12}\operatorname{tar}\Big(\dot{z}_1, \frac{F_{T012}}{C_{12}}\Big)\Big]$$

For arbitrary excitations F_2, $\dot{z}_1$, the solution $s_{20}^*(F_2, \dot{z}_1)$ must be saturated. So

$$\begin{aligned}
s_{20}^*(t) = {} & \frac{1}{F_{T012}}\Big[F_2(t) - C_{12}\operatorname{tar}\Big(\dot{z}_2(t), \frac{F_{T012}}{C_{12}}\Big)\Big] + \\
& - \operatorname{luz}\Big(\frac{1}{F_{T012}}\Big[F_2(t) - C_{12}\operatorname{tar}\Big(\dot{z}_2(t), \frac{F_{T012}}{C_{12}}\Big)\Big], 1\Big) \\
F_{T020}s_{20}^*(t) = {} & F_2(t) - C_{12}\operatorname{tar}\Big(\dot{z}_2(t), \frac{F_{T012}}{C_{12}}\Big) + \\
& - \operatorname{luz}\Big(F_2(t) - C_{12}\operatorname{tar}\Big(\dot{z}_2(t), \frac{F_{T012}}{C_{12}}\Big), F_{T020}\Big)
\end{aligned}$$

— If $\dot{z}_1 = \dot{z}_2 = 0$ the minimization task is

$$\begin{aligned}
& s_{12}^*, s_{20}^* : \qquad \min_{s_{12}^*}\Big(Q(s_{12}^*, s_{20}^*) = \frac{M_1(\ddot{z}_1(s_{12}^*, s_{20}^*))^2 + M_2(\ddot{z}_2(s_{12}^*, s_{20}^*))^2}{2}\Big) \\
& \wedge \quad s_{12}^*, s_{20}^* \in [-1,1]
\end{aligned}$$

For concrete values s_{12}^*, s_{20}^*, we have

$$M_1\ddot{z}_1^2 = \frac{(F_1 - F_{T012}s_{12}^*)^2}{M_1} \qquad M_2\ddot{z}_2^2 = \frac{(F_2 + F_{T012}s_{12}^* - F_{T020}s_{20}^*)^2}{M_2}$$

$$\begin{aligned} Q(s_{12}^*, s_{20}^*) &= \frac{(M_1+M_2)F_{T012}^2}{2M_1M_2}(s_{12}^*)^2 + \frac{F_{T020}^2}{2M_2}(s_{20}^*)^2 - \frac{F_{T012}F_{T020}}{M_2}s_{12}^*s_{20}^* + \\ &+ \frac{(M_1F_2 - M_2F_1)F_{T012}}{M_1M_2}s_{12}^* - \frac{F_2F_{T020}}{M_2}s_{20}^* + \frac{M_1F_2^2 + M_2F_1^2}{2M_1M_2} \end{aligned}$$

For $s_{12}^* \in [-1,1]$ and $s_{20}^* \in [-1,1]$ the minimization solution must fulfill (convex function minimization)

$$\begin{aligned} \frac{\partial Q(s_{12}^*, s_{20}^*)}{\partial s_{12}^*} &= \frac{(M_1+M_2)F_{T012}^2}{M_1M_2}s_{12}^* - \frac{F_{T012}F_{T020}}{M_2}s_{20}^* + \\ &+ \frac{(M_1F_2 - M_2F_1)F_{T012}}{M_1M_2} = 0 \\ \frac{\partial Q(s_{12}^*, s_{20}^*)}{\partial s_{20}^*} &= \frac{F_{T020}^2}{M_2}s_{20}^* - \frac{F_{T012}F_{T020}}{M_2}s_{12}^* - \frac{F_2F_{T020}}{M_2} = 0 \\ \frac{\partial Q^2(s_{12}^*, s_{20}^*)}{\partial (s_{12}^*)^2} &= \frac{(M_1+M_2)F_{T012}^2}{M_1M_2} > 0 \\ \frac{\partial Q^2(s_{12}^*, s_{20}^*)}{\partial (s_{20}^*)^2} &= \frac{F_{T020}^2}{M_2} > 0 \qquad \text{(fulfilled)} \end{aligned}$$

Thus, the system

$$\begin{bmatrix} \dfrac{(M_1+M_2)F_{T012}^2}{M_1M_2} & -\dfrac{F_{T012}F_{T020}}{M_2} \\ -\dfrac{F_{T012}F_{T020}}{M_2} & \dfrac{F_{T020}^2}{M_2} \end{bmatrix} \begin{bmatrix} s_{12}^* \\ s_{20}^* \end{bmatrix} = \begin{bmatrix} -\dfrac{(M_1F_2 - M_2F_1)F_{T012}}{M_1M_2} \\ -\dfrac{F_2F_{T020}}{M_2} \end{bmatrix}$$

yields for $s_{12}^* \in [-1,1]$ and $s_{20}^* \in [-1,1]$ the solution

$$s_{12}^* = \frac{F_1}{F_{T012}} \qquad s_{20}^* = \frac{F_1 + F_2}{F_{T020}}$$

For arbitrary excitations F_1, F_2, the solution $s_{12}^*(F_1, F_2)$, $s_{20}^*(F_1, F_2)$ must be saturated. Thus finally

$$\begin{aligned} s_{12}^*(t) &= \frac{F_1(t)}{F_{T012}} - \operatorname{luz}\Big(\frac{F_1(t)}{F_{T012}}, 1\Big) \\ s_{20}^*(t) &= \frac{F_1(t) + F_2(t)}{F_{T020}} - \operatorname{luz}\Big(\frac{F_1(t) + F_2(t)}{F_{T020}}, 1\Big) \end{aligned}$$

and

$$F_{T012}s_{12}^*(t) = F_1(t) - \operatorname{luz}(F_1(t), F_{T012})$$
$$F_{T020}s_{20}^* = F_1(t) + F_2(t) - \operatorname{luz}(F_1(t) + F_2(t), F_{T020})$$

Concluding, the model of the two-mass system with two frictions forces can be described in a compact form by differential equations of motion

$$M_1\ddot{z}_1 + C_{12}\operatorname{tar}\Big(\dot{z}_1 - \dot{z}_2, \frac{F_{T012}}{C_{12}}\Big) = F_1$$
$$M_2\ddot{z}_2 - C_{12}\operatorname{tar}\Big(\dot{z}_1 - \dot{z}_2, \frac{F_{T012}}{C_{12}}\Big) + C_{20}\operatorname{tar}\Big(\dot{z}_2, \frac{F_{T020}}{C_{20}}\Big) = F_2$$

where

$$s_{12}^* = \frac{F_{w12}}{F_{T012}} - \operatorname{luz}\Big(\frac{F_{w12}}{F_{T012}}, 1\Big) \qquad\qquad s_{20}^* = \frac{F_{w20}}{F_{T020}} - \operatorname{luz}\Big(\frac{F_{w20}}{F_{T020}}, 1\Big)$$

and

$$F_{w12} = \begin{cases} \dfrac{M_2F_1 - M_1F_2}{M_1 + M_2} + \dfrac{M_1C_{20}}{M_1 + M_2}\operatorname{tar}\Big(\dot{z}_2, \dfrac{F_{T020}}{C_{20}}\Big) & \text{if} \quad \dot{z}_2 \neq 0 \\ F_1 & \text{if} \quad \dot{z}_2 = 0 \end{cases}$$

$$F_{w20} = \begin{cases} F_2 + C_{12}\operatorname{tar}\Big(\dot{z}_1, \dfrac{F_{T012}}{C_{12}}\Big) & \text{if} \quad \dot{z}_1 \neq 0 \\ F_1 + F_2 & \text{if} \quad \dot{z}_1 = 0 \end{cases}$$

Displacing s_{12}^*, s_{20}^*, the model can be expressed also by variable-structure equations:

— If $\dot{z}_1 \neq \dot{z}_2 \neq 0$

$$M_1\ddot{z}_1 = F_1 - C_{12}\operatorname{tar}\Big(\dot{z}_1 - \dot{z}_2), \frac{F_{T012}}{C_{12}}\Big)$$
$$M_2\ddot{z}_2 = F_2 + C_{12}\operatorname{tar}\Big(\dot{z}_1 - \dot{z}_2, \frac{F_{T012}}{C_{12}}\Big) - C_{20}\operatorname{tar}\Big(\dot{z}_2, \frac{F_{T020}}{C_{20}}\Big)$$

— If $\dot{z}_1 = \dot{z}_2 \neq 0$

$$\begin{aligned} M_1\ddot{z}_1 = {} & \frac{M_1(F_1 + F_2)}{M_1 + M_2} - \frac{M_1C_{20}}{M_1 + M_2}\operatorname{tar}\Big(\dot{z}_2, \frac{F_{T020}}{C_{20}}\Big) + \\ & + \operatorname{luz}\Big(\frac{M_2F_1 - M_1F_2}{M_1 + M_2} + \frac{M_1C_{20}}{M_1 + M_2}\operatorname{tar}\Big(\dot{z}_2, \frac{F_{T020}}{C_{20}}\Big), F_{T012}\Big) \\ M_2\ddot{z}_2 = {} & \frac{M_2(F_1 + F_2)}{M_1 + M_2} - \frac{M_2C_{20}}{M_1 + M_2}\operatorname{tar}\Big(\dot{z}_2, \frac{F_{T020}}{C_{20}}\Big) + \\ & - \operatorname{luz}\Big(\frac{M_2F_1 - M_1F_2}{M_1 + M_2} + \frac{M_1C_{20}}{M_1 + M_2}\operatorname{tar}\Big(\dot{z}_2, \frac{F_{T020}}{C_{20}}\Big), F_{T012}\Big) \end{aligned}$$

Note, if

$$-F_{T012} \leqslant \frac{M_2F_1 - M_1F_2}{M_1+M_2} + \frac{M_1C_{20}}{M_1+M_2}\operatorname{tar}\Big(\dot{z}_2, \frac{F_{T020}}{C_{20}}\Big) \leqslant F_{T012}$$

then

$$\begin{aligned}\ddot{z}_1 &= \frac{F_1+F_2}{M_1+M_2} - \frac{C_{20}}{M_1+M_2}\operatorname{tar}\Big(\dot{z}_2, \frac{F_{T020}}{C_{20}}\Big)\\ \ddot{z}_2 &= \frac{F_1+F_2}{M_1+M_2} - \frac{C_{20}}{M_1+M_2}\operatorname{tar}\Big(\dot{z}_2, \frac{F_{T020}}{C_{20}}\Big)\end{aligned}$$

which means that $\ddot{z}_1 = \ddot{z}_2$, and the stiction state appears.
— If $\dot{z}_1 \neq 0$, $\dot{z}_2 = 0$

$$\begin{aligned}M_1\ddot{z}_1 &= F_1 - C_{12}\operatorname{tar}\Big(\dot{z}_1, \frac{F_{T012}}{C_{12}}\Big)\\ M_2\ddot{z}_2 &= \operatorname{luz}\Big(F_2 + C_{12}\operatorname{tar}\Big(\dot{z}_1, \frac{F_{T012}}{C_{12}}\Big), F_{T020}\Big)\end{aligned}$$

It means that when

$$-F_{T020} \leqslant F_2 + C_{12}\operatorname{tar}\Big(\dot{z}_1, \frac{F_{T012}}{C_{12}}\Big) \leqslant F_{T020}$$

also $\ddot{z}_2 = 0$. In other words, the bottom block is stuck to its base, and only the top block is moving according to the equation of motion

$$M_1\ddot{z}_1 = F_1 - C_{12}\operatorname{tar}\Big(\dot{z}_1, \frac{F_{T012}}{C_{12}}\Big)$$

— If $\dot{z}_1 = \dot{z}_2 = 0$

$$\begin{aligned}M_1\ddot{z}_1 &= \operatorname{luz}(F_1, F_{T012})\\ M_2\ddot{z}_2 &= -\operatorname{luz}(F_1, F_{T012}) + \operatorname{luz}(F_1+F_2, F_{T020})\end{aligned}$$

Note:

- when $-F_{T012} \leqslant F_1 \leqslant F_{T012}$ and $-F_{T020} \leqslant F_1 + F_2 \leqslant F_{T020}$ then $\ddot{z}_1 = \ddot{z}_2 = 0$ thus the blocks are stuck together and the bottom one is stuck to its base
- when $\operatorname{luz}(F_1+F_2, F_{T020}) = \operatorname{luz}(F_1, F_{T012}) \neq 0$ the bottom block is stuck, while equalization of both blocks velocities is only temporary, since $\ddot{z}_1 \neq \ddot{z}_2 = 0$.

The presented modeling of a two-mass system with two friction forces basing on the $\operatorname{luz}(\dots)$ and $\operatorname{tar}(\dots)$ mathematical apparatus and the Gauss rule gives results strictly compatible with Karnop's model (1985). Our model have a more compact description, better for analytical transformations and detailed analysis.

It is interesting that for the singular state $\dot{z}_1 = \dot{z}_2 = 0$, the model of two-element system with two friction forces can not be directly used (by setting $F_{T020} = 0$) for the description of motion of the system with cross-friction only! (Note that transformation of the two-mass elementary system's model into the single-mass elementary model was regular). Why do we have no transformation regularity in this case? We explain this in the following considerations.

When $F_{T020} = 0$ has been set a priori (before minimization)

$$
\begin{aligned}
Q(s_{12}^*, s_{20}^*) = \frac{(M_1+M_2)F_{T012}^2}{2M_1M_2}(s_{12}^*)^2 + \frac{(M_1F_2 - M_2F_1)F_{T012}}{M_1M_2}s_{12}^* + \\
+\frac{M_1F_2^2 + M_2F_1^2}{2M_1M_2}
\end{aligned}
$$

Thus we obtain

$$
s_{12}^* : \qquad F_{T012}s_{12}^* = \frac{M_2F_1 - M_1F_2}{M_1+M_2} \qquad \wedge \qquad s_{12}^* \in [-1,1]
$$

and finally, the proper model of the cross-friction system

$$
F_{T012}s_{12}^* = \frac{M_2F_1 - M_1F_2}{M_1+M_2} - \operatorname{luz}\Big(\frac{M_2F_1 - M_1F_2}{M_1+M_2}, F_{T012}\Big)
$$

When $F_{T020} = 0$ has been set after minimization, we obtain generally a different and false result $F_{T012}s_{12}^* = F_1 - \operatorname{luz}(F_1, F_{T012})$ (good result only for $F_2 = 0$).

We can draw the same conclusion when analyzing the model with $F_{T012} = 0$. When $F_{T012} = 0$ has been set before minimization

$$
Q(s_{12}^*, s_{20}^*) = \frac{F_{T020}^2}{2M_2}(s_{20}^*)^2 - \frac{F_2F_{T020}}{M_2}s_{20}^* + \frac{M_1F_2^2 + M_2F_1^2}{2M_1M_2}
$$

Thus

$$
s_{20}^* : \qquad F_2 - F_{T020}s_{20}^* = 0 \qquad \wedge \qquad s_{20}^* \in [-1,1]
$$

and finally

$$
F_{T020}s_{20}^* = F_2 - \operatorname{luz}(F_2, F_{T020})
$$

When $F_{T012} = 0$ is set a posteriori, a different result $F_{T020}s_{20}^* = F_1 + F_2 - \operatorname{luz}(F_1 + F_2, F_{T020})$ is calculated (the same only for $F_1 = 0$). As we see, the exact model of the two-mass system with two friction forces does not let regular conversion to elementary models.

Another question seems to be interesting for better understanding of the model of the two-mass system with two friction forces. The question is if the model of such a complex system can be created on the basis of simpler (here elementary) exact sub-models?

Let apply now a concept of the so-called "model decomposition-aggregation method". Such decomposition can be done in two ways.

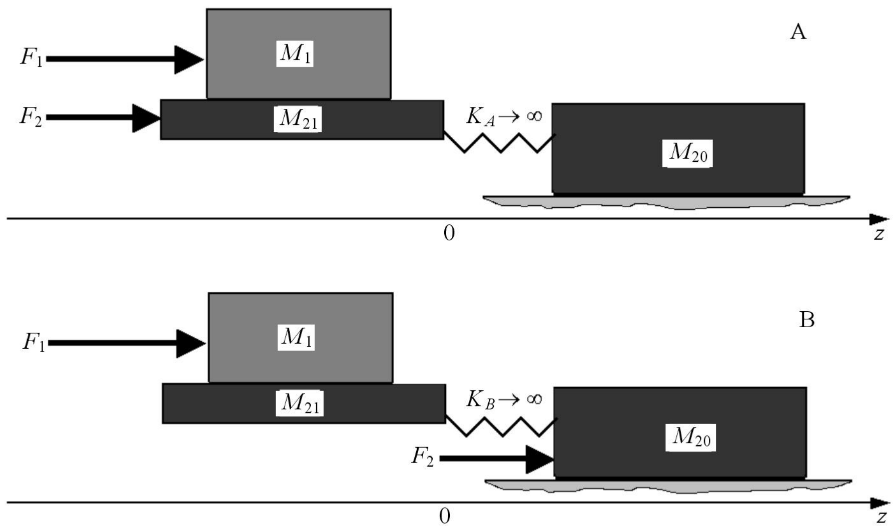

Fig. 6. Two concepts of model decomposition

In both cases, the system is replaced by equivalent three-mass systems. Of course, $M_{21} + M_{20} = M_2$. The coordinates of sub-systems are z_1, z_{21} and z_{20}. Thus our complex system is treated as a series of sub-systems containing individual frictions. For the description of these sub-systems, elementary (basing on general physical rules, i.e. on exact formulas) friction models can be used. For calculation of the final model of a complex system, we have to apply the operation $K_A \to \infty$ or $K_B \to \infty$. We will only discuss the most singular velocity state when the total stiction state appears.

When the decomposition is made according to method A, for the singular state $\dot{z}_1 = \dot{z}_{21}$, $\dot{z}_{20} = 0$, we have model description based on two elementary subsystems models. Lattering $F_A = K_A(z_{21} - z_{20})$

$$M_1\ddot{z}_1 = \frac{M_1}{M_1+M_{21}}(F_1+F_2-F_A) + \operatorname{luz}\Big(\frac{M_{21}F_1 - M_1(F_2-F_A)}{M_1+M_{21}}, F_{T012}\Big)$$
$$M_{21}\ddot{z}_{21} = \frac{M_{21}}{M_1+M_{21}}(F_1+F_2-F_A) - -\operatorname{luz}\Big(\frac{M_{21}F_1 - M_1(F_2-F_A)}{M_1+M_{21}}, F_{T012}\Big)$$

and $M_{20}\ddot{z}_{20} = \operatorname{luz}(F_A, F_{T020})$.

Setting $K_A \to \infty$ we have $z_{21} = z_{20} = z_2$, $\dot{z}_{21} = \dot{z}_{20} = \dot{z}_2 = \dot{z}_1 = 0$, $\ddot{z}_{21} = \ddot{z}_{20} = \ddot{z}_2$. By summing the second and third equation, the complex model passes to

$$M_1\ddot{z}_1 = \frac{M_1}{M_1+M_{21}}(F_1+F_2-F_A) + \operatorname{luz}\Big(\frac{M_{21}F_1 - M_1(F_2-F_A)}{M_1+M_{21}}, F_{T012}\Big)$$
$$M_2\ddot{z}_2 = \frac{M_{21}}{M_1+M_{21}}(F_1+F_2-F_A) - \operatorname{luz}\Big(\frac{M_{21}F_1 - M_1(F_2-F_A)}{M_1+M_{21}}, F_{T012}\Big) + \\ + \operatorname{luz}(F_A, F_{T020})$$

As yet $F_A = K_A(z_{21} - z_{20})$ is formally indeterminate (indeterminacy of the type "∞-0"). Admittedly the system must fulfill Gauss' rule, so the acceleration energy Q treated as a function of the unknown variable F_A should be minimal. Because of strong non-linear form of $Q(F_A)$, formal analytical minimization is very complicated. We do this with little heuristic roundabout effort, we use the method incorporated in the S-S procedure (Żardecki, 2006a). We know that minimum-minimorum of $Q(F_A)$ is warranted by the stuck state when the accelerations of masses are zero. In the state $\ddot{z}_{12} = \ddot{z}_{21} = \ddot{z}_{20} = 0$, our three-mass model is

$$M_1\ddot{z}_1 = \frac{M_1}{M_1+M_{21}}(F_1+F_2-F_A) + \operatorname{luz}\Big(\frac{M_{21}F_1 - M_1(F_2-F_A)}{M_1+M_{21}}, F_{T012}\Big) = 0$$
$$M_{21}\ddot{z}_{21} = \frac{M_{21}}{M_1+M_{21}}(F_1+F_2-F_A) - \operatorname{luz}\Big(\frac{M_{21}F_1 - M_1(F_2-F_A)}{M_1+M_{21}}, F_{T012}\Big) = 0$$
$$M_{20}\ddot{z}_{20} = \operatorname{luz}(F_A, F_{T020}) = 0$$

Because summation of the first and second equation $F_1 + F_2 - F_A = 0$, one finds

$$M_1\ddot{z}_1 = \operatorname{luz}(F_1, F_{T012}) = 0$$
$$M_{21}\ddot{z}_{21} = -\operatorname{luz}(F_1, F_{T012}) = 0$$
$$M_{20}\ddot{z}_{20} = \operatorname{luz}(F_1 + F_2, F_{T020}) = 0$$

It means that the stuck state (optimal condition for minimization of $Q(F_A)$) is warranted by the function $F_A(F_1, F_2) = F_1 + F_2$ with conditions $F_1 \in [-F_{T012}, F_{T012}]$ and $F_1 + F_2 \in [-F_{T020}, F_{T020}]$.

When $F_1 \notin [-F_{T012}, F_{T012}]$ or $F_1 + F_2 \notin [-F_{T020}, F_{T020}]$, the total stuck state vanishes. So for the aggregated model, we finally obtain the proper result

$$\begin{aligned} M_1\ddot{z}_1 &= \operatorname{luz}(F_1, F_{T012}) \\ M_2\ddot{z}_2 &= -\operatorname{luz}(F_1, F_{T012}) + \operatorname{luz}(F_1 + F_2, F_{T020}) \end{aligned}$$

Similar calculation repeated for the model decomposed according to method B (in this case $F_B(F_1, F_2) = F_1$ also gives the same final result.

The "decomposition-aggregation" method and the S-S procedure seem to be very useful for theoretical verification and analysis of multibody models with multiple friction sources. Such models can be very complicated, so affirmative proofs are desirable.

Basing on the derived mathematical model, we can analyze situations when the mass parameters go to infinity or go to zero. We discuss several non-trivial situations: (1) $M_1 \to \infty$, (2) $M_2 \to \infty$, (3) $M_1 \to 0$, (4) $M_2 \to 0$.

(1) When $M_1 \to \infty$, the top element is motionless and the moving block (mass M_2) acts under two friction forces. The state $\ddot{z}_1(t) = 0$, $\dot{z}_1(t) = 0$ results from parametrical transformation of the first model inclusion. Thus, the second model inclusion is

$$M_2\ddot{z}_2 + C_{12}\operatorname{tar}\Big(\dot{z}_2, \frac{F_{T012}}{C_{12}}\Big) + C_{20}\operatorname{tar}\Big(\dot{z}_2, \frac{F_{T020}}{C_{20}}\Big) \in F_2$$

Finally, we obtain

$$M_2\ddot{z}_2(t) = F_2(t) - (C_{12} + C_{20})\operatorname{tar}\Big(\dot{z}_2(t), \frac{F_{T012} + F_{T020}}{C_{12} + C_{20}}\Big)$$

where

$$(F_{T012} + F_{T020})s_{12}^*(t) = F_1(t) - \operatorname{luz}(F_1(t), F_{T012} + F_{T020})$$

(2) When $M_2 \to \infty$, the state $\ddot{z}_2(t) = 0$, $\dot{z}_2(t) = 0$ must be steady. After asymptotic transformation, the model is given by one equation for the mass M_1

$$M_1\ddot{z}_1(t) = F_1(t) - C_{12}\operatorname{tar}\Big(\dot{z}_1(t), \frac{F_{T012}}{C_{12}}\Big)$$

where for $\dot{z}_2(t) = 0$

$$F_{T012}s_{12}^*(t) = F_1(t) - \operatorname{luz}(F_1(t), F_{T012})$$

(3) When $M_1 \to 0$ the motion of this massless element has a kinetic character

$$C_{12}\operatorname{tar}\Big(\dot{z}_1(t) - \dot{z}_2(t), \frac{F_{T012}}{C_{12}}\Big) \in F_1(t)$$

So

$$\dot{z}_1(t) - \dot{z}_2(t) = \frac{1}{C}\,\mathrm{luz}\,(F_1(t), F_{T012})$$

The second inclusion can be written as

$$M_2\ddot{z}_2(t) + C_{20}\,\mathrm{tar}\Big(\dot{z}_2(t), \frac{F_{T020}}{C_{20}}\Big) \in F_1(t) + F_2(t)$$

So

$$M_2\ddot{z}_2(t) + C_{20}\,\mathrm{tar}\Big(\dot{z}_2(t), \frac{F_{T020}}{C_{20}}\Big) = F_1(t) + F_2(t)$$

where

$$F_{T020}s^*_{20}(t) = F_1(t) + F_2(t) - \,\mathrm{luz}\,(F_1(t) + F_2(t), F_{T020})$$

When also $-F_{T012} \leqslant F_1(t) \leqslant F_{T012}$, we have $\dot{z}_1(t) - \dot{z}_2(t) = 0$. It means the stiction state of both elements. It is independent of action of the second mass.

(4) When $M_2 \to 0$, motion of this massless element has a kinetic character. Here

$$\begin{aligned}
&M_1\ddot{z}_1(t) + C_{12}\,\mathrm{tar}\Big(\dot{z}_1(t) - \dot{z}_2(t), \frac{F_{T012}}{C_{12}}\Big) \in F_1(t)\\
&-C_{12}\,\mathrm{tar}\Big(\dot{z}_1(t) - \dot{z}_2(t), \frac{F_{T012}}{C_{12}}\Big) + C_{20}\,\mathrm{tar}\Big(\dot{z}_2(t), \frac{F_{T020}}{C_{20}}\Big) \in F_2(t)
\end{aligned}$$

This system is generally extremly complicated. Using the $\mathrm{luz}\,(\dots)$ and $\mathrm{tar}\,(\dots)$ mathematical apparatus, we can simplify it for the case when $F_2(t) = 0$. In such a case, the bottom element becomes a separation sheet for the top block, so this case is very practical.

We use the following theorem (Żardecki, 2006b): for $a \geqslant 0$, $b \geqslant 0$, $k \geqslant 0$ when $\mathrm{tar}\,(y, b) = k\,\mathrm{tar}\,(x - y, a)$ then

$$y(x) = \begin{cases} x - \dfrac{1}{k+1}\,\mathrm{luz}\,(x, ka - b) & \text{if} \quad ka \geqslant b \\[2mm] \dfrac{k}{k+1}\,\mathrm{luz}\Big(x, \dfrac{b - ka}{k}\Big) & \text{if} \quad ka \leqslant b \end{cases}$$

Therefore, because we have here

$$a = \frac{F_{T012}}{C_{12}} \qquad b = \frac{F_{T020}}{C_{20}} \qquad k = \frac{C_{12}}{C_{20}}$$

$$ka - b = \frac{C_{12}}{C_{20}}\frac{F_{T012}}{C_{12}} - \frac{F_{T020}}{C_{20}} = \frac{F_{T012} - F_{T020}}{C_{20}}$$

when

$$\operatorname{tar}\Big(\dot z_2(t), \frac{F_{T020}}{C_{20}}\Big) = \frac{C_{12}}{C_{20}} \operatorname{tar}\Big(\dot z_1(t) - \dot z_2(t), \frac{F_{T012}}{C_{12}}\Big) = 0$$

here then

$$\dot z_2(t) = \begin{cases} \dot z_1(t) - \dfrac{C_{20}}{C_{12}+C_{20}} \operatorname{luz}\Big(\dot z_1(t), \dfrac{F_{T012}-F_{T020}}{C_{20}}\Big) & \text{if} \quad F_{T012} \geqslant F_{T020} \\ \dfrac{C_{12}}{C_{12}+C_{20}} \operatorname{luz}\Big(\dot z_1(t), \dfrac{F_{T020}-F_{T012}}{C_{12}}\Big) & \text{if} \quad F_{T012} \leqslant F_{T020} \end{cases}$$

Setting this relationship to the first model inclusion, we obtain for $F_2(t) = 0$

$$M_1 \dot z_1(t) \in \begin{cases} B_1(t) & \text{if} \quad F_{T012} \geqslant F_{T020} \\ B_2(t) & \text{if} \quad F_{T012} \leqslant F_{T020} \end{cases}$$

where

$$B_1(t) = F_1(t) - C_{12} \operatorname{tar}\Big(\frac{C_{20}}{C_{12}+C_{20}} \operatorname{luz}\Big(\dot z_1(t), \frac{F_{T012}-F_{T020}}{C_{20}}\Big), \frac{F_{T012}}{C_{12}}\Big)$$

$$B_2(t) = F_1(t) - C_{12} \operatorname{tar}\Big(\dot z_1(t) - \frac{C_{12}}{C_{12}+C_{20}} \operatorname{luz}\Big(\dot z_1(t), \frac{F_{T020}-F_{T012}}{C_{12}}\Big), \frac{F_{T012}}{C_{12}}\Big)$$

In this formula, the $\operatorname{tar}(\dots)$ is undetermined for $|\dot z_1(t)| \leqslant (F_{T012} - F_{T020})/C_{20}$ if $F_{T012} \geqslant F_{T020}$ and for $\dot z_1(t) = 0$ if $F_{T012} \leqslant F_{T020}$. But in both cases $M_1 \ddot z_1(t) \in F_1(t) - F_{T012} s^*_{12}(t)$. Applying the Gauss' rule and the optimization task, we finally obtain for $F_2(t) = 0$ and arbitrary $F_1(t)$ the saturation equation $F_{T012} s^*_{12}(t) = F_1(t) - \operatorname{luz}(F_1(t), F_{T012})$, which enables one to replace the inclusion form by an equation without implicit form. The variable-structure form of this equation explains the stick-slip process for all cases of model's conditions.

5. Final remarks

In this paper, the idea and examples of application of a new method of modeling mechanical systems with freeplay and friction has been presented. The method is based on the piecewise linear $\operatorname{luz}(\dots)$ and $\operatorname{tar}(\dots)$ projections and their original mathematical apparatus. It is very useful for the description of stick-slip processes in multi-body systems which can be described by piecewise-linear equations.

The derived friction models basing on the luz (...) and tar (...) projections are strictly compatible with the legitimate Karnopp models. The models can be directly used in multi-body systems when inclusions contain simple individual tar (...) components, for example

$$M_i\ddot{z}_i(t) \in F_i(z_1(t), z_2(t), \dots, z_n(t), \dot{z}_1(t), \dot{z}_2(t), \dots, \dot{z}_n(t), t) + \\ -C_{ij}\,\mathrm{tar}\Big(\dot{z}_i(t) - \dot{z}_j(t), \frac{F_{T0ij}}{C_{ij}}\Big)$$

When the structure of a multibody system is more complicated, the synthesis of a model must be more sophisticated, as it has been presented in Section 4. But in such cases, the application of the luz (...) and tar (...) mathematical apparatus yields excellent final results – a ready-to-simulation model without implicit forms.

The piecewise-linear approximation based on the luz (...) and tar (...) projections can also be applied to friction characteristics expressing Stribeck's effect, for asymmetric characteristics an so on. Even stick-slip models have been derived here for the simplest friction characteristics, their final forms can be easily adapted to other more complicated characteristics. For example, when magnitudes of kinetic and static dry friction forces are not identical, in the variable-structure model two different parameters F_{T0K} and F_{T0S} can be applied.

The presented method has been already applied to several simulation models of systems with friction. Most of them concern car steering mechanisms (models with dry frictions in king-pins and gears) – see papers by Lozia and Żardecki (2002, 2005) as well as by Żardecki (2005a).

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Przedziałami liniowe modelowanie tarcia i zjawiska "stick-slip" w układach dyskretnych

Streszczenie

W artykule przedstawia się ideę i zastosowanie nowej metody modelowania układów mechanicznych z tarciem. Opracowana metoda bazuje na przedziałami liniowych odwzorowaniach luz (...) i tar (...) oraz ich oryginalnym aparacie matematycznym. Dzięki zastosowaniu odwzorowań luz (...) i tar (...) modele układów z tarciem mają analityczne formy doskonale wyrażające zmiennostrukturalny opis zjawiska "stick-slip". Za sprawą aparatu matematycznego luz (...) i tar (...) modele te mogą być przekształcane (np. redukowane) w sposób parametryczny, co stanowi główną zaletę metody.

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