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NUMERICAL VERIFICATION OF TWO MATHEMATICAL MODELS FOR THE HEAT TRANSFER IN A LAMINATED RIGID CONDUCTOR

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Heat transfer problems in composites with a dense periodic structure are usually investigated in the framework of certain averaged (macroscopic) mathematical models. The best known are asymptotic models of periodic heat conductors. The heuristic tolerance models take into account the effect of period lengths on the overall behaviour of a conductor. The purpose of this contribution is to compare the aforementioned models and to verify the obtained solutions to certain benchmark problems.

Key words: heat transfer, laminated conductors

1. Introduction

It is known that the direct approach to the analysis of heat conduction processes in composites with a dense periodic structure (i.e. the approach satisfying Fourier heat conduction equation in every component) leads to ill-conditioned and complicated computational problems, cf. Bensoussan *et al.* (1978). That is why some averaged (macroscopic) mathematical models for obtaining solutions to special problems have been formulated. This situation is typical for both the stationary and nonstationary processes in periodic solids and structures.

First of all we have to recall models based on the asymptotic homogenization method. In this case the field equations with highly oscillating and noncontinuous periodic coefficients, describing material structure of a periodic medium, are "approximated" by certain constant (effective) coefficients. At the same time solutions to the pertinent boundary-value problems make it possible to describe processes occuring in the periodically nonhomogeneous medium under consideration. The number of references related to the homogenization theory is very extensive; we shall mention here only basic monographs by Bensoussan *et al.* (1987), Sanchez-Palencia (1980), Bakhvalov and Panasenko (1984), Jikov *et al.* (1994). From the point of view of engineering applications the main problem of homogenization is to determine the effective moduli in the model equations, e.g. the components of the effective heat conduction tensor. A certain drawback of homogenized models is that they do not describe the effect of the period lenghts on the overall behaviour of a periodically inhomogeneous solid. This drawback has been rejected in the framework of what was called higher-order homogenization, cf. Boutin and Ariault (1993), but even for relatively simple problems this approach requires rather lengthy calculations, Fish and Wen-Chen (2001). Following Bensoussan *et al.* (1978), p. XI, we recall that a modelling question related to the given physical problem is not a mathematical one and asymptotic techniques with different scalings have to be applied to different problems.

The aforementioned drawbacks of the asymptotic homogenization technique constitute a motivation for formulations of alternative approaches to the modelling of periodic materials and structures. The overview of these approaches can be found in Woźniak and Wierzbicki (2000). Generally speaking, these approaches are based on certain *a priori* hypotheses related to the expected form of basic unknowns such as a temperature and/or a displacement field. Hypotheses of this kind are well known in formulation of different engineering theories of rods, plates and shells as certain "thin" bodies as well as in the derivation of various approximate models for the dynamic behaviour of periodically laminated solids, cf. Achenbach *et al.* (1968) and the list of references in the monograph by Woźniak and Wierzbicki (2000). In this monograph foundations of what was called the tolerance averaging technique have been summarized. This modelling technique is based on certain heuristic hypotheses which can be reasonably applied to a rather large class of engineering problems. Two main features of the tolerance averaging technique of PDEs with functional highly oscillating periodic coefficient have to be emphasized. First, the derived model equations describe an effect of the periodicity cell size on the overall behaviour of a solid under consideration (a length scale effect). Second, by neglecting in the model equations terms describing the above length scale effect, we pass to the homogenized model equations.

The tolerance averaging technique constitutes a certain generalization of what was called the microlocal parameter method, Woźniak (1987), which was applied to the modelling of micro-periodic materials in stationary problems. We have to recall a series of papers related to the elastic stratified media with cracks by Kaczyński and Matysiak (1988, 1989a,b, 1994, 2003), Kulczycki-Zyhajło and Matysiak (2004a,b), Matysiak (1989, 1991, 1992), Matysiak and Mieszkowski (1999, 2001), Matysiak and Pauk (1995), Matysiak *et al.* (1998), Matysiak and Ukhanska (1997a,b). Nonstationary processes were investigated in the framework of the tolerance averaging technique. We can mention dynamic and stability problems of "thin" periodic bodies like rods, plates and shells by Baron and Woźniak (1995), Baron (2003), Jędrysiak (1999, 2000, 2003), Michalak (1998, 2000), Michalak *et al.* (1995), Mazur-Śniady *et al.* (2004), Cielecka *et al.* (2000), and others. Problems related to media with saturated, periodically distributed inclusions were studied by Dell'Isola *et al.* (1997, 1998). Analysis of heat conduction and stress waves problems can be found in Ignaczak (1998), Ignaczak and Baczyński (1997), Wierzbicki *et al.* (1996, 2001a,b, 2002a,b). Dynamics of micro-damaged media was studied by Woźniak C. and Woźniak M. (1994) and Woźniak M. (1995). For the analysis of harmonic waves in a stratified medium, cf. Ignaczak (2003, 2004). Investigations of an elastic laminated solid were carried out by Wągrowska and Woźniak (1996), and Wągrowska (1998). A dynamic behaviour of the honeycomb elastic media was discussed in Wierzbicki and Woźniak (2000a,b). Effects of prestresing and stability on the plate behaviour were analysed by Wierzbicki and Woźniak (2002), Woźniak M. *et al.* (2004). A certain generalization of the tolerance averaging technique related to composites having different periods of inhomogeneity in a certain direction can be found in Woźniak (2002). The above review of possible applications of the tolerance averaging technique is far from being complete; for more complete review the reader is referred to Woźniak and Wierzbicki (2000).

The scope of this contribution is restricted to the nonstationary heat conduction problems in periodically laminated rigid conductors. For these problems it can be shown that the tolerance averaging technique, after formal neglecting the terms involving the length of the inhomogeneity period, leads to the homogenized model equations with the exact values of the effective moduli.

The aim of this contribution is twofold.

First, we are to answer two questions:

1. What is difference between solutions to some selected initial-boundary value problems derived respectively from the tolerance averaged model and homogenized model?

2. What is difference between the aforementioned model solutions and the pertinent solution obtained from the Fourier heat conduction equation with noncontinuous and highly oscillating periodic coefficients?

The answer to the above questions is not possible. That is why it will be restricted to some benchmark problems.

The second aim of this contribution is to emphasize the fact that the effect of the inhomogeneity period length on the macroscopic behaviour of a laminated conductor plays an important role in the analysis of many physical problems. The choice of proposed benchmark problems was motivated by this fact.

In order to make this paper self-consistent, in the first section we recall basic concepts and assumptions of the tolerance averaging technique for the heat transfer in a periodically laminated rigid conductor. It is assumed that there exists a perfect bonding between adjacent laminae which have material symmetry planes parallel to the interfaces. The problems are investigated in the framework of the linearized heat conduction theory based on the known parabolic heat transfer equation.

Notations.

Subscripts $i, j, k, \ldots$ and $\alpha, \beta, \gamma, \ldots$ run over sequences $1,2,3$ and $1,2$, respectively. Partial derivates with respect to arguments x_k are denoted by $\partial_k = \partial/\partial x_k$ and a time derivate by the overdot. We also denote $\mathbf{x} = (x_1, x_2, x_3)$.

2. Preliminary concepts

Let the physical space be parametrized by the orthogonal Cartesian coordinate system $Ox_1x_2x_3$ with the x_3-axis perpendicular to the laminae interfaces; and let t be the time coordinate. A laminated rigid conductor is assumed to occupy the region $\Pi \times (0, L)$ in the physical space where Π is a regular region on the Ox_1x_2-plane. The fragment of this conductor is shown in Fig. 1, where l is a period of inhomogeneity, $l \ll L$, and l', l'' are laminae thicknesses. A specific heat and heat conduction tensor components in pertinent laminae are denoted by c', $K'_{ij} = K'_{ji}$ and c'', $K''_{ij} = K''_{ji}$ where $K'_{\alpha 3} = K''_{\alpha 3} = 0$ due to the assumed material symmetry of the conductor. Hence $c(\cdot)$ and $K_{ij}(\cdot)$ are l-periodic piecewise constant functions of argument x_3. For an arbitrary integrable function $f(\cdot)$ defined in $\langle 0, L\rangle$ we define its averaged value by means of formula

$$\langle f\rangle(x_3) = \frac{1}{l}\int\limits_{-l/2}^{l/2} f(x_3 + z)\,dz \qquad x_3 \in \left\langle \frac{l}{2}, L - \frac{l}{2}\right\rangle \tag{2.1}$$

Obviously, if f is l-periodic function then $\langle f\rangle = \text{const}$.

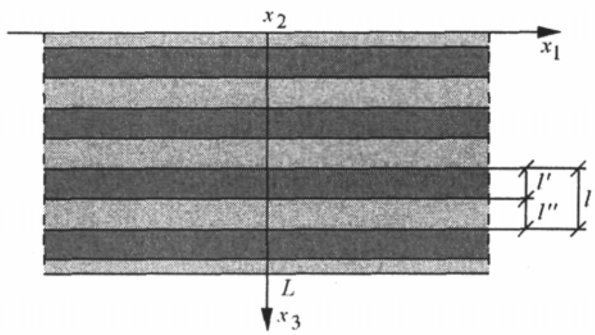

Fig. 1. Fragment of a laminated rigid conductor

By $g(\cdot)$ we denote l-periodic continuous function of argument x_3 such that $g(nl + l''/2) = \sqrt{3}l$, $g(nl + l''/2 + l') = -\sqrt{3}l$, and $g(\cdot)$ is linear in every interval $\langle nl + l''/2, nl + l''/2 + l' \rangle$, and $\langle nl + l''/2 + l', (n+1)l + l''/2 \rangle$, where $n = 0, \pm 1, \pm 2, \ldots$. The diagram of this function is shown in Fig. 2.

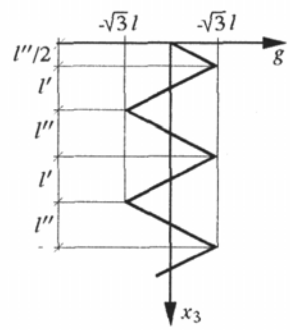

Fig. 2. l-periodic shape function $g(x_3)$

Let $F(\cdot)$ be a differentiable function depending on a certain parameter ε and defined in $\langle 0, L \rangle$. Moreover, let l, $l \ll L$, be a length parameter. Function $F(\cdot)$ will be called slowly varying (with respect to length l and a tolerance parameter ε, $0 < \varepsilon \ll 1$) provided that functions $lF'(\cdot)$ and $\varepsilon F(\cdot)$ are of the same order as related to ε. For a detailed discussion of this concept the reader is referred to Woźniak and Wierzbicki (2000). It follows that for every slowly varying function $F(\cdot)$ the following relation holds

$$(Fg)' = Fg' + O(\varepsilon) \tag{2.2}$$

Moreover, for an arbitrary integrable l-periodic function h of x_3 we also obtain

$$\langle hF \rangle(x) = \langle h \rangle F(x) + O(\varepsilon) \tag{2.3}$$

Rougly speaking, increments of an arbitrary slowly varying function F in every interval $(-l/2+x_3, l/2+x_3)$ can be neglected when compared to the maximum value of $|F(x_3)|$, $x_3 \in \langle 0, L\rangle$. Obviously, functions f and F in (2.1)-(2.3) can also depend on arguments x_1, x_2 and t.

3. Model equations

Let $\theta = \theta(\boldsymbol{x}, t)$, $\boldsymbol{x} \in \overline{\Pi} \times \langle 0, L\rangle$ be a temperature field at time t. To be more exact, θ is interpreted as the temperature increment measured from a certain constant absolute temperature. Function $\theta(\cdot)$ is assumed to be continuous in $\overline{\Pi} \times \langle 0, L\rangle$ together with its derivative $\partial_\alpha \theta(\cdot, t)$ while derivative $\partial_3 \theta(\cdot, t)$ suffers jump discontinuities on the interfaces between adjacent laminae. In every lamina temperature $\theta(\cdot)$ has to satisfy the well-known linearized Fourier heat transfer equation

$$c(x_3)\dot{\theta}(\boldsymbol{x}, t) - \partial_j(K_{ij}(x_3)\partial_i \theta(\boldsymbol{x}, t)) = 0 \tag{3.1}$$

where $c(\cdot)$, $K_{ij}(\cdot)$ are the known l-periodic functions introduced in Section 2. On all interfaces between laminae we deal with the heat flux continuity conditions

$$K_{i3}^{+}\partial_3^{+}\theta(\boldsymbol{x}, t) = K_{i3}^{-}\partial_3^{-}\theta(\boldsymbol{x}, t) \tag{3.2}$$

where ∂_3^{+}, ∂_3^{-} stand for the right-hand side and left-hand side derivatives, respectively, and K_{i3}^{+}, K_{i3}^{-} are the values of K_{i3} in the pertinent adjacent laminae. Equations (3.1), (3.2) have to be satisfied together with the appropriate initial and boundary conditions.

Bearing in mind the remarks mentioned in Introduction, we shall replace equations (3.1), (3.2) by certain model equations which have constant coefficients. Following the approach applied in Woźniak and Wierzbicki (2000), we restrict the class of temperature fields to that in which $\theta(\cdot, t)$ is described by the formula

$$\theta(\boldsymbol{x}, t) = \vartheta(\boldsymbol{x}, t) + g(x_3)\Psi(\boldsymbol{x}, t) \tag{3.3}$$

with $\boldsymbol{x} \in \overline{\Pi} \times \langle 0, L\rangle$ and $t \geqslant 0$, where $\vartheta(\cdot)$ and $\Psi(\cdot)$ are differentiable functions which are slowly varying in argument $x_3 \in \langle 0, L\rangle$. Let us observe that

$$\vartheta(\boldsymbol{x}, t) = \langle \theta \rangle(\boldsymbol{x}, t)$$
$$g(x_3)\Psi(\boldsymbol{x}, t) = \theta(\boldsymbol{x}, t) - \langle \theta \rangle(\boldsymbol{x}, t)$$

Hence ϑ is the averaged temperature, and $g\Psi$ represents fluctuations of the temperature field. This is why Ψ will be referred to as the temperature fluctuation amplitude. Functions $\vartheta(\cdot)$, $\Psi(\cdot)$ are new unknowns interrelated with temperature $\theta(\cdot)$ by formula (3.3).

In order to derive the governing equations for $\vartheta(\cdot)$ and $\Psi(\cdot)$ let us substitute the right-hand side of (3.3) into the left-hand side of (3.1) and denote by H the obtained expression. Let us apply the orthogonalization procedure by assuming that

$$\langle H\rangle(\boldsymbol{x},t)=0 \qquad \langle gH\rangle(\boldsymbol{x},t)=0$$

for every $\boldsymbol{x}\in\Pi\times(l/2,L-l/2)$ and every $t\geqslant 0$. Bearing in mind (2.2), (2.3) and introducing the tolerance averaging approximation by neglecting terms $O(\varepsilon)$, cf. Woźniak and Wierzbicki (2000), after simple manipulations we obtain finally the following system of equations

$$\begin{aligned}
&\langle c\rangle\dot{\vartheta}-\langle K_{\alpha\beta}\rangle\partial_\alpha\partial_\beta\vartheta-\langle K_{33}\rangle\partial_3\partial_3\vartheta-[K_{33}]\partial_3\Psi=0\\
&l^2\langle c\rangle\dot{\Psi}-l^2\langle K_{\alpha\beta}\rangle\partial_\alpha\partial_\beta\Psi+\{K_{33}\}\Psi+[K_{33}]\partial_3\vartheta=0
\end{aligned} \tag{3.4}$$

with constant coefficients defined by formula (2.1)

$$\langle f\rangle=\nu'f'+\nu''f'' \qquad \nu'=\frac{l'}{l} \qquad \nu''=\frac{l''}{l}$$

and by new definitions

$$\begin{aligned}
[f]&=2\sqrt{3}(f''-f')\\
\{f\}&=12\Big(\frac{f'}{\nu'}+\frac{f''}{\nu''}\Big)
\end{aligned}$$

where f stands for K_{ij}. We shall also assume that equations (3.4) have to be satisfied in $\Pi\times(0,L)$ for every $t\geqslant 0$. Boundary conditions for ϑ are formulated on $(\partial\Pi\times(0,L))\cup(\Pi\times\{0\})\cup(\Pi\times\{L\})$, where $\partial\Pi$ is the boundary of the plane region Π. Boundary conditions for Ψ have to be prescribed only on $\partial\Pi\times(0,L)$ due to the form of equation $(3.4)_2$. In nonstationary problems the initial conditions for ϑ and Ψ should be also known. Equations (3.4) together with the aforementioned boundary and initial conditions represent what is called *a tolerance model* of a periodically laminated rigid conductor. It can be shown that the heat flux continuity conditions (3.2) on the interfaces between laminae, which can be written in the form

$$\{K_{33}\}\Psi+[K_{33}]\partial_3\vartheta=0 \tag{3.5}$$

by means of $(3.4)_2$ are satisfied with an approximation of order $O(l^2)$.

The homogenized model can be treated as a special case of the tolerance model by neglecting terms $O(l^2)$ in (3.4). In this case the second of equations (3.4) reduces to the form (3.5). Hence Ψ can be eliminated from $(3.4)_1$ and after denotation

$$K^o = \frac{K'_{33}K''_{33}}{\nu' K''_{33} + \nu'' K'_{33}} \tag{3.6}$$

we obtain

$$\langle c\rangle\dot{\vartheta} - \langle K_{\alpha\beta}\rangle\partial_\alpha\partial_\beta\vartheta - K^o\partial_3\partial_3\vartheta = 0 \tag{3.7}$$

The above equation together with (3.5) and with the boundary and initial conditions for ϑ represent the homogenized model for the heat transfer in a periodically laminated rigid conductor.

Both the tolerance and homogenized models have to be considered together with formula (3.3) for temperature θ. Let us observe that tolerance models describe the effect of period length on the heat conduction in contrast to the homogenized models. At the same time, homogenized models satisfy the heat flux continuity conditions in contrast to tolerance models which satisfy these conditions only with approximation of order $O(l^2)$. It has to be remembered that solutions to initial-boundary problems formulated in the framework of the above models have a physical sense only if ϑ and Ψ are slowly varying functions of argument $x_3 \in \langle 0, L\rangle$. For more detailed discussion of the aforementioned model equations the reader is referred to Woźniak and Wierzbicki (2000).

4. Comparison of models

In order to compare the solutions related to both the tolerance and homogenized model equations we restrict considerations to problems depending only on time and on x_3-coordinate. Setting $z = x_3$ we look for functions $\vartheta(z,t)$, $\Psi(z,t)$, $z \in \langle 0, L\rangle$, $t \geqslant 0$, as the basic unknowns. Let us also define $k = K_{33}$. Under these restrictions formula (3.3) reduces to the form

$$\theta(z,t) = \vartheta(z,t) + g(z)\Psi(z,t) \tag{4.1}$$

The related tolerance model is governed by equations

$$\begin{aligned} &\langle c\rangle\dot{\vartheta} - \langle k\rangle\vartheta'' - [k]\Psi' = 0 \\ &l^2\langle c\rangle\dot{\Psi} + \{k\}\Psi + [k]\vartheta' = 0 \end{aligned} \tag{4.2}$$

and the homogenized model is described by equations

$$\langle c\rangle\dot{\vartheta} - k^o\vartheta'' = 0 \qquad \Psi = -\frac{[k]}{\{k\}}\vartheta' \tag{4.3}$$

where

$$k^o = \frac{k'k''}{\nu'k'' + \nu''k'}$$

is the effective heat conduction modulus. Subsequently we shall also assume that $l' = l''$ and hence $\nu' = \nu'' = 0.5$.

Let us begin with the analysis of the tolerance model. To this end equations (4.2) will be considered together with conditions

$$\begin{aligned} &\vartheta(z,0) = A\sin\frac{\pi z}{L} \qquad && \Psi(z,0) = B\cos\frac{\pi z}{L} \qquad && z \in \langle 0, L\rangle \\ &\vartheta(0,t) = \vartheta(L,t) = 0 && t \geqslant 0 \end{aligned}$$

Define

$$\lambda = \Big(\frac{l}{L}\Big)^2 \qquad \tau = \frac{t}{T}$$

where T is a certain time parameter. We look for solutions to (4.2) in the form

$$\begin{aligned} \vartheta(z,t) &= \vartheta_o u(\tau)\sin\frac{\pi z}{L} \\ \Psi(z,t) &= \frac{\vartheta_o}{L}v(\tau)\cos\frac{\pi z}{L} \end{aligned} \tag{4.4}$$

where ϑ_o is the constant reference temperature and $u(\cdot)$, $v(\cdot)$ are dimensionless functions. After denotations

$$\alpha = \frac{\langle k\rangle}{\langle c\rangle}\Big(\frac{\pi}{L}\Big)^2 T \qquad \beta = \frac{[k]}{\langle c\rangle}\frac{\pi T}{L^2} \qquad \gamma = \frac{\{k\}}{\langle c\rangle}\frac{T}{L^2}$$

we obtain from (4.2) the following system of the tolerance model equations for the dimensionless time-dependent functions $u(\tau)$, $v(\tau)$

$$\begin{aligned} &\dot{u}(\tau) + \alpha u(\tau) + \beta v(\tau) = 0 \\ &\lambda\dot{v}(\tau) + \gamma v(\tau) + \beta u(\tau) = 0 \end{aligned} \tag{4.5}$$

Let the initial conditions will be assumed either in the form

$$u(0) = 1 \qquad v(0) = 0 \tag{4.6}$$

in which temperature fluctuations are equal to zero, or will be given by

$$u(0) = 0 \qquad v(0) = 1 \tag{4.7}$$

where the averaged temperature is equal to zero.

Computations have been carried out for different values of λ and those of the inhomogeneity coefficient $\varkappa$ defined by

$$\varkappa = \frac{k'}{k''} - 1$$

and under assumption

$$\frac{k''}{c' + c''} = \text{const}$$

The results of computations are shown in Fig. 3 and Fig. 4 for conditions (4.6), and in Fig. 5 and Fig. 6 for conditions (4.7).

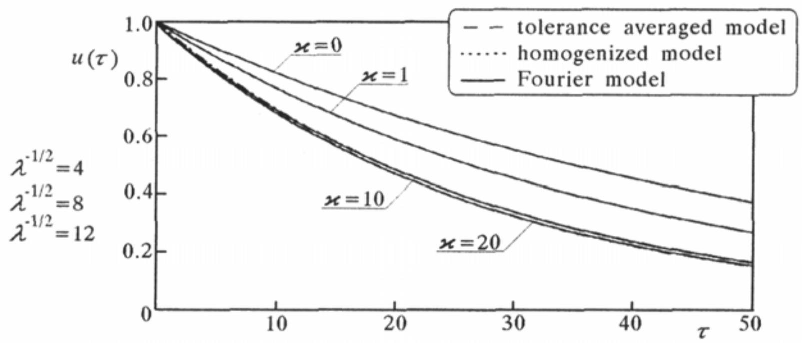

Fig. 3. The dimensionless function $u(\tau)$ describing the averaged part for initial conditions defined by (4.6)

Passage to the homogenized model equations (4.3) can be realized by setting $\lambda = 0$ in (4.5). Denoting

$$\alpha^o = \alpha - \frac{\beta^2}{\gamma}$$

we obtain the homogenized model equations

$$\dot{u}(\tau) + \alpha^o u(\tau) = 0 \qquad v(\tau) = -\frac{\beta}{\gamma} u(\tau) \tag{4.8}$$

with the initial condition

$$u(0) = 1 \tag{4.9}$$

The results of computations are shown in Fig. 3 and Fig. 4.

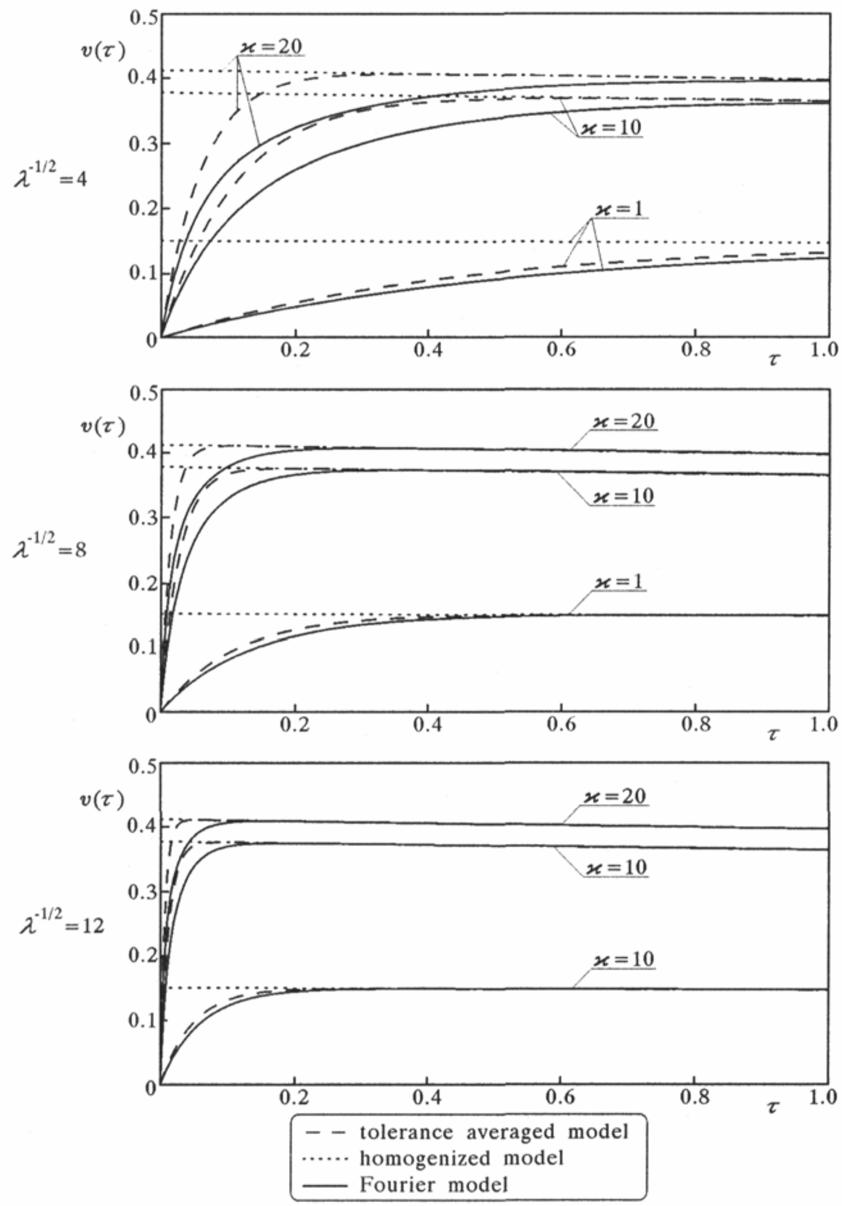

Fig. 4. The dimensionless function $v(\tau)$ describing the fluctuation part for initial conditions defined by (4.6)

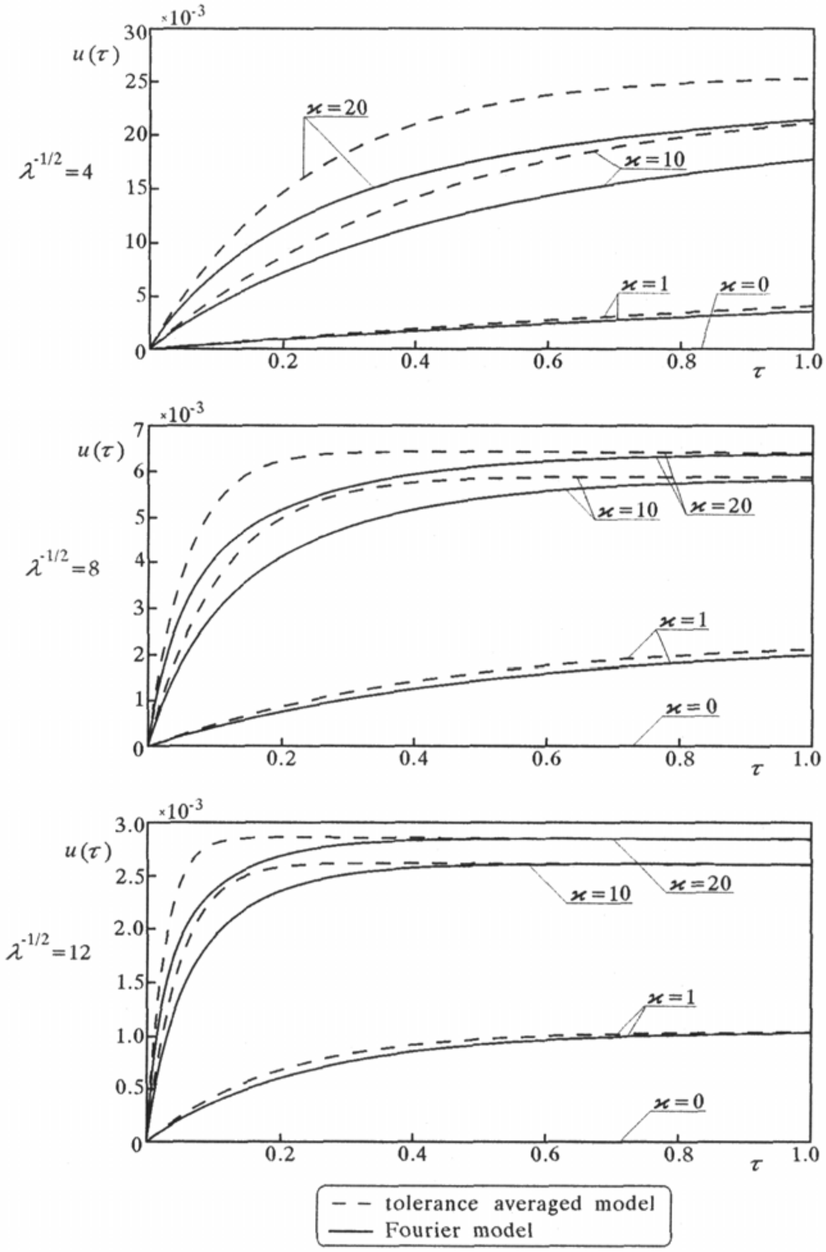

Fig. 5. The dimensionless function $u(\tau)$ describing the averaged part for initial conditions defined by (4.7)

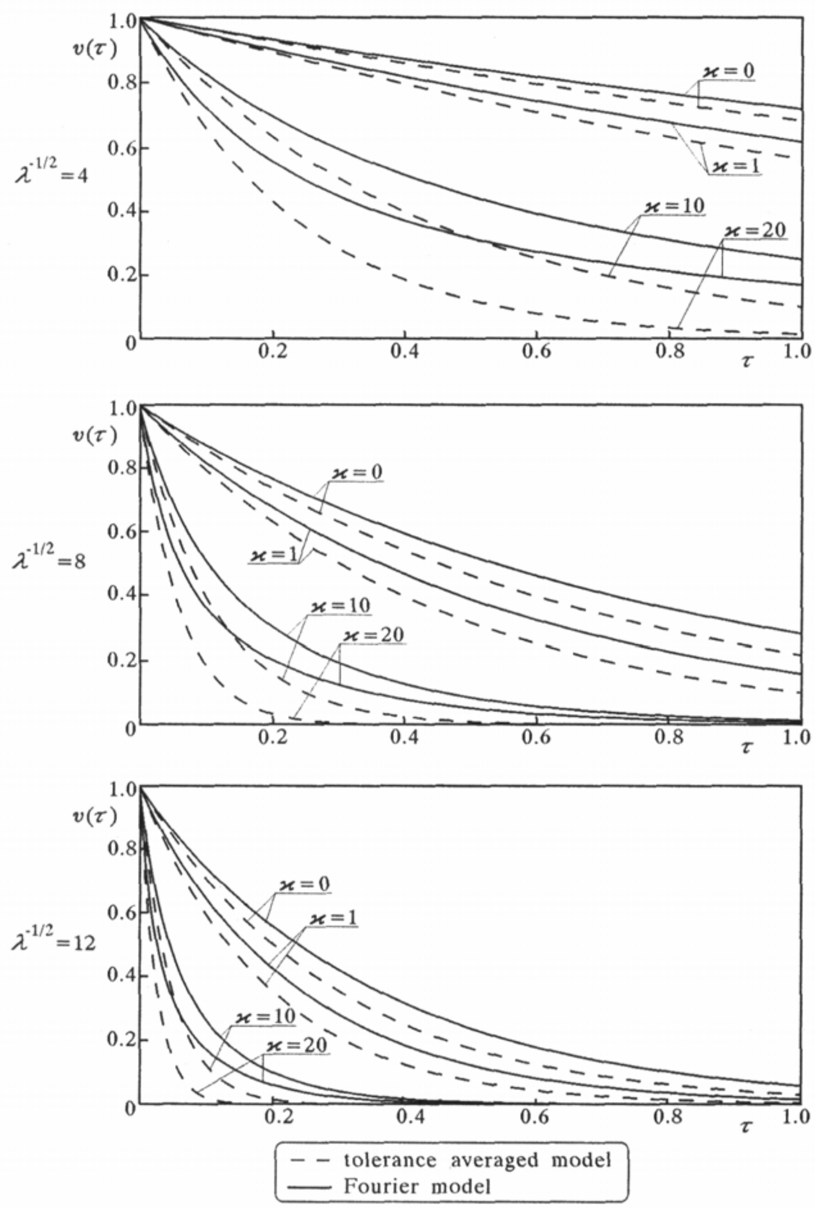

Fig. 6. The dimensionless function $v(\tau)$ describing the averaged part for initial conditions defined by (4.7)

5. Verification of the obtained results

In order to verify the results derived in Section 4 we have to solve initial-boundary value problems similar to these investigated above but in the framework of the exact Fourier model. To this end we shall use equation (3.1) which for temperature field $\theta(z,t)$, $z \in \langle 0, L\rangle$, $t \geqslant 0$ takes the form

$$c(z)\dot{\theta}(z,t) - k(z)\theta''(z,t) = 0$$

which has to hold in every lamina. On the interfaces between the adjacent laminae from (3.2) we obtain the heat flux continuity conditions

$$k'_{+}\theta'_{+}(z,t) = k''_{-}\theta'_{-}(z,t)$$

The above problem will be solved numerically using a finite difference method, Crank (1975). To this end we shall approximate temperature field θ_{FDM} by (4.1), bearing in mind (4.4). The aforementioned approximation will be realized using the least square method. According to this method we shall look for the values of dimensionless functions $u(\tau)$, $v(\tau)$ by minimizing the least square error defined by

$$\int_0^L (\theta_{FDM} - \theta)^2 \, dz$$

where θ is a temperature field in the form (4.1). The results of computations are shown in Fig. 3 - Fig. 6.

6. Conclusions

In this contribution, two mathematical models of a rigid periodically laminated heat conductors have been verified by direct numerical solutions to certain benchmark problems. It was shown that both, the homogenized and tolerance averaged models exactly approximate the averaged part of a temperature field. The quality of this approximation is not noticeable depending on the ratio l/L, cf. Fig. 3. A fluctuation of temperature in the homogenized model depends on the averaged part only. Thus, the homogenized model does not comply with initial condition of the fluctuation temperature field. It follows that the homogenized model is useless in situations in which initial conditions of the averaged part of temperature have to be taken into account. The tolerance averaged model provides perfect coherence for both, the averaged and fluctuation parts of the initial condition. Results of computations are presented in Fig. 4 - Fig. 6. It is also shown that reciprocal impact of averaged and fluctuation fields is stronger in the tolerance averaging model than that of the Fourier model.

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Numeryczna weryfikacja dwóch modeli matematycznych przepływu ciepła dla kompozytów warstwowych

Streszczenie

Do rozwiązywania zagadnień przepływu ciepła w kompozytach o gęstej strukturze periodycznej stosuje się zazwyczaj pewne uśrednione (makroskopowe) modele matematyczne. Najbardziej znanymi modelami ośrodków periodycznych są modele asymptotyczne. Alternatywne heurystyczne modele uwzględniają wpływ periodyki na makroskopowe zachowanie ośrodka. Celem artykułu jest porównanie wspomnianych modeli i weryfikacja uzyskanych rozwiązań numerycznych dla pewnego zagadnienia brzegowo-początkowego.

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