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VIBRATIONS AND STABILITY OF COLUMNS LOADED BY FOUR-SIDE SURFACES OF CIRCULAR CYLINDERS

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The paper presents a new type of specific load realised through components making a side surface of four circular cylinders having pair-wise equal radii. This is a conservative load representing a follower force directed towards the positive pole. Theoretical considerations are made related to the determination of boundary conditions with installation of needle rolling bearings or rigid elements in the receiving head. The course of the natural vibration frequency against the external load is determined. The results of theoretical research are compared to results of experimental examinations. The rigidity of the substitute rotational spring is defined, considering "the rigidity impact" of the free end of the column.

Key words: elastic column, specific load, divergence instability, natural frequency

1. Introduction

Only a conservative load is considered in the paper, without considering publications related to non-conservative systems. Systems subjected to conservative loads (compare Gajewski and Życzkowski, 1969; Leipholtz, 1974; Timoshenko and Gere, 1963; Ziegler, 1968) lose their stability by buckling. The system change from the static into non-static condition occurs when the value of natural vibration frequency reaches zero ($\omega = 0$) at the so called divergence critical force. Solution to the problem of free vibration of slender systems exposed to external compressive forces leads to a definite course of curves in the plane load (P) – natural frequency (ω), called the characteristic curves. Leipholz (1974) proved that all the characteristic curves of divergence system always have a negative slope.

In Tomski *et al.* (1994), the authors for the first time presented results of theoretical, numerical and experimental research concerning a new type of load, called the specific load.

The specific load of slender systems gave a new (not shown by other authors) course of the characteristic curves. The function $P(\omega)$ of such systems has the following course:

- if the load $P \in \langle 0, P_c)$ (P_c is the critical load), angle of the curve tangent $P(\omega)$ may be a positive, equal to zero or negative value
- if $P \approx P_c$, the slope of the curve in the $P\omega$-plane is always negative
- the change of the natural vibration form (from the first to the second one and inversely) takes place along the curve which determines the $P(\omega)$ function for the basic frequency ($M1$, $M2$ – denote the first and second form of vibration, respectively).

The system characterised by such a course of the characteristic curve in the $P\omega$-plane was named the divergence pseudoflutter system by Bogacz *et al.* (1998).

The specific load is defined in the following way:

- general loading with a force directed to the pole – loading with a force concentrated at the free end of the column directed to the fixed point (pole) located on the undeformed axis of the column and causing a bending moment which is linearly dependent on this force (Tomski, 2004; Tomski *et al.*, 1994, 1995, 1996, 1999a; Tomski and Szmidla, 2004)
- loading with a follower force directed towards the pole – is realised with a concentrated force the direction of which agrees with the tangent to the deflection angle of the column and crosses the fixed point (pole) situated on the undeformed axis (Tomski, 2004; Tomski *et al.*, 1998, 1999b, 2004c, 2006)
- loading through a stretched component of finite bending rigidity – loading with a force transferred to the compressed rods of the column by an additional stretched component which is rigidly connected to the end of the system (Tomski *et al.*, 1997, 2004b, 2005, 2006).

For each discussed type of load there is a design solution realised by structures consisting of linear components (Tomski *et al.*, 1994, 1995, 1996, 1998, 2004) or circular components (Tomski, 2004; Tomski *et al.*, 1995, 1999, 2004c, 2006; Tomski and Szmidla, 2004). The systems subjected to the specific load depending on geometric parameters of the loading structures, may be of the divergence pseudoflutter type, and also of the divergence type.

Through experimental examinations, it was found that selection of the intermediate component in the receiving head (rolling bearing, slide bearing, rigid component) may have considerable influence on the value of the basic experimental natural frequency of the system compared to the same value obtained through numerical simulations (for the applied mathematical model), see Tomski *et al.* (1999a, 2001, 2002, 2003, 2006). The examinations concerning the influence of components used in the loading head on natural vibration of slender systems with specific load and Euler's load were included in the following elaborations:

- with the generalised load with a force directed towards the positive pole – Tomski *et al.* (1999a, 2003)
- loaded with a follower force directed towards the positive pole – Tomski *et al.* (2002, 2003, 2006) and negative pole – Tomski *et al.* (2001)
- loaded with a force directed towards the positive pole – Tomski *et al.* (1999a, 2003)
- with Euler's load – Tomski *et al.* (1999a, 2001, 2002, 2003).

Making use of frequency values obtained from the approximation, the rigidity of the substitute rotational spring (taken into account in the boundary conditions) was determined against the external load (Tomski *et al.*, 1999a, 2003, 2006) by comparing the obtained results with those obtained for Euler's load. It was demonstrated that the way of loading the column has important influence on the stiffening of the system free end. Changes in the flexural rigidity for individual stages of the column gave similar results.

2. Formulation of the problem

This paper presents theoretical, numerical and experimental examinations of the system shown in Fig. 1. The examination of this system includes determination of:

- free vibration (characteristic curves)
- critical force
- influence of components used in the receiving head: needle rolling bearings (variant F_1) or rigid elements (variant F_2) on the system free vibration.

The structure of the loading system consists of a forcing head and receiving head. The forcing head is made of circular components of radii R and centres in points O_{RL} and O_{RP} (Fig. 1). The receiving heads are composed of circular components of the radii r and centres O_{rL} and O_{rP}. The segments of the loading heads have two common points (points of tangency). The poles O_{RL} and O_{RP} are located at the constant distance equal to $(R-r)$ from centres O_{rL} and O_{rP}.

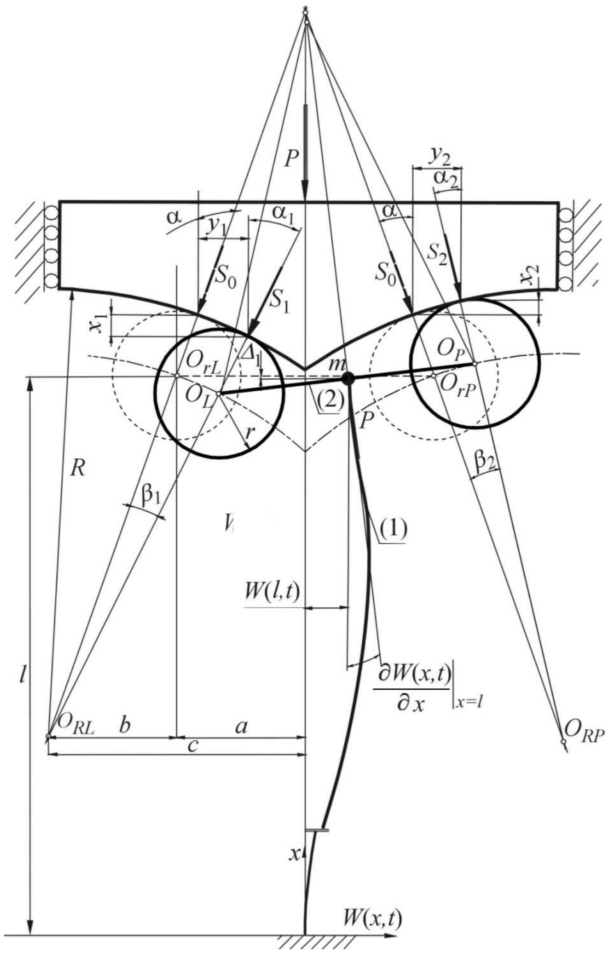

Fig. 1. The physical model of the system

In the structure of the load receiving heads, there is a rigid element (2) installed orthogonally to the free end of the column (1), the length of which $2a$ is strictly connected with location of the centers of elements having the r radius. Location of the centres of the forcing head circular components having the R radii in relation to the neutral axis of the system is defined as c. There is a $c > a$ relation between the c and a values, and the analysis is carried out in the range up to $2a$.

2.1. Description of geometrical relations in the system

The location of the receiving system with the centres of circles of the r radius in the points O_{rL} and O_{rP} is the initial configuration. The final configuration, resulting from the location of the receiving element, is determined by location of the O_L and O_P centres of these circles.

The system geometry and components of the external load P are described by the following quantities:
a) related to the undeformed condition of the column:

S_0	–	components of the external load for the undeformed condition of the column
S_{0x}, S_{0y}	–	horizontal and vertical components of the S_0 forces
α	–	angle formed by the components of the P load with the neutral axis of the system in the undeformed condition
a	–	location of the centres of the r circles in relation to the neutral axis of the system
c	–	location of the centres of the R circles in relation to the neutral axis of the system
b	–	quantity determining the mutual location of the centres of circular components $(b = c - a)$, of the loading head

b) related to the deformed condition of the system:

S_1, S_2	–	components of the external load for the deformed condition of the column
S_{1x}, S_{2x}	–	vertical components of the S_1 and S_2 forces
S_{1y}, S_{2y}	–	horizontal components of the S_1 and S_2 forces
α_1, α_2	–	angles of the S_1 and S_2 forces in relation to the neutral axis of the system
β_1, β_2	–	angles between the load components, for the initial (undeformed) and deformed condition of the column.

The geometrical parameters characterising the system under consideration fulfil the following relation

$$\sin\alpha = \frac{b}{R-r} \tag{2.1}$$

The S_0 forces and their vertical S_{0x} and horizontal components S_{0y} equal to

$$S_0 = \frac{P}{2}\frac{1}{\cos\alpha} \qquad S_{0x} = \frac{P}{2} \qquad S_{0y} = \frac{P}{2}\tan\alpha \tag{2.2}$$

For the deformed condition of the column, the external load P is distributed into two components S_1 and S_2, and their angles in relation to the undeformed axis of the system are

$$\alpha_i = \alpha - \beta_i(-1)^i \qquad \text{for} \quad i = 1, 2 \tag{2.3}$$

In order to determine the vertical and horizontal components of the S_1 and S_2 forces and corresponding displacements, the following equations have been determined $(i = 1, 2)$

$$
\begin{aligned}
\sin\alpha_i &= \frac{1}{R-r}\Big[a+b-a\cos\frac{\partial W(x,t)}{\partial x}\Big|_{x=l} - W(l,t)(-1)^i\Big] \\
\cos\alpha_i &= -\frac{a}{R-r}(-1)^i\sin\frac{\partial W(x,t)}{\partial x}\Big|_{x=l} + \cos\alpha\Big[1-\frac{a^2}{2b^2}\sin^2\frac{\partial W(x,t)}{\partial x}\Big|_{x=l}\Big]
\end{aligned}
\tag{2.4}
$$

The β angles are small, therefore, it has been assumed that $\sin\beta_1 = \sin\beta_2 = \sin\beta = \beta$, and the following was determined

$$
\beta = -\frac{a}{(R-r)\sin\alpha}\sin\frac{\partial W(x,t)}{\partial x}\Big|_{x=l} = -\frac{a}{b}\sin\frac{\partial W(x,t)}{\partial x}\Big|_{x=l}
\tag{2.5}
$$

The components of the external load for the deformed condition are expressed as follows: $S_1 = S_2 = S_0$. The vertical and horizontal components of these forces, after applying relations (2.4), and eliminating the non-linear elements, amount to

$$
\begin{aligned}
S_{ix} &= \frac{1}{2}P\Big[1-(-1)^i\frac{a}{b}\tan\alpha\sin\frac{\partial W(x,t)}{\partial x}\Big|_{x=l}\Big] \\
S_{iy} &= \frac{1}{2}P\frac{1}{(R-r)\cos\alpha}\Big[a+b-(-1)^iW(l,t)+(-1)^ia\cos\frac{\partial W(x,t)}{\partial x}\Big|_{x=l}\Big]
\end{aligned}
\tag{2.6}
$$

and the following relation is satisfied

$$
S_{1x} + S_{2x} = P
\tag{2.7}
$$

The displacements for the y_1 and y_2 horizontal components, after considering relations (2.4), are determined as follows

$$
\begin{aligned}
y_i &= -a(-1)^i\Big[1-\cos\frac{\partial W(x,t)}{\partial x}\Big|_{x=l}\Big] + W(x,t)\Big|_{x=l} + \\
&-(-1)^i\Big\{r\sin\alpha + \frac{r}{R-r}\Big[(-1)^i(a+b) - W(l,t) + a\cos\frac{\partial W(x,t)}{\partial x}\Big|_{x=l}\Big]\Big\}
\end{aligned}
\tag{2.8}
$$

The displacements x_1 and x_2 of appropriate vertical components are

$$
\begin{aligned}
x_i &= -a\Big(\frac{1}{R-r}+1\Big)\sin\frac{\partial W(x,t)}{\partial x}\Big|_{x=l} - \mathit{\Delta}_1(-1)^i + \\
&-(-1)^i\frac{a^2r}{2b^2}\cos\alpha\sin^2\frac{\partial W(x,t)}{\partial x}\Big|_{x=l}
\end{aligned}
\tag{2.9}
$$

where the $\mathit{\Delta}_1$ is a displacement resulting from shortening of the axis of the column caused by bending, and amounts to

$$
\mathit{\Delta}_1 = \frac{1}{2}\int\limits_0^l \Big[\frac{\partial W(x,t)}{\partial x}\Big]^2\,dx
\tag{2.10}
$$

The geometrical relations between the elements of the loading structures lead to a relation between the transverse displacement and the deflection angle of the column free end in the following way

$$W(l,t) = -\frac{a(R-r)}{b}\cos\alpha\sin\frac{\partial W(x,t)}{\partial x}\Big|_{x=l} \tag{2.11}$$

3. Mechanical energy of the system. Boundary conditions

The kinetic energy T for the considered system in variant (F_1) is as follows

$$\begin{aligned} T = T_1 + T_2 + T_3 = \frac{1}{2}\rho_0 A\int\limits_0^l\Big[\frac{\partial W(x,t)}{\partial t}\Big]^2\,dx + \frac{1}{2}m\Big[\frac{\partial W(x,t)}{\partial t}\Big|_{x=l}\Big]^2 + \\ +\frac{1}{2}B\Big[\frac{\partial^2 W(x,t)}{\partial x\partial t}\Big|_{x=l}\Big]^2 \end{aligned} \tag{3.1}$$

where B is the moment of inertia of the mass placed at the end of the column calculated with respect to the axis perpendicular to the plane of vibration.

The relations for the potential energy of the system are determined as follows:

— energy of elastic strain

$$V_1 = \frac{1}{2}EJ\int\limits_0^l\Big[\frac{\partial^2 W(x,t)}{\partial x^2}\Big]^2\,dx \tag{3.2}$$

— potential energy of the horizontal components of the P force

$$V_2 = \frac{1}{2}[(S_{1y} + S_{0y})y_1 - (S_{2y} + S_{0y})y_2] \tag{3.3}$$

— potential energy of the vertical components of the P force

$$V_3 = \frac{1}{2}[(S_{2x} + S_{0x})x_2 - (S_{1x} + S_{0x})x_1] \tag{3.4}$$

The formulation of the problem is carried out with the use of Hamilton's principle (Goldstein, 1950), i.e.

$$\delta\int\limits_{t_1}^{t_2}\Big(T - \sum_{k=1}^{3} V_k\Big)\,dt = 0 \tag{3.5}$$

The commutation of integration (with respect to x and t) and variational calculus have been used within Hamilton's principle (3.5). The equation of motion, after taking into account the commutation of variation and differentiation operators and after integrating kinetic and potential energies of the system, is obtained in the form

$$EJ\frac{\partial^4 W(x,t)}{\partial x^4} + P\frac{\partial^2 W(x,t)}{\partial x^2} + \rho_0 A\frac{\partial^2 W(x,t)}{\partial t^2} = 0 \tag{3.6}$$

Equation (3.6) is a differential equation of column motion. Considering the *a priori* known geometrical boundary conditions of the considered system in relation (3.5) regarding the fixed point

$$W(0,t) = \frac{\partial W(x,t)}{\partial x}\Big|_{x=0} = 0 \tag{3.7}$$

allows for determination of the missing boundary condition at the free end of the column, which is necessary for the solution of the boundary problem (Podgórska-Brzdękiewicz, 2004)

$$\begin{aligned}&\frac{\partial^3 W(x,t)}{\partial x^3}\Big|_{x=l} + \frac{b}{a(R-r)\cos\alpha}\Big[\frac{\partial^2 W(x,t)}{\partial x^2}\Big|_{x=l} + \frac{B}{EJ}\frac{\partial^3 W(x,t)}{\partial t^2 \partial x}\Big|_{x=l}\Big] +\\ &-\frac{m}{EJ}\frac{\partial^2 W(l,t)}{\partial t^2} + \frac{P}{EJ}\Big[1 + \frac{a}{b} + \frac{Rb(a+b)}{(R-r)^3\cos^2\alpha}\Big]\frac{\partial W(x,t)}{\partial x}\Big|_{x=l} = 0\end{aligned} \tag{3.8}$$

For columns of variant (F_2), boundary condition (3.8) should be modified by introducing the rigidity of the equivalent rotational spring C which models the stiffening of the system. The described value is determined in the following way

$$C_1 = g_1 + g_2 P \tag{3.9}$$

where g_1 and g_2 coefficients define the stiffening of the system caused by:

g_1 – "preliminary tension" in the loading head, without participation of the external loading,

g_2 – influence of the external load.

The modified boundary condition is described as follows

$$\begin{aligned}&\frac{\partial^3 W(x,t)}{\partial x^3}\Big|_{x=l} + \frac{b}{a(R-r)\cos\alpha}\Big[\frac{\partial^2 W(x,t)}{\partial x^2}\Big|_{x=l} + \frac{B}{EJ}\frac{\partial^3 W(x,t)}{\partial t^2 \partial x}\Big|_{x=l}\Big] +\\ &-\frac{m}{EJ}\frac{\partial^2 W(l,t)}{\partial t^2} + \frac{P}{EJ}\Big[1 + \frac{a}{b} + \frac{Rb(a+b)}{(R-r)^3\cos^2\alpha} + \frac{C_1}{P}\Big]\frac{\partial W(x,t)}{\partial x}\Big|_{x=l} = 0\end{aligned} \tag{3.10}$$

4. Solution to the boundary problem

Considering a symmetric distribution of the flexural rigidity and the mass per unit length, the equations of motion for the reviewed system after distinguishing the function variables $W_i(x,t)$ in relation to time t and space x in the form of

$$W_i(x,t) = y_i(x)\cos(\omega t) \qquad i = 1,2 \tag{4.1}$$

are as follows

$$(EJ)_i y_i^{\rm IV}(x) + (S)_i y_i^{\rm II}(x) - (\rho_0 A)_i \omega^2 y_i(x) = 0 \qquad (S)_1 + (S)_2 = P \tag{4.2}$$

where $(S)_i$ is the internal force in the ith rod of the system.

The boundary conditions in the fixing point and at the free end of the column (in (F_1) variant), after distinguishing the variables, have the following form

$$\begin{aligned}
&y_1(0) = y_2(0) = 0 \qquad\qquad y_1^{\rm I}(0) = y_2^{\rm I}(0) = 0\\
&y_1(l) = y_2(l) \qquad\qquad y_1^{I}(l) = y_2^{I}(l)\\
&y_1(l) = \frac{a}{b}(R-r)\cos\alpha\, y_1^{\rm I}(l)\\
&y_1^{\rm III}(l) + y_2^{\rm III}(l) - \frac{b}{a(R-r)\cos\alpha}\Big[y_1^{\rm II}(l) + y_2^{\rm II}(l) - \frac{B\omega^2}{(EJ)_1}y_1^{\rm I}(l)\Big] +\\
&\quad + k^2 y_1^{\rm I}(l)\Big[1 + \frac{a}{b} + \frac{Rb(a+b)}{(R-r)^3\cos^2\alpha}\Big] + \frac{m\omega^2}{(EJ)_1}y_1(l) = 0
\end{aligned} \tag{4.3}$$

General solutions to equations (4.2) are as follows

$$y_i(x) = D_{1i}\cosh(\alpha_i x) + D_{2i}\sinh(\alpha_i x) + D_{3i}\cos(\beta_i x) + D_{4i}\sin(\beta_i x) \tag{4.4}$$

where D_{ni} are integration constants $(n = 1,2,3,4)$, and

$$\alpha_i^2 = -\frac{1}{2}k_i^2 + \sqrt{\frac{1}{4}k_i^4 + \varOmega_i^2} \qquad \beta_i^2 = \frac{1}{2}k_i^2 + \sqrt{\frac{1}{4}k_i^4 + \varOmega_i^2} \tag{4.5}$$

where

$$\varOmega_i^2 = \frac{(\rho_0 A)_i\omega^2}{(EJ)_i} \qquad k_i = \sqrt{\frac{(S)_i}{(EJ)_i}} \tag{4.6}$$

Substituting solutions (4.4) into boundary conditions (4.3), yields a transcendental equation for eigenvalues of the considered system.

5. The experimental research

5.1. Experimental stand

The experimental research was carried out on a test stand designed and built at the Institute of Mechanics and Machine Design Fundamentals of the Technical University of Częstochowa. A constructional diagram of the stand is shown in Fig. 2 (Tomski *et al.*, 2004a).

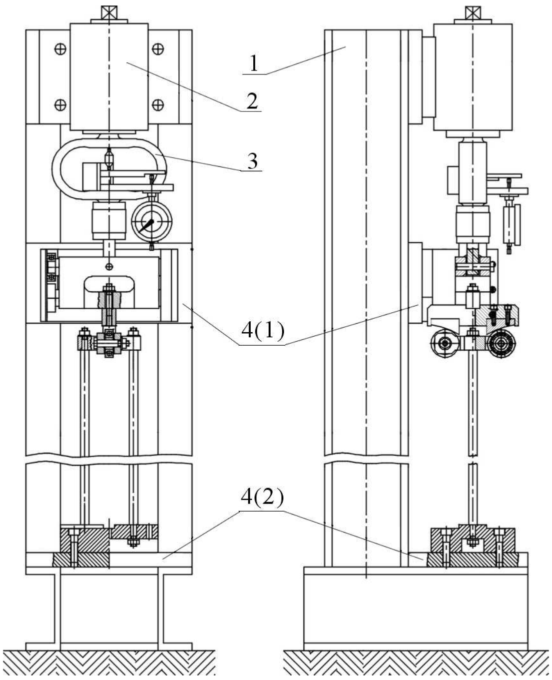

Fig. 2. The test stand for examination of natural frequencies of slender systems with fixed columns (Tomski *et al.*, 2004a)

The experimental stand consists of support frame 1 to which head 2 is fixed. The head has a screw system the movement of which loads the examined column. The loading force is measured by dynamometer 3. The requested boundary conditions are realised by appropriate column supports fastened to plates 4(1) and 4(2).

Plate 4.2 has a support mounted to it realising appropriate boundary conditions of the column fixing. Plate 4.1 has an element mounted to it, where the loading head moves in the longitudinal direction. Investigations of natural frequencies are performed with the use of a two-channel vibration analyser made by Brüel & Kjaer (Denmark). Vibration of the columns is induced by a hammer. The acceleration of the individual measuring point was measured by means of an accelerometric sensor. The signal from the sensor is transmitted to the analyser.

5.2. Loading structure of the column

The constructional scheme of the column loading head is presented in Fig. 3. It is composed of part G forcing the load and part H receiving this load. The forcing head G consists of element (3) where cubes (4) are characterised by circular outlines of the radius R. Element (3) is joint with movable system (5). The receiving head H is composed of cube (2) in which pins (6.1) and (6.2) are located on which circular outline elements (7.1) and (7.2) of the radius r are mounted.

The circular outlines of elements (4) and (7) making a part of the forcing and receiving head respectively, have the same symbol ($R > 0$ and $r > 0$).

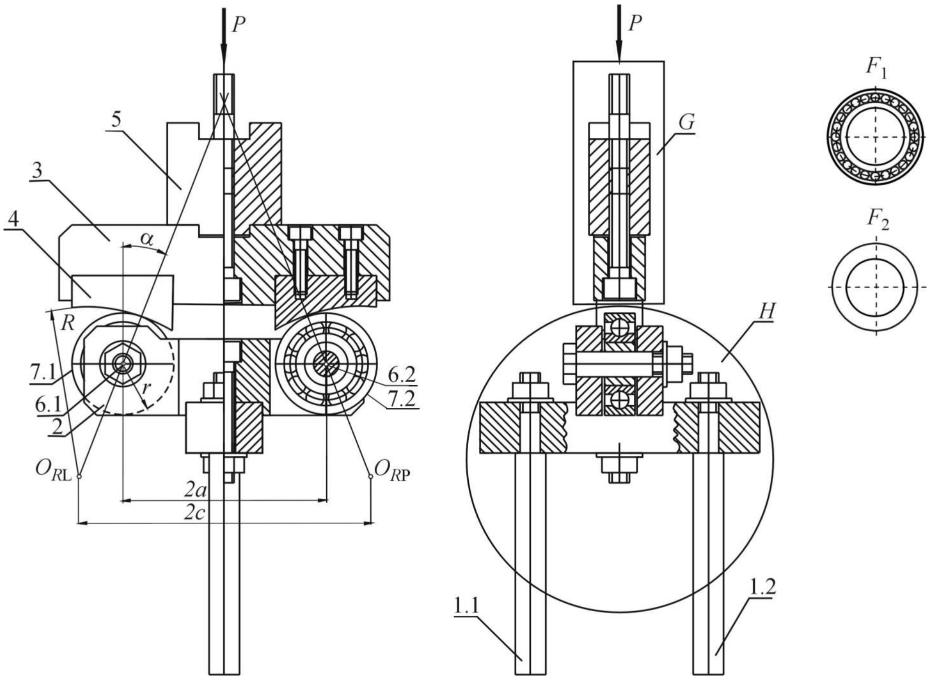

Fig. 3. Structural scheme of the loading head of the column

The column consists of two rods (6.1 and 6.2) with the bending rigidity $(EJ)_1$ and $(EJ)_2$ respectively, and the mass per unit length $(\rho_0 A)_1$ and $(\rho_0 A)_2$, and $(EJ)_1 = (EJ)_2$, $(\rho_0 A)_1 = (\rho_0 A)_2$, $(EJ)_1 + (EJ)_2 = EJ$, $(\rho_0 A)_1 + (\rho_0 A)_2 = \rho_0 A$, where: E – longitudinal modulus of elasticity of the rod material, J – central axial moment of inertia of the column rod, ρ_0 – material density, A – cross-section area. The lengths of the column rods are equal to l. The rods of the column have the same cross-sections and are made of the same material. The rods and their physical and geometrical parameters are distinguished by 1, 2 indices, which are only needed to calculate symmetric natural frequencies and to determine corresponding forms of vibration. Hence, we can assume a global bending rigidity EJ and elementary mass of the column $\rho_0 A$ in the following considerations.

5.3. Results of experimental examinations and numerical calculations

Based on the solution of the boundary condition (for the (F_1) column), numerical calculations of the natural frequency course against the external load were performed. Then they were verified experimentally on a test stand (Fig. 2) for the considered design solutions of the receiving heads, i.e. using needle bearings (F_1) and for the rigid components (F_2). The physical and geometrical parameters of the systems as well as the values characterising the forcing and receiving heads were distinguished by introduced designations A, B and C as presented in Table 1.

Table 1. Physical and geometrical parameters of the considered systems

Column	EJ [Nm^2]	$\rho_0 A$ [kg/m]	l [m]	R [m]	b [m]	α [°]	m [kg]	B [kg m^2]
$A(F_1), A(F_2)$	152.68	0.631	0.7	0.05	0.01	19.47	0.615	$5.5 \cdot 10^{-4}$
$B(F_1), B(F_2)$	152.68	0.631	0.7	0.08	0.01	9.6	0.615	$5.5 \cdot 10^{-4}$
$C(F_1), C(F_2)$	152.68	0.631	0.7	0.14	0.02	9.6	0.615	$5.5 \cdot 10^{-4}$

$a = 0.04\,\mathrm{m}$, $r = 0.02$

The obtained results of the experimental testing and numerical calculations are shown in Fig. 4 - Fig. 6. The lines represent results of numerical calculations (for the column F_1). The points present the obtained results of the experimental testing with the needle bearings in the receiving head (columns $A(F_1)$, $B(F_1)$ and $C(F_1)$) or rigid components ($A(F_2)$, $B(F_2)$ and $C(F_2)$). The results are limited to the first four basic natural frequencies ($M1$ - $M4$) and three additional frequencies ($M2^e$, $M3^e$, $M4^e$) characterised by symmetry of vibrations.

Installation of the needle bearings in the receiving head (columns $A(F_1)$, $B(F_1)$ and $C(F_1)$) gave a good conformity of the results of numerical calculations and experimental testing. It was found that the loading of the column with a head where the rigid components take part in taking the load (columns $A(F_2)$, $B(F_2)$ and $C(F_2)$) cause a considerable increase in the basic natural frequency of the system compared to its value obtained through numerical simulations and experimental tests with the needle bearings.

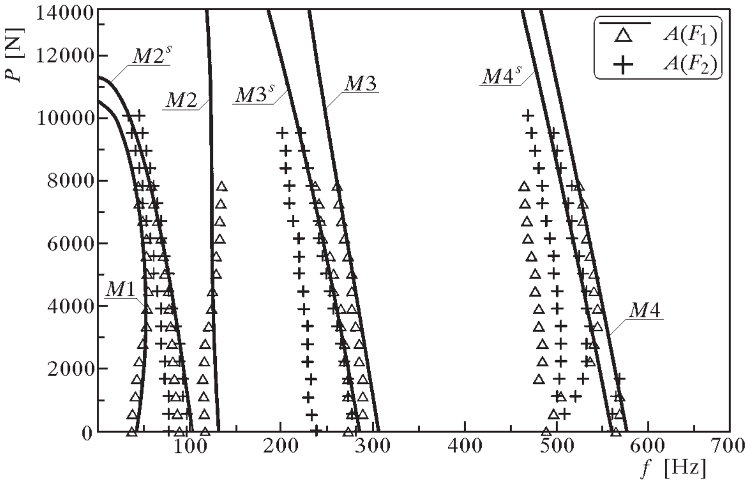

Fig. 4. Curves in the plane: load - natural frequency for columns $A(F_1)$, $A(F_2)$

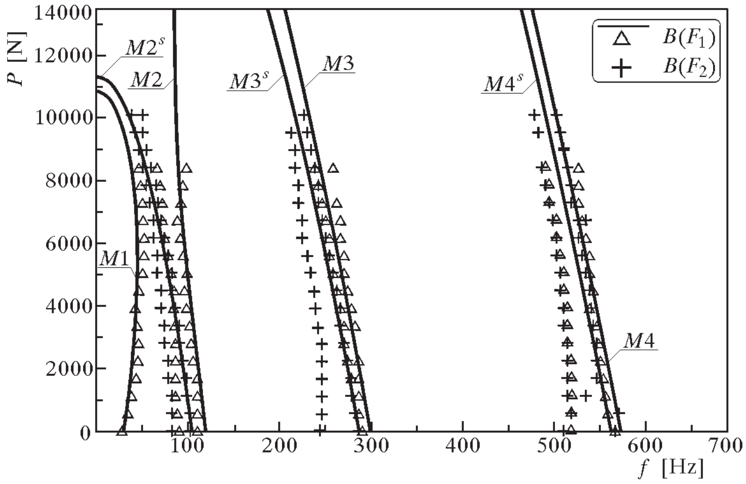

Fig. 5. Curves in the plane: load - natural frequency for columns $B(F_1)$, $B(F_2)$

Considering the results of the experimental testing, the course of changes of characteristic values against the external load is approximated in order to obtain a theoretical characteristic curve with possibly highest correlation coefficient and lowest standard deviation. Appropriate results were determined in dimensionless co-ordinates where λ^* is a parameter of the external load, and Ω^* defines a dimensionless parameter of the system natural frequency

$$\lambda^* = \frac{Pl^2}{EJ} \qquad\qquad \Omega^* = \frac{\rho_0 A\omega^2 l^4}{EJ} \tag{5.1}$$

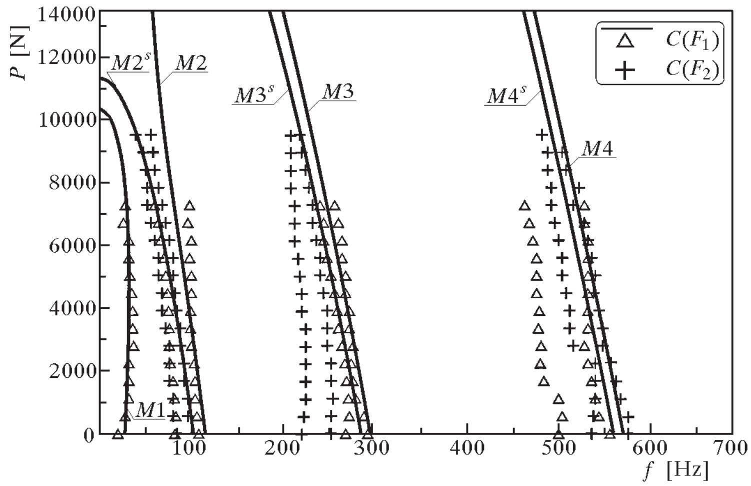

Fig. 6. Curves in the plane: load - natural frequency for columns $C(F_1)$, $C(F_2)$

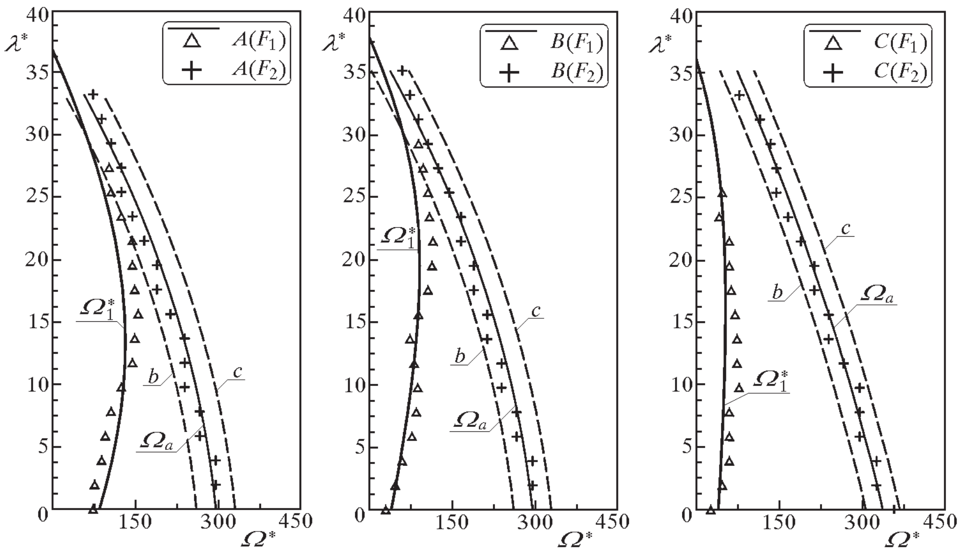

Fig. 7. Curves in the plane: load - natural frequency for columns (a) $A(F_1)$, $A(F_2)$, (b) $B(F_1)$, $B(F_2)$, (c) $C(F_1)$, $C(F_2)$

The curves Ω_a obtained from the approximation are presented in Fig. 7, and their equations including the determined correlation coefficients and standard deviations are shown in Table 2. The curves marked with indices b, c determine the triple standard deviations of the individual experimental results from the expected value.

Table 2. Equations of approximating functions, values of correlation coefficients and standard deviations

Column	Approximating function	Correlation coefficient	Standard deviation
$A(F_2)$	$\Omega^*(\lambda^*) = 289.3 - 2.18\lambda^* - 0.15(\lambda^*)^2$	0.982	11.64
$B(F_2)$	$\Omega^*(\lambda^*) = 295.3 - 2.38\lambda^* - 0.14(\lambda^*)^2$	0.984	11.29
$C(F_2)$	$\Omega^*(\lambda^*) = 357.4 - 10.4\lambda^* - 0.11(\lambda^*)^2$	0.985	10.25

By making use of the results obtained from the linear regression and introduced rigidity of the substitute rotational spring in the boundary conditions, the external load for the systems under consideration is determined. The results of calculations are presented in Fig. 8 within the scope of the external load realised for the experimental measurements.

The rigidity of spring C_1 is expressed in a dimensionless form

$$c_1^* = \frac{C_1 l}{EJ} \tag{5.2}$$

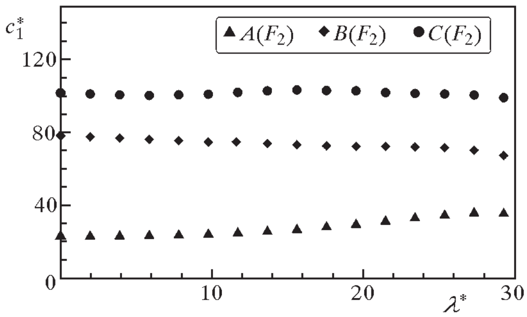

Fig. 8. Change of the nondimensional parameter of the rotational spring rigidity c_1^* against the external load parameter

The rigidity coefficient of the spring modelling the stiffening of the system free end depends on the geometric parameters of the forcing and receiving heads. The influence of the external force loading the system on the range of change of the spring rigidity for a given geometry of the forcing head is small. The discrepancies between some constant value of the c_1^* coefficient (at constant geometry) observed in Fig. 8 may result from the error of the experimental tests. The measurement results are influenced by inaccuracies of the system and simplifying assumptions resulting from:

- accuracy of physical and geometrical parameters assumed in the calculations and characterising the system
- assumption of infinite rigidity of the forcing and receiving head components

- assumption of infinite rigidity of the system fastening
- influence of properties of the testing stand on the examined system, etc.

For the applied mathematical model of the considered system, using the needle bearings in the receiving head (type F_1), numerical calculations were made concerning the influence of selected parameters of loading structures on the stability and free vibration of the system. The value of the critical load and parameters of the forcing and receiving heads are expressed in dimensionless coordinates

$$\begin{array}{lll} \lambda_c^* = \dfrac{P_c l^2}{EJ} & R^* = \dfrac{R}{l} & r^* = \dfrac{r}{l} \\ \Delta r^* = \dfrac{R - r}{l} & a^* = \dfrac{a}{l} & c^* = \dfrac{b}{a} \end{array} \tag{5.3}$$

Figure 9 presents the influence of change in the forcing head radius R^* on the critical load λ_c^* for some selected values of the parameter c^*.

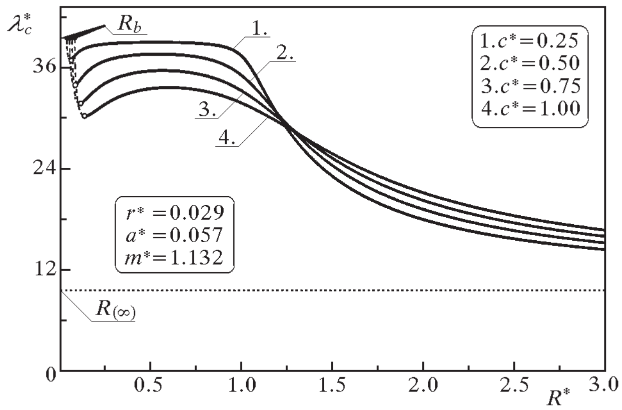

Fig. 9. Change of the load critical parameter λ_c^* against the column radius R^* of the forcing head for various values of the parameter c^*

The curves of the critical load changes are characterised by the presence of the maximum value of the critical load parameter. The values related to geometry of the loading heads are interdependent of geometrical relationships (compare (2.1)), the radii R^*, r^* have to satisfy the relation resulting from the structure of elements realising the load ($R^* \geqslant r^* + b$).

If the radius R^* of the forcing head increases to very high values ($R^* \to \pm\infty$), the critical force reaches the same value – point $R_{(\infty)}$. Satisfying $\Delta r^* = b^*$ (point R_b), the critical parameter of load corresponds to the system characterised by rigid fastening at $x = l$ (compare Table 3). Values of the natural frequency against the external load were numerically calculated.

Table 3. Boundary conditions for columns having the limit values of the parameter Δr^*

$\Delta r^* \to \infty$	$\dfrac{\partial W(x,t)}{\partial x}\Big\vert_{x=l} = 0$	$\dfrac{\partial^3 W(x,t)}{\partial x^3}\Big\vert_{x=l} - \dfrac{m}{EJ}\dfrac{\partial^2 W(l,t)}{\partial t^2} = 0$
$\Delta r^* \to b^*$	$W(l,t) = 0$	$\dfrac{\partial W(x,t)}{\partial x}\Big\vert_{x=l} = 0$

Determination of the nature of changes is limited to two basic natural frequencies in a dimensionless form Ω_t^* $(t = 1, 2)$ and additional symmetrical natural frequency Ω_2^{*s} (Tomski *et al.*, 2001, 2004) against the dimensionless parameter of load λ^* (compare (5.1)) for selected parameters of the forcing and receiving heads and a constant value of the concentrated mass m at the free end of the system. The following is assumed

$$m^* = \frac{m}{\rho_0 A l} \tag{5.4}$$

For the columns loaded with heads having the limit values of the parameter Δr^*, Table 3 presents the boundary conditions at their free end.

Figures 10 and 11 present the influence of changes of the forcing head geometry on the course of curves in the plane: loading parameter λ^* - natural frequency parameter Ω^*.

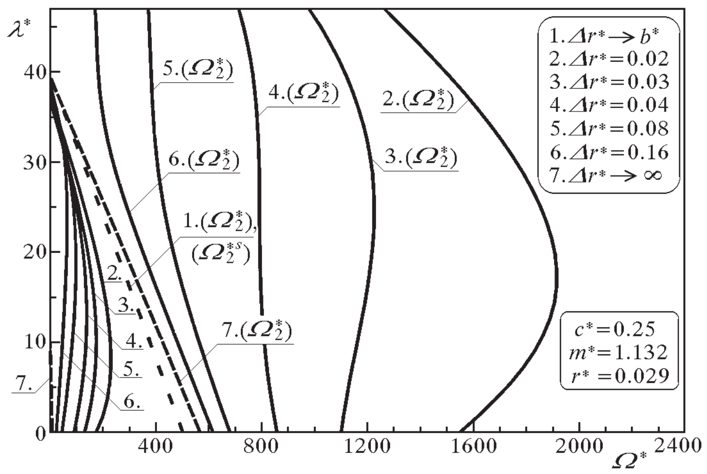

Fig. 10. Curves in the plane: load parameter λ^* - natural frequency parameter Ω^* for a column with $r^* = 0.029$ and $c^* = 0.25$

The discussed diagrams have been obtained for constant values of the radius parameter of the receiving head r^*, constant parameter a^* characterising

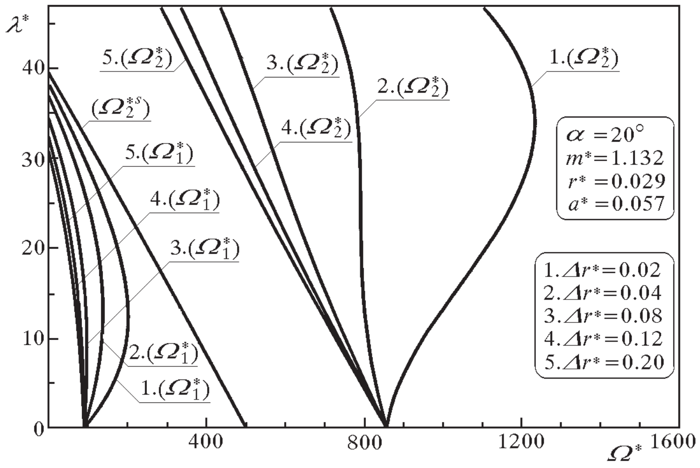

Fig. 11. Curves in the plane: load parameter λ^* - natural frequency parameter Ω^* for a column with $\alpha = 20°$

the length of the rigid component of the receiving head aa well as a constant value of the concentrated mass m^*. In Fig. 10, the course of characteristic curves is limited by broken curves (1) and (7) made for the systems having the limit values of the parameter Δr^*.

The critical load is presented by points on the first natural frequency curve of the system for $\Omega_1^* = 0$. The course of the additional natural frequency Ω^{*s} corresponded by the symmetrical form having a constant value independent of geometry of the forcing and receiving heads is characteristic for the presented diagrams.

The presented curves in the plane: loading parameter λ^* - natural frequency parameter Ω^* allow one to classify the considered columns among two types of systems, i.e. divergent or divergent pseudoflutter ones.

6. Final remarks

Based on theoretical considerations, experimental tests and results of numerical simulations related to stability and free vibration of considered columns loaded by heads made of circular outline cylinders, the following conclusions can be made:

- geometric relations between the forcing and receiving structures lead to determination of geometric relations between the transverse displacement and the deflection angle of the free end of the column

- selection of components used in the receiving head has influence on the course of the basic experimental natural frequency giving a good conformity of the results from numerical calculations and experimental tests with the use of needle bearings referring to the same value obtained through numerical simulations
- installation of rigid components in the loading head causes a considerable increase of the basic natural frequency and requires modification of the boundary conditions by introducing some rigidity of the substitute rotational spring that models the stiffening effect of the system free end
- depending on the course of curves shown in the loading - natural frequency plane, the considered systems can be classified as one of the two possible types: i.e. as divergent or divergent pseudoflutter systems.

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Drgania i stateczność kolumn obciążonych poprzez cztery pobocznice walców kołowych

Streszczenie

W pracy prezentuje się nowy typ obciążenia swoistego zrealizowanego poprzez elementy tworzące pobocznicę czterech walców kołowych o promieniach parami równych. Jest to obciążenie, które reprezentuje siłę śledzącą skierowaną do bieguna dodatniego. Przeprowadza się rozważania teoretyczne dotyczące sformułowania warunków brzegowych przy zastosowaniu w głowicy przejmującej obciążenie łożysk tocznych igiełkowych lub elementów sztywnych. Określa się przebieg częstości drgań własnych w funkcji obciążenia zewnętrznego. Wyniki badań teoretycznych porównuje się z wynikami badań eksperymentalnych. Wyznacza się sztywność zastępczej sprężyny rotacyjnej uwzględniającej „usztywnienie" nieutwierdzonego końca kolumny.

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