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STEADY OSEEN'S FLOW PAST A DEFORMED SPHERE: AN ANALYTICAL APPROACH

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In a recent authors' paper, the general expression of Stokes drag experienced by a deformed sphere in both longitudinal and transverse flow situations was calculated in terms of the deformation parameter up to the second order. In this paper, Oseen's correction to the axial Stokes drag on the deformed sphere is presented by using Brenner's formula in general, first and then applied to prolate and oblate the deformed spheroid up to the second order of the deformation parameter. Numerical values of Oseen's correction is obtained with respect to the deformation parameter and Reynolds number. The corresponding variations are depicted in figures. Some particular cases of a needle shaped body and flat circular disk are considered and found to be in good agreement with those existing in the literature. The important applications are also highlighted.

Key words: Stokes drag, Oseen's drag, axially symmetric arbitrary body, axial flow.

1. Introduction

The problem of great importance in the hydrodynamics of low Reynolds number flows is the drag or resistance experienced by a particle moving uniformly through an infinite fluid. Since the appearance of Stokes's approximate solution for the flow of a viscous fluid past a sphere (Stokes, 1851), very well known as Stokes law, numerous attempts have been made, both to generalize the problem by changing the shape of the body, and to improve the calculation by including the effect of inertia terms which were neglected in the original calculation. Oseen (1927) tackled this type of problem involving a correction to Stokes drag extensively. Oseen provided solutions for the flow past various bodies at a small Reynolds number Re. and calculated the force to the first order in Re, one term more than would be given by the Stokes approximation. By the inclusion of the effect of the inertia terms, Oseen improved the flow picture far from the body where the Stokes approximation is inadequate, but near the body the difference between the two solutions is of an order of smallness which is outside the accuracy of either approximation. Oseen's calculation for the force thus requires some further justification, for flow past a sphere, by the work of Kaplun (1957), Kaplun and Lagerstrom (1957) and Proudman and Pearson (1957). Oseen failed to calculate correctly the velocity field, his result for the drag on the sphere, namely

$$D = D_0\Big(1 + \frac{3}{8}\mathrm{Re}\Big) \tag{1.1}$$

where D_0 is the Stokes drag, is in fact valid because the correction to the velocity field makes no contribution to the total force on the sphere. Almost a similar problem has been considered by Chang (1960) for the axially symmetric Stokes flow of a conducting fluid past a body of revolution in the presence of a uniform magnetic field. An equation identical to that cited above, except for the dimensionless Hartmann number M appears in place of the Reynolds number Re. Chang (1960) gave a formal proof of this relation as well as similar results by Chang (1961) based on matching' the fundamental solution of the Stokes equations to the fundamental solution of his magnetohydrodynamics differential equation. These fundamental solutions correspond physically to the fields resulting from a point force concentrated at the origin, and are termed the inner' and outer' solutions, respectively. The ideas used in these papers were based on the work of Lagerstrom and Cole (1955) and Proudman and Pearson (1957). The structure of Oseen's equation and its fundamental solution are extremely similar to those for the corresponding magnetohydrodynamics. They do, however, differ in one important respect. The first-order Stokes (inner) equation (Proudman and Pearson, 1957, eq. 3.40), which eventually gives rise to the drag term of $O(\mathrm{Re})$, contains an inhomogeneous forcing term. This is lacking in the analogous first-order inner equation of Chang (1960), equation (6b), which is the source of the drag term of order $O(M)$ in the magnetohydrodynamic problem. Moreover, Proudman and Pearson (1957) have fully discussed the matching' of the Oseen and Stokes equations for the particular case of a spherical particle. Perhaps for these reasons, Brenner (1961) skipped the proof of his own formula (given in eq. 2.7) regarding Oseen's drag on axially symmetric bodies. According to him, the combined work of Proudman and Pearson (1957) and Chang (1960) assured the existence of a general relation of the form of equation 2.7. It is interesting to note here that Chang's (1960) result is restricted to axially symmetric flows because of the requirement that there be sufficient symmetry to preclude the existence of an electric field. While, on the other hand, Brenner's (1961) result is limited only by the requirement that the Stokes drag on the particle (and thus the Oseen drag) be parallel to its direction of motion. Krasovitskaya *et al.* (1970) proposed a formula based on Oseen's correction for calculating the settling of solid particles of powdered materials with enhanced accuracy in carrying out sedimentation analysis. Dyer and Ohkawa (1992) have used the Oseen drag in acoustic levitation. These two works are the main practical applications of Oseen's correction which was not possible with the Stokes drag.

The main aim of the present investigation is to provide a general criterion for the use of Oseen's equations as an approximate representation of the full Navier-Stokes equations. For the better understanding of the reader, the author's previously well proved conjecture (Datta and Srivastava, 1999) is described briefly first followed by Brenner's method (Brenner, 1961) in Section 2.

2. Method

Let us consider an axially symmetric body of characteristic length L placed along its axis (x-axis, say) in a uniform stream U of a viscous fluid of density ρ_1 and kinematic viscosity ν. When the Reynolds number UL/ν is small, the steady motion of incompressible fluid is governed by Stokes equations (Happel and Brenner, 1964)

$$\mathbf{0} = -\frac{1}{\rho_1}\,\mathrm{grad}\,p + \nu\nabla^2 u \qquad \mathrm{div}\,\mathbf{u} = 0 \tag{2.1}$$

subject to the no-slip boundary condition.

We have taken up the class of those axially symmetric bodies which possess continuously turning tangent, placed in the uniform stream U along the axis of symmetry (which is x-axis), as well as constant radius b of maximum circular cross-section at the mid of the body. This axi-symmetric body is obtained by the revolution of meridional plane curve (depicted in Fig. 1) about axis of symmetry, which obeys the following limitations:

i. Tangents at the points A, on the x-axis , must be vertical,

ii. Tangents at the points B, on the y-axis , must be horizontal,

iii. The semi-transverse axis length b must be fixed.

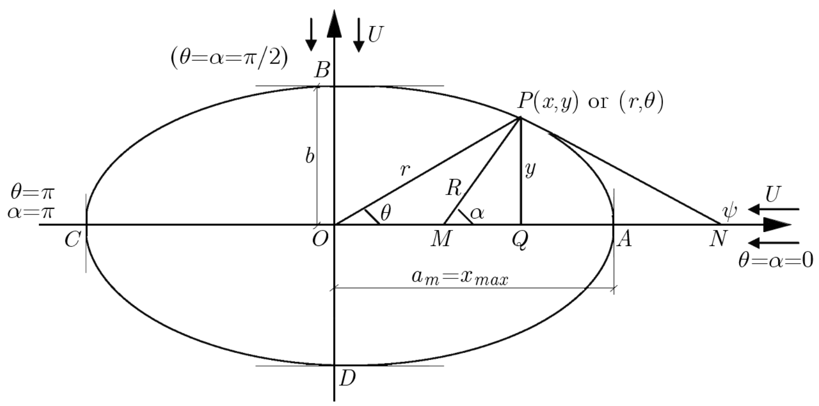

Fig. 1. Geometry of axially symmetric body

The point P on the curve may be represented by the Cartesian coordinates (x, y) or polar coordinates (r, θ), respectively, PN and PM are the length of tangent and normal at the point P. The symbol R stands for the intercepting length of the normal between the point on the curve and the point on the axis of symmetry, and the symbol α is the slope of the normal PM which can be vary from 0 to π.

2.1. Axial flow

The expression of the Stokes drag on such a type of axially symmetric bodies placed in an axial flow (uniform flow parallel to the axis of symmetry) is given by Datta and Srivastava (1999)

$$F_x = \frac{1}{2}\frac{\lambda b^2}{h_x} \qquad \text{where} \qquad \lambda = 6\pi\mu U$$
$$h_x = \frac{3}{8}\int\limits_0^\pi \mathrm{Re}\sin^3\alpha\, d\alpha \tag{2.2}$$

where the suffix x has been introduced to assert that the force is in the axial direction.

Brenner (1961) proposed a general expression of Oseen's resistance of a particle of arbitrary shape after having inspiration from the work of Oseen (1927), Proudman and Pearson (1957), and Chang (1960) in terms of the general expression of the Stokes drag on the same body placed under a uniform axial flow along the axis of symmetry. This expression is

$$\frac{F}{F_x} = 1 + \frac{F_x}{16\pi\mu c U}\mathrm{Re} + O(\mathrm{Re}^2) \tag{2.3}$$

where c is any characteristic particle dimension, $\mathrm{Re} = \rho Uc/\mu$ is the particle Reynolds number and F_x is the Stokes drag. The author has already applied the D-S conjecture, given in equations (2.2), to evaluate the Stokes drag on axially symmetric particles like sphere, spheroid (prolate and oblate), deformed sphere, cycloidal body, cassini oval, hypocycloidal body, cylindrical capsule with semi spherical ends and complicated egg-shaped body consisting of semi spherical and semi spheroidal ends. These results are already published (please see for details, Datta and Srivastava, 1999; Srivastava, 2000; Srivastava, 2012). Now, the author claims that by utilizing the D-S conjecture, one can always improve the expression of the Stokes drag on an axially symmetric body placed in a uniform axial flow parallel to the axis of symmetry with the help of Brenner's formula (2.3). The author also claims here that the improved form of the Stokes drag in the light of Brenner's formula must be the solution to steady Oseen's equation in which the linear term $(\mathbf{U}\cdot\operatorname{grad})\mathbf{u}$ appears on the left hand side in comparison to the Stokes equations, where the non-linear inertia term $(\mathbf{u}\cdot\operatorname{grad})\mathbf{u}$ is absent. This argument may be checked experimentally, which is out of scope of this paper.

3. Formulation of the problem

Let us consider an axially symmetric arbitrary body of the characteristic length L placed along its axis (x-axis, say) in a uniform stream U of a viscous fluid of density ρ_1 and kinematic viscosity ν. When the particle Reynolds number UL/ν is small, the steady motion of an incompressible fluid around the axially symmetric body is governed by the Stokes equations (Happel and Brenner, 1964)

$$\mathbf{0} = \frac{1}{\rho_1}\operatorname{grad} p + \nu\nabla^2\mathbf{u} \qquad \operatorname{div}\mathbf{u} = 0 \tag{3.1}$$

subject to the no-slip boundary condition.

This equation is the reduced form of the complete Navier-Stokes equations neglecting the inertia term $(\mathbf{u}\cdot\operatorname{grad})\mathbf{u}$ which is unimportant in the vicinity of the body where the viscous term dominates (Stokes approximation). Solution to this equation (3.1), called the Stokes law, $6\pi\mu Ua$, for a slowly moving sphere having radius a is valid only in the vicinity of the body which breaks down at a distance far away from the body. This break-down of the Stokes solution at a far distance from the body is known as Whitehead's paradox. It was Oseen in 1910, who pointed out the origin of Whitehead's paradox and suggest a scheme for its resolution (see Oseen, 1927). In order to rectify the difficulty, Oseen went on to make the following additional observations.

In the limit, where the particle Reynolds number $\rho Ua/\mu \to 0$, the Stokes approximation becomes invalid only when $r/a \to \infty$. But at such enormous distances, the local velocity $\mathbf{v}$ differs only imperceptibly from a uniform stream of the velocity $\mathbf{U}$. Thus, Oseen was inspired to suggest that the inertial term $(\mathbf{u}\cdot\operatorname{grad})\mathbf{u}$ could be uniformly approximated by the term $(\mathbf{U}\cdot\operatorname{grad})\mathbf{u}$. By such arguments, Oseen proposed that uniformly valid solutions of the problem of steady streaming flow past a body at small particle Reynolds numbers could be obtained by solving the linear equations

$$(\mathbf{U}\cdot\operatorname{grad})\mathbf{u} = -\frac{1}{\rho}\operatorname{grad} p + \nu\nabla^2\mathbf{u} \qquad \operatorname{div}\mathbf{u} = 0 \tag{3.2}$$

known as Oseen's equation. Oseen obtained an approximated solution to his equations for flow past a sphere, from which he obtained the Stokes drag formula (Happel and Brenner, p. 44, eq. (2-6.5), 1964)

$$F = 6\pi\mu aU\Big(1 + \frac{3}{8}N_{\mathrm{Re}} + O(N_{\mathrm{Re}}^2)\Big) \tag{3.3}$$

where $N_{\mathrm{Re}} = \rho Ua/\mu$ is the body Reynolds number.

Now, we find the general solution to Oseen's equations (3.2) for a deformed sphere under no-slip boundary conditions by use of D-S conjecture (2.2) and corrected up to the first order of Reynolds number Re by using Brenner's formula (2.3). Later on, this expression is applied to the perturbed prolate and oblate spheroid which falls under the class of deformed sphere having polar equation (4.1). Some particular cases of a slender spheroid and a flat circular disk are also discussed in the limiting situations.

4. Solution

We consider the axially symmetric body defined by

$$r = a\Big(1 + \varepsilon\sum_{k=0}^{\infty} d_k P_k(\mu)\Big) \qquad \mu = \cos\theta \tag{4.1}$$

where (r,θ) are spherical polar coordinates, ε is the small deformation parameter, d_k are design or shape factors and $P_k(\mu)$ are Legendre's functions of the first kind. For small values of the parameter ε, equation (4.1) represents the deformed sphere.

The expression for the axial Stokes drag, by D-S conjecture (Datta and Srivastava, 1999), comes out to be (for details, see Srivastava *et al.*, 2012)

$$F_x = 6\pi\mu Ua\mathcal{C} \tag{4.2}$$

where

$$\begin{aligned}\mathcal{C} = \Big\{1 + \varepsilon\Big(d_0 - \frac{1}{5}d_2 + \frac{3}{8}d_4 + \ldots\Big) + \varepsilon^2\Big[\Big(2d_0^2 + \frac{89}{100}d_2^2 + \frac{9}{64}d_4^2 + \ldots\Big)\\ +2d_0\Big(-\frac{21}{10}d_2 + \frac{3}{8}d_4 + \ldots\Big) + 2d_2\Big(\frac{3}{8}d_4 + \ldots\Big) + \ldots\Big] + O(\varepsilon^3)\Big\}\end{aligned}$$

Now, the general expression of Oseen's correction may be obtained for same deformed sphere by substituting the value of Stokes drag (4.2) in Brenner's formula (2.7) as

$$\frac{F}{F_x} = 1 + \frac{6\pi\mu Ua}{16\pi\mu Ua}\mathcal{C}\mathrm{R} + O(\mathrm{Re}^2) = 1 + \frac{3}{8}\mathcal{C}\mathrm{Re} + O(\mathrm{Re}^2\Big) \tag{4.3}$$

where $\mathrm{Re} = \rho Ua/\mu$ is the particle Reynolds number. This expression immediately reduces to the case of sphere(Chester, 1962) in the limiting case as $\varepsilon \to 0$. Now, we use this general expression over the prolate and oblate deformed spheroid to evaluate Oseen's correction.

5. Flow past prolate spheroid

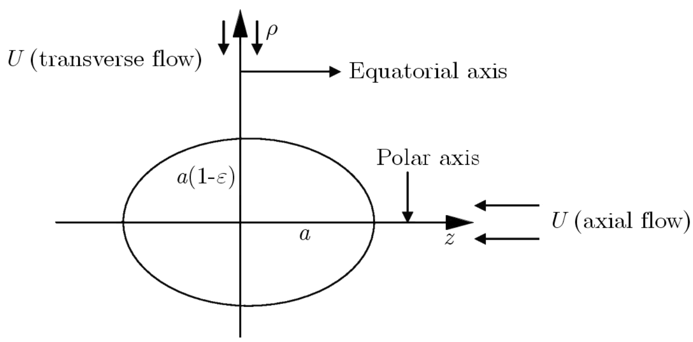

Fig. 2. Prolate spheroid in meridional two-dimensional plane (ρ, z)

We consider the prolate spheroid as it belongs to the class of axi-symmetric deformed sphere, whose Cartesian equation is

$$\frac{x^2 + y^2}{a^2(1-\varepsilon)^2} + \frac{z^2}{a^2} = 1 \tag{5.1}$$

where the equatorial radius is $a(1-\varepsilon)$ and the polar radius is a, in which the deformation parameter ε is positive and sufficiently small that squares and higher powers of it may be neglected. Its polar equation, up to the order of $O(\varepsilon)$, by using the transformation is

$$z = r\cos\theta \qquad \rho = r\sin\theta \qquad r = a(1 - \varepsilon\sin^2\theta) \tag{5.2}$$

It can be written in linear combination of Legendre's functions of the first kind (Happel and Brenner, 1964; Senchenko and Keh, 2006)

$$r = a\Big[1 + \varepsilon\Big(-\frac{2}{3}P_0(\mu) + \frac{2}{3}P_2(\mu)\Big)\Big] \qquad \mu = \cos\theta \tag{5.3}$$

On comparing this equation with the polar equation of deformed sphere (3.1), the appropriate design factors for this prolate spheroid are

$$d_0 = -\frac{2}{3} \qquad d_1 = 0 \qquad d_2 = \frac{2}{3} \qquad d_k = 0 \qquad \text{for} \qquad k \geqslant 3 \tag{5.4}$$

with

$$P_0(\mu) = 1 \qquad P_1(\mu) = \mu \qquad P_2(\mu) = \frac{3\mu^2 - 1}{2} \qquad \mu = \cos\theta \tag{5.5}$$

Also, polar equation $(5.2)_3$ of the prolate spheroid may be written in terms of Gegenbauer's functions of the first kind by using the following well known relation as

$$\begin{aligned} \mathcal{I}_k(\mu) &= \frac{P_{k-2}(\mu) - P_k(\mu)}{2k-1} \qquad && k \geqslant 2, \quad \mu = \cos\theta \\ r &= a[1 - 2\varepsilon\mathcal{I}_2(\mu)] && \mu = \cos\theta \end{aligned} \tag{5.6}$$

Now, we find the expression of the Stokes drag on this axi-symmetric prolate body placed in both axial or longitudinal flow (in which uniform stream is parallel to the polar axis or axis of symmetry) with the aid of general expression of Oseen's drag (4.3).

5.1. Axial flow

With the aid of equations (5.3) and (5.4), the expression of the axial Stokes drag on the prolate spheroid can be written with the help of (4.2), and comes out to be

$$F_z = 6\pi\mu Ua\Big(1 - \frac{4}{5}\varepsilon + \frac{709}{225}\varepsilon^2 + \ldots\Big) \tag{5.7}$$

which matches with that given by Datta and Srivastava (1999) only up to the first order. By using (5.7), Oseen's correction may be obtained from Brenner's formula (2.3) as

$$\frac{F}{F_z} = 1 + \frac{3}{8}\Big(1 - \frac{4}{5}\varepsilon + \frac{709}{225}\varepsilon^2 + \ldots\Big)\mathrm{Re} + O(\mathrm{Re}^2) \tag{5.8}$$

where $\mathrm{Re} = \rho_1 Ua/\mu$ is the particle Reynolds number.

The conjectured expression for the axial and transverse Stokes drag on axially symmetric bodies proposed in Datta and Srivastava (1999) holds good for the spheroid and provide the accurate closed form solution. For the convenience, we write both expressions of drag on the prolate spheroid (Happel and Brenner, 1964; Chwang and Wu, 1975)

$$\begin{aligned} F_{\|} &= 16\pi\mu U_{\|} ae^3\Big[(1+e^2)\log\frac{1+e}{1-e} - 2e\Big]^{-1} \\ F_{\perp} &= 32\pi\mu U_{\perp} ae^3\Big[2e + (3e^2 - 1)\log\frac{1+e}{1-e}\Big]^{-1} \end{aligned} \tag{5.9}$$

where e is the eccentricity of the prolate spheroid.

The expression of the Stokes drag in axial flow (5.8) on the prolate spheroid may also be deduced from the exact expression of the axial Stokes drag $(5.9)_1$ on the prolate spheroid for flow parallel to its axis of revolution in terms of the deformation parameter ε up to the first order.

For the considered prolate spheroid, the eccentricity is $e = \sqrt{1 - (b/a)^2}$, with $b = a(1-\varepsilon)$, which further gives us the relation between the eccentricity e and deformation parameter ε as

$$e = \sqrt{1 - (1-\varepsilon)^2} = \sqrt{2\varepsilon - \varepsilon^2} = \sqrt{2\varepsilon} \tag{5.10}$$

(leaving the square term).

The exact expression of the Stokes drag F_z, in second powers of the deformation parameter ε, would be of the closed form from $(5.9)_1$

$$F_z = 6\pi\mu Ua\Big(1 - \frac{4}{5}\varepsilon + \frac{2}{175}\varepsilon^2 + \dots\Big) \tag{5.11}$$

This expression is in good agreement to that presented by Chang and Keh (2009, page 205, eq. 36a). The first two terms of this expression also confirm our result (5.8), up to the order of $O(\varepsilon)$, of the axial drag on the prolate spheroid lying in the class of the deformed sphere.

On considering the polar equation of the prolate spheroid as a deformed sphere with the same eccentricity defined as above and related with the deformation parameter ε by (5.14) as

$$r = a\Big(1 - \varepsilon\sin^2\theta - \frac{3}{2}\varepsilon^2\sin^2\theta\cos^2\theta\Big) \tag{5.12}$$

the expression of the axial Stokes drag containing the second order terms in e and ε can be achieved by independent application of D-S conjecture (Srivastava *et al.*, 2012) as

$$F_z = 6\pi\mu Ua\Big(1 - \frac{2}{5}e^2 - \frac{17}{175}e^4 + \dots\Big) = 6\pi\mu Ua\Big(1 - \frac{4}{5}\varepsilon + \frac{2}{175}\varepsilon^2 + \dots\Big) \tag{5.13}$$

It is interesting to note that there is discrepancy at the third term in the expressions of the Stokes drag (5.8) and last (5.13). The reason behind this discrepancy lies in the fact that polar equation (4.1) contains terms only up to first order, while polar equation (5.15) contains terms up to the second order in the deformation parameter.

On applying Brenner's formula (2.3), by using Stokes drag (5.10), Oseen's correction comes out to be

$$\frac{F}{F_z} - 1 = \frac{3}{8}\Big(1 - \frac{2}{5}e^2 - \frac{17}{175}e^4 + \dots\Big)\mathrm{Re} + O(\mathrm{Re}^2) = \frac{3}{8}\Big(1 - \frac{4}{5}\varepsilon + \frac{2}{175}\varepsilon^2 + \dots\Big)\mathrm{Re} + O(\mathrm{Re}^2) \tag{5.14}$$

5.2. Particular case

5.2.1. Slender elongated spheroid or needle shaped body

Last expression (5.14) is true not only for a small deformation parameter (near zero value) ε but surprisingly provides good agreement in the limiting situation for $e = \varepsilon = 1$, which is the cased of a needle shaped body.

On substituting $\varepsilon = 1$ in equation (5.14), Oseen's correction comes out to be

$$\frac{F}{F_z} - 1 = 0.079275\mathrm{Re} \tag{5.15}$$

for an elongated spheroid which is not far away from the zero value obtained from asymptotic expressions given by Happel and Brenner (1964) and Chwang and Wu (part 4, 1975) for Reynolds number $\mathrm{Re} \ll 1$.

6. Flow past oblate spheroid

We consider the oblate spheroid as it belongs to the class of the axi-symmetric deformed sphere, whose Cartesian equation is

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{a^2(1 - \varepsilon')^2} = 1 \tag{6.1}$$

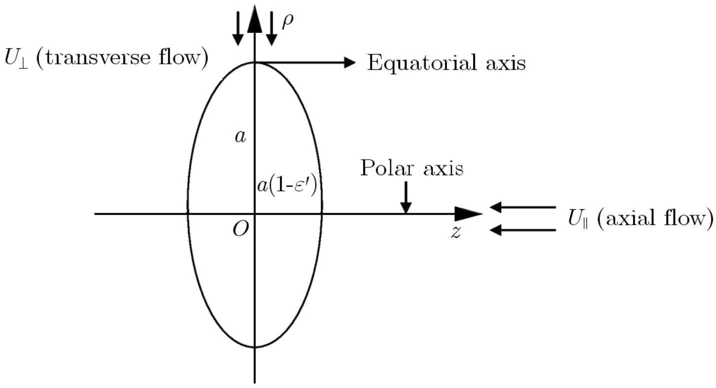

Fig. 3. Oblate spheroid in meridional two-dimensional plane (z, ρ)

where the equatorial radius is a and polar radius is $a(1-\varepsilon)$, in which the deformation parameter ε is positive and sufficiently small that squares and higher powers of it may be neglected. Its polar equation (by using $z = r\cos\theta$, $\rho = r\sin\theta$, $\rho = \sqrt{x^2+y^2}$, up to the order of $o(\varepsilon)$) is

$$r = a(1-\varepsilon'\cos^2\theta) \tag{6.2}$$

it can be written in linear combination of Legendre's functions of the first kind (Happel and Brenner, 1964; Senchensko and Keh, 2009)

$$r = a\Big[1-\varepsilon'\Big(\frac{1}{3}P_0(\mu)+\frac{2}{3}P_2(\mu)\Big)\Big] \qquad \mu=\cos\theta \tag{6.3}$$

On comparing this equation with polar equation of deformed sphere (3.1), the appropriate design factors for this oblate spheroid are

$$d_0 = -\frac{1}{3} \qquad d_1 = 0 \qquad d_2 = -\frac{2}{3} \qquad d_k = 0 \qquad \text{for} \qquad k \geqslant 3 \tag{6.4}$$

with

$$P_0(\mu) = 1 \qquad P_1(\mu) = \mu \qquad P_2(\mu) = \frac{3\mu^2-1}{2} \qquad \mu = \cos\theta \tag{6.5}$$

Also, polar equation (6.3) of the oblate spheroid may be written in terms of Gegenbauer's functions of the first kind by using the following well known relation

$$\begin{aligned}
\mathcal{I}_k(\mu) &= \frac{P_{k-2}(\mu)-P_k(\mu)}{2k-1} && k \geqslant 2, \quad \mu=\cos\theta \\
r &= a[1-\varepsilon'+2\varepsilon'\mathcal{I}_2(\mu)] && \mu=\cos\theta
\end{aligned} \tag{6.6}$$

or, if we put $d = a(1-\varepsilon')$, we have

$$r = d[1+2\varepsilon'\mathcal{I}_2(\mu)] \tag{6.7}$$

Now, we find the expression for the Stokes drag on this axi-symmetric oblate body placed in both axial flow (in which uniform stream is parallel to the polar axis or axis of symmetry) situations with the aid of general expressions of Oseen's drag (4.3).

6.1. Axial flow

With the aid of equations (6.4) and (6.5), the expression of the axial Stokes drag can be written up to the order of $O(\varepsilon')$ with the help of (4.2), and comes out to be

$$F_z = 6\pi\mu Ua\Big(1 - \frac{\varepsilon'}{5} - \frac{71}{225}\varepsilon'^2 + \ldots\Big) \tag{6.8}$$

This matches with that given in the paper of Datta and Srivastava (1999) only up to the first order. By using (6.8), Oseen's correction may be obtained from Brenner's formula (2.3) as

$$\frac{F}{F_z} - 1 = \frac{3}{8}\Big(1 - \frac{1}{5}\varepsilon' - \frac{71}{225}\varepsilon'^2 + \ldots\Big)\mathrm{Re} + O(\mathrm{Re}^2) \tag{6.9}$$

where $\mathrm{Re} = \rho_1 Ua/\mu$ is the particle Reynolds number.

The conjectured expression for the axial and transverse Stokes drag on axially symmetric bodies proposed by Datta and Srivastava (1999) holds good for the spheroid and provides the accurate closed form solution. For the convenience, we write both expressions of drag on the oblate spheroid (Happel and Brenner, 1964; Chwang and Wu, 1975)

$$\begin{aligned} F_{\|} &= 8\pi\mu U_{\|} ae^3\big[e\sqrt{1-e^2} - (1-2e^2)\sin^{-1}e\big]^{-1} \\ F_{\perp} &= 16\pi\mu U_{\perp} ae^3\big[-e\sqrt{1-e^2} + (1+2e^2)\sin^{-1}e\big]^{-1} \end{aligned} \tag{6.10}$$

where e is the eccentricity of the spheroid.

The expression of Stokes drag in axial flow (6.8) on the oblate spheroid may also be deduced from the exact expression for axial Stokes drag $(6.10)_2$ on a oblate spheroid for flow parallel to its axis of revolution by utilizing the appropriate relation between the eccentricity e and deformation parameter ε'.

For the considered oblate spheroid, the eccentricity is $e = \sqrt{1-(b/a)^2}$, with $b = a(1-\varepsilon')$, which further gives us the relation between the eccentricity e and deformation parameter ε' as

$$e = \sqrt{1-(1-\varepsilon')^2} = \sqrt{2\varepsilon' - \varepsilon'^2} = \sqrt{2\varepsilon'} \tag{6.11}$$

(leaving the square term).

The exact expression of the Stokes drag F_z, in second powers of the deformation parameter ε, would be of the dosed form from solution $(6.10)_1$

$$F_z = 6\pi\mu Ua\Big(1 - \frac{1}{5}\varepsilon' + \frac{2}{175}\varepsilon'^2 + \ldots\Big) \tag{6.12}$$

This expression is in good agreement to that presented by Chang and Keh (2009). The first two terms of this expression also confirm our result (6.8), up to the order of $O(\varepsilon)$, of the axial drag on the oblate spheroid lying in the class of the deformed sphere.

On considering the polar equation of the oblate spheroid as a deformed sphere with the same eccentricity defined as above and related with the deformation parameter ε' as

$$r = a\Big(1 - \varepsilon'\cos^2\theta - \frac{3}{2}\varepsilon'^2\sin^2\theta\cos^2\theta\Big) \tag{6.13}$$

The expressions of the axial and transverse Stokes drag containing the second order terms in e and ε' can be achieved by independent application of equation (2.2) and (2.3) (Datta and Srivastava, 1999) as

$$F_z = 6\pi\mu Ua\Big(1 - \frac{1}{10}e^2 - \frac{31}{1400}e^4 + \frac{9}{1400}e^6 + \ldots\Big) = 6\pi\mu Ua\Big(1 - \frac{1}{5}\varepsilon' + \frac{2}{175}\varepsilon'^2 + \ldots\Big) \tag{6.14}$$

It is interesting to note that there is discrepancy at the third term in the expressions of the Stokes drag (6.8) and last (6.14). The reason behind this discrepancy lies in the fact that polar equation (4.1) contains terms only up to the first order, while polar equation (6.13) contains terms up to the second order in the deformation parameter.

On applying Brenner's formula (2.3), by using Stokes drag (6.14), Oseen's correction comes out to be

$$\begin{aligned} \frac{F}{F_z}-1&=\frac{3}{8}\Big(1-\frac{1}{10}e^2-\frac{31}{1400}e^4+\frac{9}{1400}e^6+\ldots\Big)\mathrm{Re}+O(\mathrm{Re}^2)\\ &=\frac{3}{8}\Big(1-\frac{1}{5}\varepsilon'+\frac{2}{175}\varepsilon'^2+\ldots\Big)\mathrm{Re}+O(\mathrm{Re}^2) \end{aligned} \tag{6.15}$$

where $\mathrm{Re}=\rho_1 Ua/\mu$ is particle Reynolds number.

6.2. Particular case

6.2.1. Flat circular disk

Las expression (6.15) is true not only for a small deformation parameter (near zero value) ε but surprisingly provides good agreement in the limiting situation for $e=\varepsilon=1$, which is the cased of a flat circular disk (broad side on case).

On substituting $\varepsilon=1$ in equation (6.15), Oseen's correction reduces to the form

$$\frac{F}{F_z}-1=0.3042856\mathrm{Re} \tag{6.16}$$

for a flat circular disk which is not far away from the zero value obtained from asymptotic expressions given by Happel and Brenner (1964) for Reynolds numbers $\mathrm{Re}\ll 1$.

7. Numerical discussion

The numerical values of Oseen's correction F/F_z-1 for prolate and oblate deformed spheroids with respect to the deformation parameter ε (0 to 1) for various values of the Reynolds number Re (0 to 100) are calculated and presented in Tables 1 and 2. According to both tables, Oseen's correction reduces to zero for $\mathrm{Re}=0$ confirming the Stokes drag on the corresponding deformed body (see Chang and Keh, 2009). In every other possibility of the Reynolds number Re, Oseen's correction decreases significantly with the increasing deformation parameter ε for the prolate deformed spheroid. On the other hand, Oseen's correction decreases very slowly with the increasing deformation parameter ε for the oblate deformed spheroid. The reason for this anomaly is due to the broader part of the oblate deformed spheroid facing the axial flow. For specific values of the deformation parameter ve, Oseen's correction increases with respect to increasing values of the Reynolds number Re. The corresponding variation of Oseen's correction with respect to the deformation parameter ε for various values of the Reynolds number Re are shown in Figs. 4 and 5.

8. Conclusion

The general expression of Oseen's correction to the Stokes drag on a deformed sphere is provided with the help of D-S conjecture (Datta and Srivastava, 1999) followed by Brenner's formula (Brenner, 1961). Later on, this expression is utilized to evaluate Oseen's drag on a prolate and oblate deformed spheroid and verified for the first two terms. It is also found that this correction may be modified when we apply D-S conjecture independently of these perturbed bodies whose

Table 1. Numerical values of Oseen's correction $F/F_z - 1$ with respect to deformation parameter ε of deformed prolate spheroid calculated from Eq. (5.14) and depicted in Fig. 4

ε	Re											
	0	1	5	10	20	30	40	50	60	70	80	90	100
0	0	0.3750	1.8750	3.7500	7.5000	11.2500	15.0000	18.7500	22.5000	26.2500	30.0000	33.7500	37.5000
0.1	0	0.3450	1.7252	3.4504	6.9009	10.3513	13.8017	17.2521	20.7026	24.1530	27.6034	31.0539	34.5043
0.2	0	0.3152	1.5758	3.1517	6.3034	9.4551	12.6069	15.7586	18.9103	22.0620	25.2137	28.3654	31.5171
0.3	0	0.2854	1.4269	2.8539	5.7077	8.5616	11.4154	14.2693	17.1231	19.9770	22.8309	25.6847	28.5386
0.4	0	0.2557	1.2784	2.5568	5.1137	7.6706	10.2274	12.7843	15.3411	17.8980	20.4549	23.0117	25.5686
0.5	0	0.2261	1.1304	2.2607	4.5214	6.7821	9.0429	11.3036	13.5643	15.8250	18.0857	20.3464	22.6071
0.6	0	0.1965	0.9827	1.9654	3.9309	5.8963	7.8617	9.8271	11.7926	13.7580	15.7234	17.6889	19.6543
0.7	0	0.1671	0.8355	1.6710	3.3420	5.0130	6.6840	8.3550	10.0260	11.6970	13.3680	15.0390	16.7100
0.8	0	0.1377	0.6887	1.3774	2.7549	4.1323	5.5097	6.8871	8.2646	9.6420	11.0875	12.3969	13.7743
0.9	0	0.1085	0.5424	1.0847	2.1694	3.2541	4.3389	5.4236	6.5083	7.5930	8.6777	9.7624	10.8471
1.0	0	0.0793	0.3964	0.7928	1.5857	2.3786	3.1714	3.6071	4.7571	5.5500	6.3428	7.1357	7.9286

Table 2. Numerical values of Oseen's correction $F/F_z - 1$ with respect to deformation parameter ε' of deformed oblate spheroid calculated from Eq. (6.15) and depicted in Fig. 5

ε'	Re											
	0	1	5	10	20	30	40	50	60	70	80	90	100
0	0	0.3750	1.8750	3.7500	7.5000	11.2500	15.0000	18.7500	22.5000	26.2500	30.0000	32.7500	37.5000
0.1	0	0.3675	1.8377	3.6754	7.3508	11.0263	14.7017	18.3771	22.0526	25.7280	29.4034	32.0789	36.7543
0.2	0	0.3602	1.8008	3.6017	7.2034	10.8051	14.4069	18.0086	21.6103	25.2120	28.8137	31.4154	36.0171
0.3	0	0.3528	1.7644	3.5288	7.0577	10.5866	14.1154	17.6443	21.1731	24.7020	28.2309	30.7597	35.2886
0.4	0	0.3456	1.7284	3.4568	6.9137	10.3706	13.8274	17.2843	20.7411	24.1980	27.6549	30.1117	34.5686
0.5	0	0.3385	1.6928	3.3857	6.7714	10.1571	13.5429	16.9286	20.3143	23.7000	27.0857	29.4714	33.8571
0.6	0	0.3315	1.6577	3.3154	6.6308	9.9463	13.2617	16.5771	19.8926	23.2080	26.5234	28.8389	33.1543
0.7	0	0.3246	1.6230	3.2460	6.4920	9.7380	12.9840	16.2300	19.4760	22.7220	25.9680	28.2140	32.4600
0.8	0	0.3177	1.5887	3.1774	6.3548	9.5323	12.7097	15.8870	19.0646	22.2420	25.4194	27.5969	31.7743
0.9	0	0.3109	1.5548	3.1097	6.2194	9.3291	12.4389	15.5486	18.6583	21.7680	24.8777	26.9874	31.0971
1.0	0	0.3042	1.5214	3.0429	6.0857	9.1286	12.1714	15.2143	18.2571	21.3000	24.3429	26.3857	30.4286

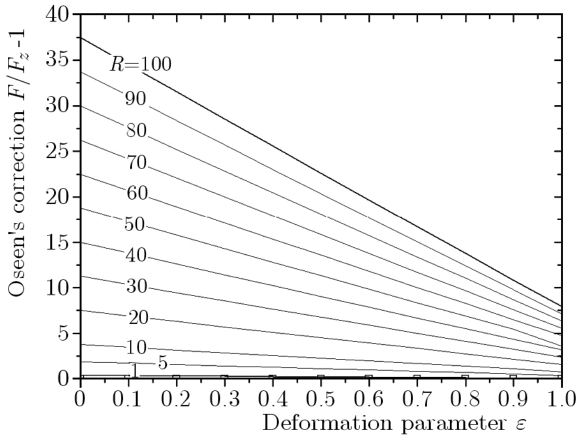

Fig. 4. Variation of Oseen's correction with respect to the deformation parameter ε for various values of the Reynolds number $\text{Re} = \rho UA/mu$ for the deformed prolate spheroid

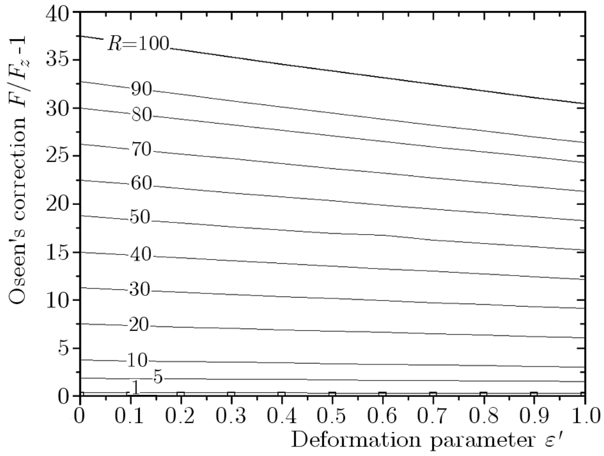

Fig. 5. Variation of Oseen's correction with respect to the deformation parameter ε' for various values of the Reynolds number $\mathrm{Re} = \rho UA/mu$ for the deformed oblate spheroid

polar equations contain second order terms in the deformation parameter followed by Brenner's formula. The modified expressions for these bodies are given in equations (5.14) and (6.15) in terms of the eccentricity e and deformation parameter ε both. The numerical values of drag correction are presented in Tables 1 and 2, and the corresponding variations are shown in Figs. 4 and 5. It may be seen from the tables and figures that in the limiting cases for the needle shaped body and the flat circular disk, the drag is in agreement with that known in literature, which may further be checked experimentally. These values are useful in prediction of optimum drag profiles in low Reynolds number flows. In the end, it is concluded that for a small deformation parameter, Oseen's correction increases with the increasing Reynolds number. Other important applications of Oseen's correction are in calculation of the settling of solid particles of powdered materials with enhanced accuracy in carrying out sedimentation analysis and in acoustic levitation. These two works are the main practical applications of Oseen's correction, which was not possible with the Stokes drag.

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