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TORSIONAL VIBRATIONS OF DISCRETE-CONTINUOUS SYSTEMS WITH LOCAL NONLINEARITY HAVING HARD TYPE CHARACTERISTICS

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In the paper, nonlinear vibrations of torsionally deformed multi-mass mechanical systems are studied. The systems consist of shafts connected by an arbitrary number of rigid bodies together with a local nonlinearity having the characteristic of a hard type. It is proposed to describe this characteristic by irrational functions. In the considerations, an approach using the wave solution to equations of motion is applied. Detailed numerical calculations are given for a two-mass and a three-mass system. They focus on nonlinear effects in the considered systems.

Key words: nonlinear oscillations, dynamics of mechanical systems, discrete-continuous models

1. Introduction

In the paper, nonlinear torsional vibrations of discrete-continuous mechanical systems with a local nonlinearity having the characteristic of a hard type are studied. This characteristic is described by irrational functions. The considered systems consist of shafts with circular cross-sections connected by means of rigid bodies. In the case of torsional deformations, the equations of motion of shafts are classical wave equations (cf. Pielorz, 2003).

Vibrations of nonlinear discrete systems with nonlineariries having hard type characteristics are considered mainly with polynomials of the third degree (cf. Hagedorn, 1981; Szemplińska-Stupnicka, 1990). The present paper concerns discrete-continuous systems with a local nonlinearity. Local nonlinearities are suggested by engineering solutions (cf. Thomson, 1981), and here irrational functions are proposed for their description. These functions contain the cubic nonlinearity.

Considerations in the present paper are similar to those given in Pielorz (1995). The differences are in the assumed functions describing local nonlinearities.

The method proposed for the determination of solutions in arbitrary cross-sections of shafts in the system leads to equations with a retarded argument. The numerical analysis is performed for a two-mass and a three-mass system and is focussed on the effect of the local nonlinearity on angular displacements in selected cross-sections.

2. Governing equations

The discrete-continuous model of a multi-mass torsional system under consideration is shown in Fig. 1. It is assumed that the x-axis is parallel to the main axis of the system, and its origin coincides with the position of the left-hand end of the first shaft in an undisturbed state at the time instant $t = 0$. The i-th shaft, $i = 1, 2, \dots, N$, is characterized by the length l_i, density ρ, shear modulus G and polar moment of inertia I_{0i}. The i-th rigid body connecting appropriate shafts is characterized by the mass moment of inertia J_i. A single local nonlinearity is located in the cross-section $x = 0$, and it can represent mechanical properties of various elements, such as clutches and gears, having nonlinear characteristics. The rigid body J_1 is loaded by the external loading $M(t)$. Damping in the system is taken into account by an equivalent external and internal damping in the selected cross-sections x_{0i} with coefficients d_i, D_i in the form

$$M_{di}(t) = -d_i\theta_{i,t}(x_{0i}, t) \qquad\qquad M_{Di}(t) = GJ_{0i}D_i\theta_{i,xt}(x_{0i}, t) \tag{2.1}$$

where θ_i are displacements of the i-th shaft. Moreover, it is assumed that the displacements and velocities of shaft cross-sections are equal to zero at the time instant $t = 0$.

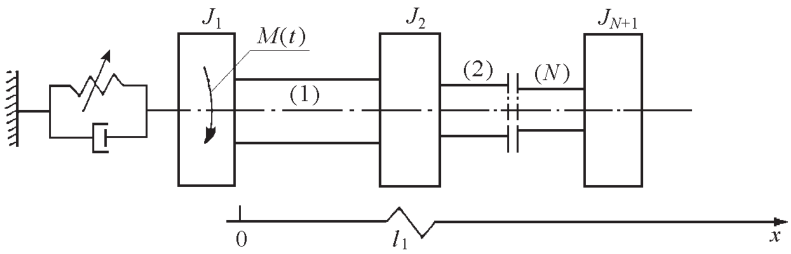

Fig. 1. A nonlinear discrete-continuous model of a torsional system

In the cross-section $x = 0$, the local nonlinearity with the hard type characteristic is located. It is described by functions

$$M_{sp}(t) = \begin{cases} k_1\theta_1(0,t) - k_w(-\theta_1(0,t))^w & \text{for} \quad \theta_1 \leqslant 0 \\ k_1\theta_1(0,t) + k_w(\theta_1(0,t))^w & \text{for} \quad \theta_1 \geqslant 0 \end{cases} \tag{2.2}$$

where k_1 and k_w appear at the linear and nonlinear terms, respectively, the exponent w is assumed to be a real number greater than 1, and in the case of a hard characteristic k_w is a positive constant. The linear case corresponds to $k_w = 0$.

Local nonlinearities and the functions of type (2.2) for their description are justified by numerous experimental studies (cf. Boiler and Seeger, 1987; Szolc, 2003; Thomson, 1981). The constants k_1 and k_w are usually determined experimentally. Functions (2.2) contain the third degree polynomial used in Pielorz (1995), and here the results by Pielorz (1995) are generalized by introducing irrational functions.

On the above assumptions and upon the introduction of the following dimensionless quantities

$$\begin{array}{llll} \overline{x} = \dfrac{x}{l_1} & \overline{t} = \dfrac{ct}{l_1} & \overline{\theta}_i = \dfrac{\theta_i}{\theta_0} & \overline{d}_i = \dfrac{d_i l_1}{J_1 c} \\ \overline{D}_i = \dfrac{D_i c}{l_1} & \overline{k}_1 = \dfrac{k_1 l_1^2}{J_1 c^2} & \overline{k}_w = \dfrac{k_w \theta_0^2 l_1^2}{J_1 c^2} & K_r = \dfrac{I_{01}\rho l_1}{J_1} \\ E_i = \dfrac{J_1}{J_i} & \overline{M} = \dfrac{M l_1^2}{J_1 c^2 \theta_0} & \overline{M}_{sp} = \dfrac{M_{sp} l_1^2}{J_1 c^2 \theta_0} & \overline{l}_i = \dfrac{l_i}{l_1} \\ B_i = \dfrac{I_{0i}}{I_{01}} & & & \end{array} \tag{2.3}$$

the determination of angular displacements θ_i is reduced to solving N equations

$$\theta_{i,tt} - \theta_{i,xx} = 0 \qquad i = 1,2,\dots,N \tag{2.4}$$

with the zero initial conditions

$$\theta_i(x,0) = \theta_{i,t}(x,0) = 0 \qquad i = 1,2,\dots,N \tag{2.5}$$

and with the following nonlinear boundary conditions:
— for $x = 0$

$$M(t) - \theta_{1,tt} + K_r(D_1\theta_{1,xt} + \theta_{1,x}) - d_1\theta_{1,t} - M_{sp}(t) = 0 \tag{2.6}$$

— for $x = \sum_{k=1}^{i} l_k$, $i = 1,2,\dots,N-1$

$$\begin{aligned} &\theta_i(x,t) = \theta_{i+1}(x,t) \\ &-\theta_{i,tt} - K_rB_iE_{i+1}(D_i\theta_{i,xt} + \theta_{i,x}) + K_rB_{i+1}E_{i+1}(D_{i+1}\theta_{i+1,xt} + \theta_{i+1,x}) + \\ &\qquad -E_{i+1}d_{i+1}\theta_{i,t} = 0 \end{aligned} \tag{2.7}$$

— for $x = \sum_{k=1}^{N} l_k$

$$-\theta_{N,tt} - K_r B_N E_{N+1}(D_N \theta_{N,xt} + \theta_{N,x}) - E_{N+1} d_{N+1} \theta_{N,t} = 0 \tag{2.8}$$

where θ_0 is a fixed angular displacement, the bars denoting dimensionless quantities are omitted for convenience, and the comma denotes partial differentiation.

Solutions to equations of motion (2.4) are sought in the form

$$\theta_i(x,t) = f_i(t-x) + g_i\Big(t + x - 2\sum_{k=1}^{i-1} l_k\Big) \qquad i = 1,2,\dots,N \tag{2.9}$$

where the functions f_i and g_i represent waves caused by the external loading $M(t)$ propagating in the i-th shaft in the positive and negative directions of the x-axis, respectively. They are continuous and equal to zero for negative arguments.

Substituting solutions (2.9) into boundary conditions (2.6)-(2.8), the following set of equations for the unknown functions f_i and g_i is obtained

$$\begin{aligned}
&r_{N+1,1} g''_N(z) + r_{N+1,2} g'_N(z) = r_{N+1,3} f''_N(z - 2l_N) + r_{N+1,4} f'_N(z - 2l_N) \\
&g_i(z) = f_{i+1}(z - 2l_i) + g_{i+1}(z - 2l_i) - f_i(z - 2l_i) \qquad i = 1,2,\dots,N-1 \\
&r_{11} f''_1(z) = M(z) + r_{12} g''_1(z) + r_{13} f'_1(z) + r_{14} g'_1(z) - M_{sp}(z) \\
&r_{i1} f''_i(z) + r_{i2} f'_i(z) = r_{i3} g''_i(z) + r_{i4} g'_i(z) + r_{i5} f''_{i-1}(z) + r_{i6} f'_{i-1}(z) \\
&\qquad\qquad i = 2,3,\dots,N
\end{aligned} \tag{2.10}$$

where $(i = 2,3,\dots,N)$

$$\begin{aligned}
r_{11} &= K_r D_1 + 1 & r_{12} &= K_r D_1 - 1 \\
r_{13} &= -K_r - d_1 & r_{14} &= K_r - d_1 \\
r_{i1} &= K_r E_i (B_i D_i + B_{i-1} D_{i-1}) + 1 & r_{i2} &= E_i [K_r (B_i + B_{i-1}) + d_i] \\
r_{i3} &= K_r E_i (B_i D_i - B_{i-1} D_{i-1}) - 1 & r_{i4} &= E_i [K_r (B_i - B_{i-1}) - d_i] \\
r_{i5} &= 2K_r B_{i-1} E_i D_{i-1} & r_{i6} &= 2K_r B_{i-1} E_i \\
r_{N+1,1} &= K_r B_N E_{N+1} D_N + 1 & r_{N+1,2} &= E_{N+1}(K_r B_N + d_{N+1}) \\
r_{N+1,3} &= K_r B_N E_{N+1} D_N - 1 & r_{N+1,4} &= E_{N+1}(K_r B_N - d_{N+1})
\end{aligned} \tag{2.11}$$

Equations (2.10) are solved numerically by means of the Runge-Kutta method. These equations are similar to those in Pielorz (1995). Differences lay in the description of M_{sp}.

3. Numerical results

The main aim of numerical calculations is to investigate the influence of the local nonlinearity with the hard-type characteristic described by irrational functions (2.2) on the behaviour of the considered discrete-continuous systems. This is done on the basis of the amplitude-frequency curves for angular displacements of the two-mass and the three-mass torsional systems for selected parameters such as the exponent w, the coefficient k_w, the amplitude of the external moment and damping coefficients which are connected with the nonlinear effects.

The external moment $M(t)$ appearing in equations (2.10) can be arbitrary. Similarly to the investigations of nonlinear discrete systems, it is assumed in the form of the following harmonic function $M(t) = M_0 \sin(pt)$, where p is the dimensionless loading frequency, M_0 is the loading amplitude and the discussion is focussed on the solutions in steady states.

Numerical results given below are exemplary. Some comparable calculations given in Pielorz (1995) for the special case of a discrete-continuous model with the local nonlinearity shown in Fig. 1, confirm the correctness of the results obtained by means of the wave approach. They are done for a system consisting of a single shaft with the right-hand end being fixed, using the wave method and the method of separation of variables, neglecting the internal damping and assuming that the cubic nonlinearity is of the hard type.

During the determination of amplitude frequency curves for angular displacements, a jump phenomenon is observed, i.e., there exist values for the frequency of the external moment $M(t)$ where displacement amplitudes jump from higher to lower values.

Equations (2.10) are solved from $z = 0$ in two manners: with the zero initial conditions for each p and with nonzero initial conditions in such a way that the last values for solutions become new initial conditions for the next value of p. Equations (2.10) are equations with a retarded argument, so these new initial conditions ought to be known in appropriate intervals of the lengths $2l_i$ as it is required in (2.10). Amplitude jumps take place for smaller values of frequencies p in the case of zero initial conditions for each p. It should be pointed out that the jump phenomenon is typical for nonlinear discrete systems (cf. Hagedorn, 1981; Szemplińska-Stupnicka, 1990).

3.1. Two-mass system

The two-mass system is characterized by the following parameters

$$\begin{array}{lll} K_r = 0.05 & k_1 = 0.05 & N = 1 \\ l_1 = 1 & l_2 = 1 & E_2 = 0.8 \end{array} \tag{3.1}$$

The remaining parameters appearing in (2.10) can vary.

Diagrams in Fig. 2 and Fig. 3 show that using the wave approach one can determine displacement amplitudes in arbitrary cross-sections. They are plotted in five cross-sections $x = 0$, 0.25. 0.5, 0.75, 1.0 for $w = 3.25$, $k_w = 0.005$, $M_0 = 1.0$, $d_0 = d_i = D_i = 0.1$. The diagrams in Fig. 2 contain two resonant regions ($\omega_1 = 0.126$, $\omega_2 = 0.351$), while in Fig. 3 the third resonant region ($\omega_3 = 3.171$). From diagrams in Fig. 2 it follows that the jump phenomenon is observed in the second resonant region for all considered cross-sections, and the both jumps occur for the same values of the frequency p in these cross-sections. In the first resonant region, the maximal amplitudes decrease from the cross-section $x = 1.0$ to the cross-section $x = 0$, while in the second resonant region from $x = 0$ to $x = 1.0$. In the third region, the maximal amplitude occurs in $x = 0.5$, while the smallest one in $x = 1.0$ and no nonlinear effects are noticed there.

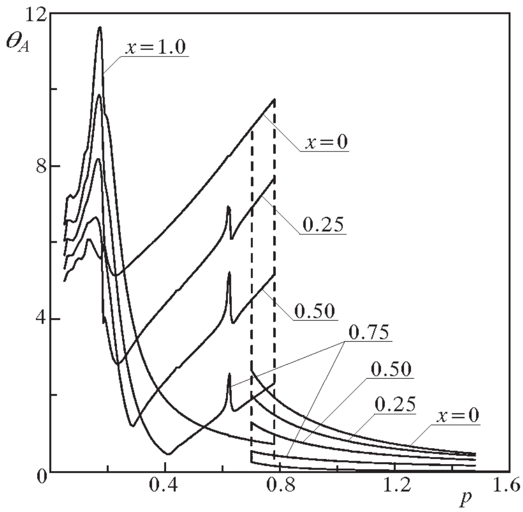

Fig. 2. Amplitude-frequency curves for angular displacements in $x = 0$, 0.25, 0.5, 0.75, 1.0 for the two-mass system with $w = 3.25$, $k_w = 0.005$ and $M_0 = 1$

Nonlinear effects are caused directly by the nonlinear moment $M_{sp}(t)$ described by functions (2.2). They are also connected with the amplitude M_0 of the external moment and with the external and internal damping. In the further numerical analysis, the effect of the exponent w, the parameter k_w standing by the nonlinear term in (2.2), the amplitude M_0 of the external moment and damping are investigated. All results concern the solutions in the cross-section $x = 0$, where the nonlinear discrete element is introduced and the first two resonant regions because for $p > 1.5$ no nonlinear effects are observed.

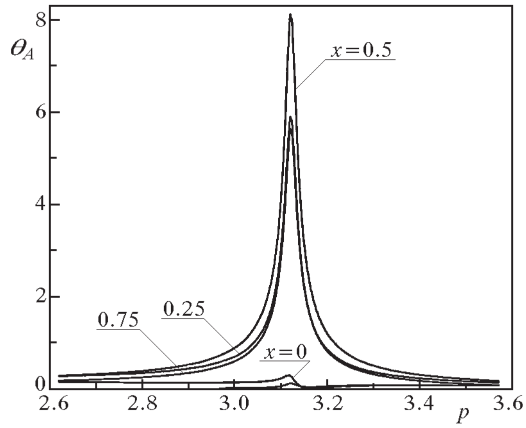

Fig. 3. Amplitude-frequency curves for angular displacements in the third resonant region for the two-mass system with $w = 3.25$, $k_w = 0.005$ and $M_0 = 1$

Amplitude-frequency curves for $w = 1.9$, 2.25, 2.5, 2.75, 3.0, 3.25 with $k_w = 0.005$, $M_0 = 1.0$, $d_0 = d_i = D_i = 0.1$ are plotted in Fig. 4. From these diagrams it follows that the maximal displacement amplitudes decrease with the increase of the exponent w and the distances between jumps increase with the increase of w. No amplitude jumps are observed for $w = 1.9$ and 2.25.

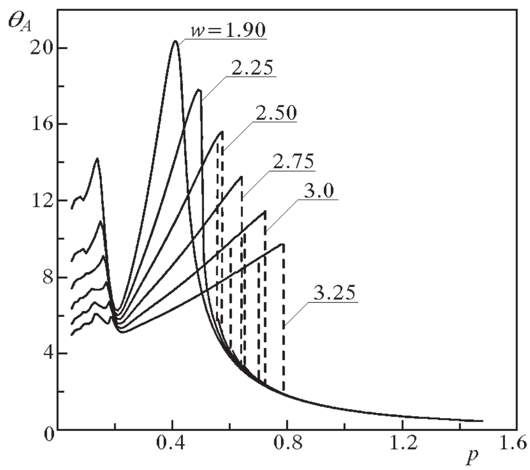

Fig. 4. The effect of w for the two-mass system in $x = 0$ with $k_w = 0.005$ and $M_0 = 1$

The effect of the coefficient k_w is shown in Fig. 5 for $w = 3.25$, $M_0 = 1.0$, $d_0 = d_i = D_i = 0.1$. It is assumed that $k_w = 0.0001$, 0.001, 0.003, 0.005, 0.01. One can see that the maximal amplitudes decrease with the increase of k_w

while distances between amplitude jumps increase with the increase of k_w. No jumps are observed when $k_w = 0.0001$.

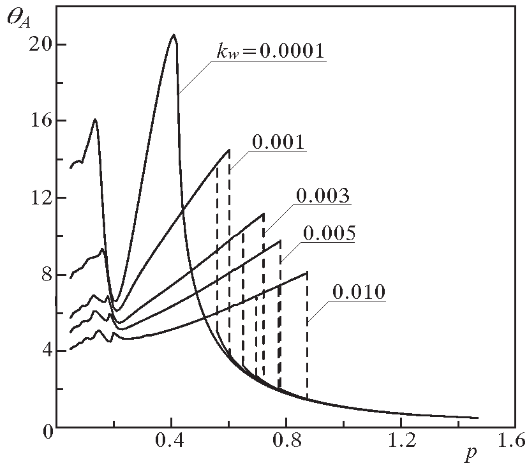

Fig. 5. The effect of k_w for the two-mass system in $x = 0$ with $w = 3.25$ and $M_0 = 1$

Amplitude-frequency curves for angular displacements plotted in Fig. 6 for $M_0 = 0.1$, 0.25, 0.5, 1.0 with $w = 3.25$, $k_w = 0.005$ and $d_0 = d_i = D_i = 0.1$ inform on the effect of the amplitude of the external moment. From these curves it follows that the maximal amplitudes and jump distances increase with the increase of M_0. No jumps are noticed for $M_0 = 0.1$.

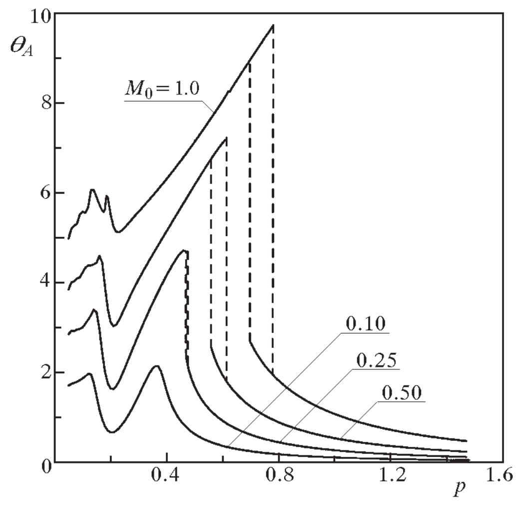

Fig. 6. The effect of M_0 for the two-mass system with $w = 3.25$ and $k_w = 0.005$

In Fig. 7, the effect of damping coefficients is presented for $d_0 = d_i = D_i =$ 0.05, 0.1, 0.2, 0.3 with $w = 3.25$, $k_w = 0.005$, $M_0 = 1.0$. It is seen that the maximal amplitudes in the second resonant region and the distances between amplitude jumps decrease with the increase of d_0.

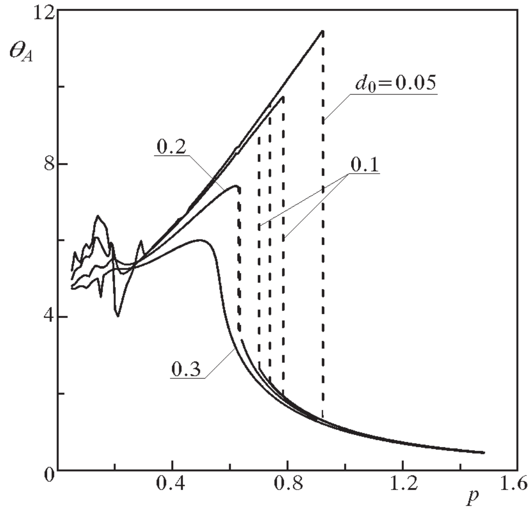

Fig. 7. The effect of damping for the two-mass system

3.2. Three-mass system

Similar numerical calculations as for the two-mass system are performed for the three mass system using equations (2.10) with the following basic parameters

$$\begin{array}{lll} K_r = 0.05 & k_1 = 0.05 & N = 2 \\ l_1 = l_2 = 1 & E_2 = E_3 = 0.8 & B_2 = 1 \end{array} \tag{3.2}$$

In Fig. 8 and Fig. 9, the amplitude-frequency curves for angular displacements in the cross-sections $x = 0, 0.5, 1.0, 1.5, 2.0$ of the three-mass system are plotted with $w = 3.25$, $k_w = 0.005$, $M_0 = 1.0$ and $d_0 = d_i = D_i = 0.1$. Diagrams in Fig. 8 contain three resonant regions ($\omega_1 = 0.089$, $\omega_2 = 0.261$, $\omega_3 = 0.376$), while the diagrams in Fig. 9 the fourth resonant region ($\omega_4 = 3.156$). From Fig. 8 it follows that in the first resonant region the maximal amplitudes decrease with the decrease of x, in the third resonant region they decrease with the increase of x, while in the second resonant region no regularity of this type occurs. The jump phenomenon is observed in the third resonant region with the same distances between jumps taking place at the same frequencies p of the external moment for all considered cross-sections. From Fig. 9 it follows that no nonlinear effects are observed in the fourth resonant region, and that the highest amplitude occurs in the cross-section $x = 0.5$, while the smallest one in $x = 2.0$.

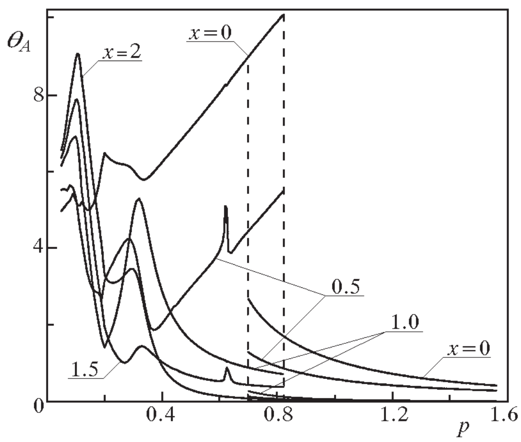

Fig. 8. Amplitude-frequency curves for angular displacements in $x = 0$, 0.5, 1.0, 1.5, 2.0 for the three-mass system with $w = 3.25$, $k_w = 0.005$ and $M_0 = 1$

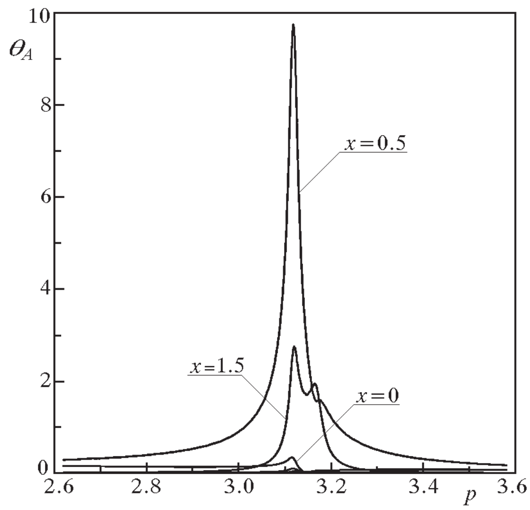

Fig. 9. Amplitude-frequency curves for angular displacements in the fourth resonant region for the three-mass system with $w = 3.25$, $k_w = 0.005$ and $M_0 = 1$

Further numerical results concern the cross-section $x = 0$ where the local nonlinearity is introduced and three resonant regions because for $p > 1.5$ no nonlinear effects are seen.

The effect of the exponent w is shown in Fig. 10 where amplitude-frequency curves are plotted for $w = 1.9$, 2.25, 2.5, 2.75, 3.0, 3.25 with $k_w = 0.005$, $M_0 = 1.0$, $d_0 = d_i = D_i = 0.1$. From diagrams in Fig. 10 it follows that the maximal amplitudes decrease with the increase of w and the jump distances decrease with the decrease of w. No jumps are noticed for $w = 1.9$ and 2.25.

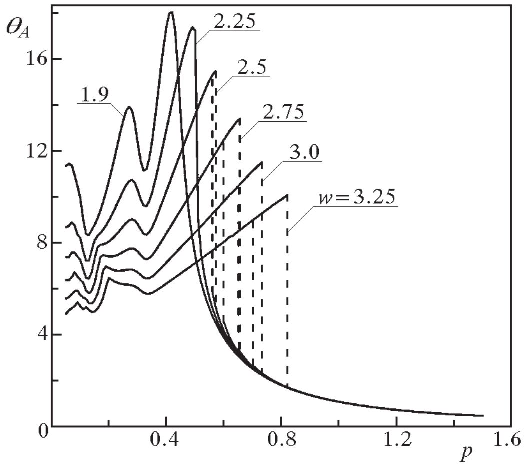

Fig. 10. The effect of w for the three-mass system in $x = 0$ with $k_w = 0.005$ and $M_0 = 1$

In Fig. 11, amplitude-frequency curves are presented for $k_w = 0.0001$, 0.001, 0.003, 0.005, 0.01 with $w = 3.25$, $M_0 = 1.0$ and $d_0 = d_i = D_i = 0.1$. They indicate that the maximal displacement amplitudes decrease with the increase of the coefficient k_w standing by the nonlinear term in irrational functions (2.2) and that the distances of jumps increase with the increase of k_w. No jumps occur for $k_w = 0.0001$.

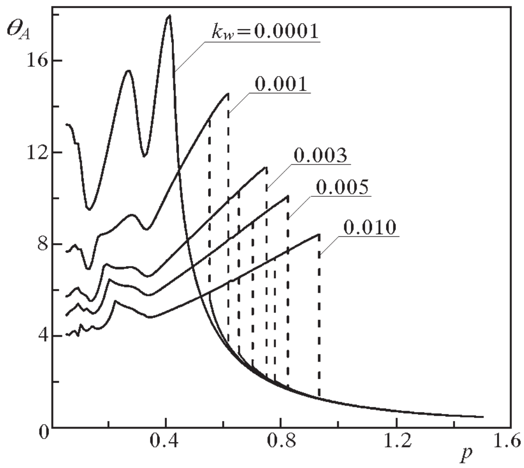

Fig. 11. The effect of k_w for the three-mass system in $x = 0$ with $w = 3.25$ and $M_0 = 1$

Diagrams in Fig. 12 plotted for $M_0 = 0.1$, 0.25, 0.5, 1.0 and for $w = 3.25$, $k_w = 0.005$, $d_0 = d_i = D_i = 0.1$ show the effect of the amplitude of the external moment on amplitudes of angular displacements. From them it follows that the maximal amplitudes and jump distances decrease with the decrease of M_0. For $M_0 = 0.1$ and 0.25 no jumps occur.

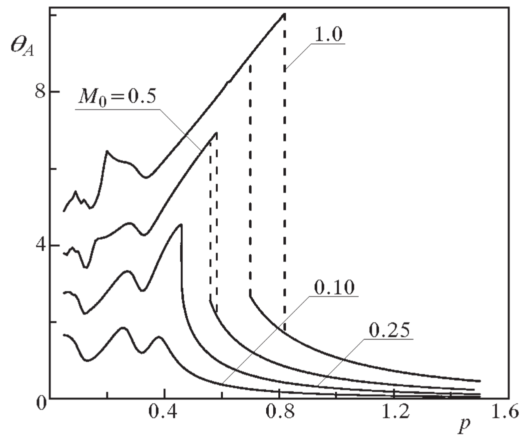

Fig. 12. The effect of M_0 for the three-mass system in $x = 0$ with $w = 3.25$ and $k_w = 0.005$ and $M_0 = 1$

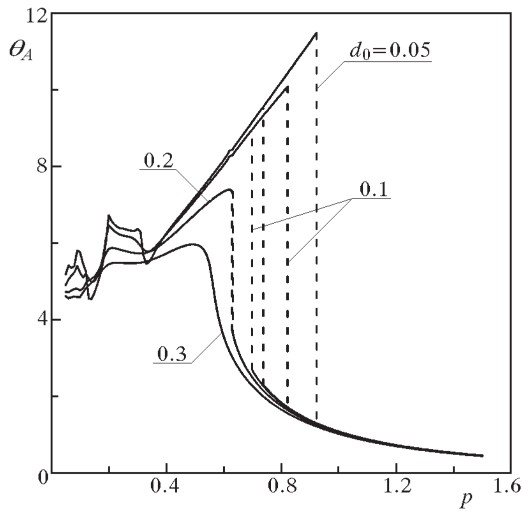

Fig. 13. The effect of damping for the three-mass system in $x = 0$ with $w = 3.25$, $k_w = 0.005$ and $M_0 = 1$

The effect of damping is shown in Fig. 13 for $d_0 = d_i = D_i = 0.05$, 0.1, 0.2, 0.3 with $w = 3.25$, $k_w = 0.005$, $M_0 = 1.0$. From Fig. 13 it follows that the maximal amplitudes and the distances of jumps decrease with the increase of damping. For $d_0 = 0.3$ no jumps are noticed.

4. Final remarks

From the above discussion, it follows that irrational functions can be included for the description of local nonlinearities with the characteristic of a hard type in multi-mass discrete-continuous torsional systems investigated by means of the method using wave solutions to equations of motion. The irrational functions describe weak and strong nonlinearities together with a linear case. The use of them give more possibilities for the description of appropriate experimental data.

Numerical calculations show the effect of parameters occurring in the irrational functions on the behaviour of the two- and three-mass systems. One can notice that nonlinear effects in the form of the jump phenomenon are observed in the second resonant region for the two-mass system, and in the third resonant region for the three-mass system.

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Drgania skrętne układów dyskretno-ciągłych z lokalną nieliniowością o charakterystyce typu twardego

Streszczenie

Praca dotyczy nieliniowych drgań wielomasowych układów odkształcanych skrętnie złożonych z wałów połączonych bryłami sztywnych. W układach tych uwzględniono lokalną nieliniowość o charakterystyce typu twardego. Nieliniowość ta została opisana za pomocą funkcji niewymiernych. W rozważaniach zastosowano podejście wykorzystujące rozwiązanie falowe równań ruchu. Obliczenia numeryczne wykonano dla układu dwumasowego i układu trzymasowego. Skoncentrowano się w nich na efektach nieliniowych w rozpatrywanych układach.

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