JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
43, 4, pp. 825-840, Warsaw 2005

ON THE MATERIAL AND SPATIAL DESCRIPTION OF RIGID FIELDS[1]

Marek Rudnicki

Faculty of Civil Engineering, Warsaw University of Technology
e-mail: marr@siwy.il.pw.edu.pl

Rigid kinematics is re-examined applying the material, spatial and mixed description without use of any coordinate system. Some tensor representations for the displacement gradient, velocity gradient, and acceleration gradient are obtained. Special cases of rotation about fixed as well as nearly fixed axis are distinguished. Assumption of small rotation is also taken into account. Relative motion of a particle is briefly described.

Key words: kinematics, rigid body, material and spatial description, relative motion

1. Introduction

The most part of applied mechanics is study of deforming bodies. Nevertheless, rigid fields play a significant role in mechanics. Firstly, every homogeneous deformation may be, in general, composed of stretch, translation, and rotation, the last two of which being the evident parts of rigid deformation. Next, an infinitesimal rigid displacement is involved in the fundamental theorem of virtual work (cf. Gurtin, 1981). In theory of plasticity, the rigidplastic model of materials is applied. Last but not least, a rigid body itself may be a satisfactory idealization in many branches of mechanics, e.g., in celestial mechanics, in theory of machines, mechanisms and devices (position analysis, collapse), in robotics (cf. Nwokah and Hurmuzlu, 2002). The notion of rigidity is exploited within both classical kinematics as well as theory of microstructure bodies. "The assumption of rigidity was the key step..." (cf. Uicker *et al.*, 2003).

[1]See also: Material and spatial description of rigid fields, XIII Conference on "Theoretical Foundations of Civil Engineering", Dnepropetrovsk-Warsaw, June 2005, Polish-Ukrainian Transactions, 571-578

The material and spatial description are two well-known basic approaches applied in continuum mechanics. However, the simultaneous use of both of them is not frequent. Typically, as far as rigid continua are concerned, the spatial description is used (cf., e.g., Gurtin and Williams, 1976; Wang, 1979). Some authors employ a vector (indicial) notation without any use of tensors (cf. Easthope, 1964).

The main purpose of this work is to provide a refreshing comparison of the material and spatial description by an example of the simplest material. We carry out both descriptions parallel one to another. Appropriate mixed relations are also provided. In order to simplify the notations, we do not use any coordinate system in the physical space. Although the indicial notation "kills two birds with one stone", i.e., coordinate generality and specificity (cf. Papastavridis, 1998), at the same time can overshadow to some extent the subject being considered. The paper is entirely confined to "kinematics which unencumbered by physical restrictions can provide the preliminary light" (see Truesdell and Toupin, 1960), however, no doubt, some theoretical aspects of rigid body dynamics are still worthy of attention (cf. Sławianowski, 2004).

2. Deformation

A rigid deformation on any set of points can be extended to form a global rigid deformation on all of the physical space. Thus, consider an orientation-preserving isometry $\chi : \mathcal{E} \to \mathcal{E}$, where $\mathcal{E}$ is an oriented three-dimensional Euclidean point space called a physical space endowed with the translation space $\mathcal{V}$. The deformation gradient is then a constant mapping $\operatorname{grad}\chi : \mathcal{E} \to \mathcal{L}(\mathcal{E}, \mathcal{V}\otimes\mathcal{V})$, where at every point $\operatorname{grad}\chi$ is a proper orthogonal tensor.

The vector-valued displacement field is defined by

$$\begin{aligned} \boldsymbol{u}^m(A) &= \chi(A) - A = \boldsymbol{u}^s(\chi(A)) \\ \boldsymbol{u}^s(A) &= A - \chi^{-1}(A) = \boldsymbol{u}^m(\chi^{-1}(A)) \end{aligned} \tag{2.1}$$

where A is a general point of the physical space, and superscripts m and s indicate the material and spatial description, respectively. By virtue of (2.1), the corresponding displacement gradients are

$$\operatorname{grad}\boldsymbol{u}^m = \mathbf{R} - \mathbf{1} \qquad\qquad \operatorname{grad}\boldsymbol{u}^s = \mathbf{1} - \mathbf{R}^\top \tag{2.2}$$

where $\mathbf{1} \in \mathcal{V}\otimes\mathcal{V}$ stands for the identity tensor.

Given any point P, so-called base point (cf. Easthope, 1964), we can decompose the mapping χ as follows

$$\chi = \kappa_P \circ \vartheta_P \tag{2.3}$$

where $\vartheta_P : \mathcal{E} \to \mathcal{E}$ is rotation with P fixed, while $\kappa_P : \mathcal{E} \to \mathcal{E}$ is translation with the translation vector equal to the translation vector of point P, i.e.,

$$\vartheta_P(P) = P \qquad\qquad \kappa_P(P) = \chi(P) \tag{2.4}$$

Precisely, for every $A \in \mathcal{E}$

$$\vartheta_P(A) = P + \mathbf{R}[A - P] \qquad\qquad \kappa_P(A) = A + \boldsymbol{w}_P \tag{2.5}$$

where

$$\boldsymbol{w}_P = \boldsymbol{u}^m(P) \qquad\qquad \mathbf{R} = \operatorname{grad}\chi(A) = \operatorname{grad}\vartheta_P(A) \tag{2.6}$$

Generally, it takes six coordinates to characterize a particular χ, namely three ones to characterize $\boldsymbol{w}_P$ and next three to characterize $\mathbf{R}$. Making use of (2.5) and (2.6), we replace (2.3) by

$$\chi(A) = P + \boldsymbol{w}_P + \mathbf{R}[A - P] = \vartheta_P(A) + \boldsymbol{u}^m(P) = \vartheta_P(A) + \kappa_P(A) - A \tag{2.7}$$

Hence

$$\boldsymbol{u}^m(A) = \boldsymbol{u}^m(A; \vartheta_P) + \boldsymbol{u}^m(A; \kappa_P) \tag{2.8}$$

where the first term in the right hand side of (2.8) represents the displacement due to rotation ϑ_P, whereas the second term represents the displacement due to translation κ_P, i.e.

$$\boldsymbol{u}^m(A; \vartheta_P) = \vartheta_P(A) - A \qquad\qquad \boldsymbol{u}^m(A; \kappa_P) = \kappa_P(A) - A \tag{2.9}$$

The inverse of (2.3) is

$$\chi^{-1} = (\kappa_P \circ \vartheta_P)^{-1} = (\vartheta_P)^{-1} \circ (\kappa_P)^{-1} \tag{2.10}$$

where

$$(\kappa_P)^{-1}(A) = A - \boldsymbol{w}_P \qquad\qquad (\vartheta_P)^{-1}(A) = P + \mathbf{R}^\top[A - P] \tag{2.11}$$

After obvious transformations, we obtain (cf. (2.7))

$$\begin{aligned}\chi^{-1}(A) &= P + \mathbf{R}^\top[A - P - \boldsymbol{w}_P] = P - \mathbf{R}^\top\boldsymbol{w}_P + \mathbf{R}^\top[A - P] = \\ &= (\vartheta_P)^{-1}(A) + \mathbf{R}^\top[(\kappa_P)^{-1}(A) - A]\end{aligned} \tag{2.12}$$

Hence

$$\boldsymbol{u}^s(A) = A-(\vartheta_P)^{-1}(A)+\mathbf{R}^\top[A-(\kappa_P)^{-1}(A)] = (\mathbf{1}-\mathbf{R}^\top)[A-P]+\mathbf{R}^\top\boldsymbol{w}_P \tag{2.13}$$

On comparing (2.8) and (2.13), and taking (2.5) into account, both forms of the displacement are related with each other by

$$\boldsymbol{u}^s = \mathbf{R}^\top\boldsymbol{u}^m \qquad\qquad \boldsymbol{u}^m = \mathbf{R}\boldsymbol{u}^s \tag{2.14}$$

whereas the corresponding displacement gradients transform to each other by

$$\begin{aligned} \operatorname{grad}\boldsymbol{u}^m &= -(\operatorname{grad}\boldsymbol{u}^s)^\top = \mathbf{R}\operatorname{grad}\boldsymbol{u}^s = (\operatorname{grad}\boldsymbol{u}^s)\mathbf{R} \\ \operatorname{grad}\boldsymbol{u}^s &= -(\operatorname{grad}\boldsymbol{u}^m)^\top = \mathbf{R}^\top\operatorname{grad}\boldsymbol{u}^m = (\operatorname{grad}\boldsymbol{u}^m)\mathbf{R}^\top \end{aligned} \tag{2.15}$$

We observe that the symmetric parts of $(2.15)_1$ and $(2.15)_2$ are opposite, whereas the skew parts coincide, i.e.

$$\begin{aligned} &\operatorname{sym}\ \operatorname{grad}\boldsymbol{u}^m + \operatorname{sym}\ \operatorname{grad}\boldsymbol{u}^s = 0 \\ &\operatorname{skw}\ \operatorname{grad}\boldsymbol{u}^m = \operatorname{skw}\ \operatorname{grad}\boldsymbol{u}^s \end{aligned} \tag{2.16}$$

3. Motion

Let $\mathcal{T} = [a,b) \subset \mathcal{R}$ be a set of real numbers which correspond to time instants. Rigid motion during the time interval $\mathcal{T}$ is a mapping $\chi:\ \mathcal{E}\times\mathcal{T}\to\mathcal{E}$, where $\chi(A,t)$ is a position of a point A at the time t. Analogously, the displacements as well as other resulting fields are functions of two arguments. The mapping χ when restricted to a given time t constitutes global deformation at the time t, and when restricted to a given point A describes motion of the point A.

In the subsequent analysis, the rotation tensor is a key quantity. Following the orthogonality of $\mathbf{R}$ and taking into account that the time differentiating commutes with the transpose operation, we have

$$\begin{aligned} \partial_t\mathbf{R}^\top &= -\mathbf{R}^\top(\partial_t\mathbf{R})\mathbf{R}^\top \\ \partial_t^2\mathbf{R}^\top &= \mathbf{R}^\top\Big(2(\partial_t\mathbf{R})\mathbf{R}^\top(\partial_t\mathbf{R})\mathbf{R}^\top - (\partial_t^2\mathbf{R})\mathbf{R}^\top\Big) \end{aligned} \tag{3.1}$$

where ∂_t means time differentiation.

Material description

The velocity $\boldsymbol{v}^m$ and the acceleration $\boldsymbol{a}^m$ are determined by

$$\boldsymbol{v}^m = \partial_t \boldsymbol{u}^m \qquad\qquad \boldsymbol{a}^m = \partial_t \boldsymbol{v}^m = \partial_t^2 \boldsymbol{u}^m \tag{3.2}$$

Knowing that the partial time differentiation commutes with the gradient operation, the corresponding velocity gradients are

$$\begin{aligned} \operatorname{grad} \boldsymbol{v}^m &= \partial_t \operatorname{grad} \boldsymbol{u}^m = \partial_t \mathbf{R} \\ \operatorname{grad} \boldsymbol{a}^m &= \partial_t \operatorname{grad} \boldsymbol{v}^m = \partial_t^2 \operatorname{grad} \boldsymbol{u}^m = \partial_t^2 \mathbf{R} \end{aligned} \tag{3.3}$$

Note that the moving centrode in planar motion consists of points for which $\boldsymbol{v}^m = \mathbf{0}$.

Spatial description

Unlike in the material description, the velocity $\boldsymbol{v}^s$ and the acceleration $\boldsymbol{a}^s$ are defined with the use of appropriate field gradients, i.e.,

$$\boldsymbol{v}^s = \partial_t \boldsymbol{u}^s + (\operatorname{grad} \boldsymbol{u}^s)\boldsymbol{v}^s \qquad\qquad \boldsymbol{a}^s = \partial_t \boldsymbol{v}^s + (\operatorname{grad} \boldsymbol{v}^s)\boldsymbol{v}^s \tag{3.4}$$

where ∂_t means time differentiation with the point fixed. Taking the gradient of (3.4), results in

$$\begin{aligned} \operatorname{grad} \boldsymbol{v}^s &= \partial_t \operatorname{grad} \boldsymbol{u}^s + (\operatorname{grad} \boldsymbol{u}^s) \operatorname{grad} \boldsymbol{v}^s \\ \operatorname{grad} \boldsymbol{a}^s &= \partial_t \operatorname{grad} \boldsymbol{v}^s + (\operatorname{grad} \boldsymbol{v}^s) \operatorname{grad} \boldsymbol{v}^s \end{aligned} \tag{3.5}$$

Making use of (3.1), the first and the second time rates of the displacement gradient are

$$\begin{aligned} \partial_t \operatorname{grad} \boldsymbol{u}^s &= \mathbf{R}^\top (\partial_t \mathbf{R}) \mathbf{R}^\top \\ \partial_t^2 \operatorname{grad} \boldsymbol{u}^s &= \mathbf{R}^\top \Big((\partial_t^2 \mathbf{R}) \mathbf{R}^\top - 2(\partial_t \mathbf{R}) \mathbf{R}^\top (\partial_t \mathbf{R}) \mathbf{R}^\top \Big) \end{aligned} \tag{3.6}$$

Rearranging $(3.4)_1$, and next carrying the gradient operation, after the use of $(3.6)_1$, we simplify $(3.4)_1$ and $(3.5)_1$ to

$$\boldsymbol{v}^s = \mathbf{R} \partial_t \boldsymbol{u}^s \qquad\qquad \operatorname{grad} \boldsymbol{v}^s = \mathbf{R} \partial_t \operatorname{grad} \boldsymbol{u}^s = (\partial_t \mathbf{R}) \mathbf{R}^\top \tag{3.7}$$

Introducing $(3.7)_1$ into $(3.4)_2$, and $(3.7)_2$ into $(3.5)_2$ we find

$$\begin{aligned} \boldsymbol{a}^s &= \mathbf{R} \partial_t^2 \boldsymbol{u}^s + 2(\partial_t \mathbf{R}) \partial_t \boldsymbol{u}^s \\ \operatorname{grad} \boldsymbol{a}^s &= \mathbf{R} \partial_t^2 \operatorname{grad} \boldsymbol{u}^s + 2(\partial_t \mathbf{R}) \partial_t \operatorname{grad} \boldsymbol{u}^s = (\partial_t^2 \mathbf{R}) \mathbf{R}^\top \end{aligned} \tag{3.8}$$

With the aid of (3.1), the velocity gradient and the acceleration gradient can be easily decomposed into the symmetric and skew-symmetric parts, i.e.,

$$\begin{aligned}
&\operatorname{sym}\operatorname{grad}\boldsymbol{v}^s = 0 \qquad\qquad \operatorname{skw}\operatorname{grad}\boldsymbol{v}^s = (\partial_t\mathbf{R})\mathbf{R}^\top\\
&\operatorname{sym}\operatorname{grad}\boldsymbol{a}^s = (\operatorname{grad}\boldsymbol{v}^s)\operatorname{grad}\boldsymbol{v}^s = (\partial_t\mathbf{R})\partial_t\operatorname{grad}\boldsymbol{u}^s = (\partial_t\mathbf{R})\mathbf{R}^\top(\partial_t\mathbf{R})\mathbf{R}^\top\\
&\operatorname{skw}\operatorname{grad}\boldsymbol{a}^s = \partial_t\operatorname{grad}\boldsymbol{v}^s = \mathbf{R}\partial_t^2\operatorname{grad}\boldsymbol{u}^s + (\partial_t\mathbf{R})\partial_t\operatorname{grad}\boldsymbol{u}^s =\\
&\qquad = (\partial_t^2\mathbf{R} - (\partial_t\mathbf{R})\mathbf{R}^\top(\partial_t\mathbf{R}))\mathbf{R}^\top
\end{aligned} \tag{3.9}$$

In a consequence of $(3.9)_1$, the following identities are satisfied

$$\begin{aligned}
(\operatorname{grad}\boldsymbol{v}^s)\boldsymbol{v}^s &= -\frac{1}{2}\operatorname{grad}(\boldsymbol{v}^s \bullet \boldsymbol{v}^s)\\
\operatorname{tr}[(\operatorname{grad}\boldsymbol{v}^s)\operatorname{grad}\boldsymbol{v}^s] &= -\operatorname{grad}\boldsymbol{v}^s \bullet \operatorname{grad}\boldsymbol{v}^s
\end{aligned} \tag{3.10}$$

where the symbol " $\bullet$ " denotes the inner product of two vectors or two tensors (cf. Gurtin 1981).

Note that the fixed centrode in planar motion consists of points for which $\boldsymbol{v}^s = \mathbf{0}$.

Mixed relations

The "cross" formulas relating velocity and displacement, i.e., the velocity in the material form and the displacement in the spatial form or vice versa, after using (2.14), can be written down as follows

$$\begin{aligned}
&\boldsymbol{v}^m = \mathbf{R}\partial_t\boldsymbol{u}^s + (\operatorname{grad}\boldsymbol{v}^m)\boldsymbol{u}^s \qquad\qquad \boldsymbol{v}^s = \partial_t\boldsymbol{u}^m - (\operatorname{grad}\boldsymbol{v}^s)\boldsymbol{u}^m\\
&\boldsymbol{a}^m = \mathbf{R}\partial_t^2\boldsymbol{u}^s + 2(\operatorname{grad}\boldsymbol{v}^m)\partial_t\boldsymbol{u}^s + (\operatorname{grad}\boldsymbol{a}^m)\boldsymbol{u}^s\\
&\boldsymbol{a}^s = \partial_t^2\boldsymbol{u}^m - (\operatorname{grad}\boldsymbol{a}^s)\boldsymbol{u}^m
\end{aligned} \tag{3.11}$$

The gradients of the material and spatial forms of the velocity and acceleration are related by, respectively

$$\begin{aligned}
\operatorname{grad}\boldsymbol{v}^m &= (\operatorname{grad}\boldsymbol{v}^s)\mathbf{R} \qquad\qquad \operatorname{grad}\boldsymbol{v}^s = (\operatorname{grad}\boldsymbol{v}^m)\mathbf{R}^\top\\
\operatorname{grad}\boldsymbol{a}^m &= (\operatorname{grad}\boldsymbol{a}^s)\mathbf{R} \qquad\qquad \operatorname{grad}\boldsymbol{a}^s = (\operatorname{grad}\boldsymbol{a}^m)\mathbf{R}^\top
\end{aligned} \tag{3.12}$$

Subtraction of the material and spatial form of the velocity and the acceleration yields, respectively

$$\begin{aligned}
\boldsymbol{v}^m - \boldsymbol{v}^s &= (\operatorname{grad}\boldsymbol{v}^m)\boldsymbol{u}^s = (\operatorname{grad}\boldsymbol{v}^s)\boldsymbol{u}^m\\
\boldsymbol{a}^m - \boldsymbol{a}^s &= (\operatorname{grad}\boldsymbol{a}^m)\boldsymbol{u}^s = (\operatorname{grad}\boldsymbol{a}^s)\boldsymbol{u}^m
\end{aligned} \tag{3.13}$$

4. Representations

4.1. Vector-valued fields

The spatial dependence of kinematical fields has the form (the second argument t is omitted)

$$\begin{aligned}
\boldsymbol{u}^m(A) &= \boldsymbol{u}^m(P) + (\mathbf{R} - \mathbf{1})(A - P) & \boldsymbol{u}^s(A) &= \boldsymbol{u}^s(P) + (\mathbf{1} - \mathbf{R}^\top)(A - P)\\
\boldsymbol{v}^m(A) &= \boldsymbol{v}^m(P) + \partial_t\mathbf{R}(A - P) & \boldsymbol{v}^s(A) &= \boldsymbol{v}^s(P) + \partial_t\mathbf{R}\mathbf{R}^\top(A - P)\\
\boldsymbol{a}^m(A) &= \boldsymbol{a}^m(P) + \partial_t^2\mathbf{R}(A - P) & \boldsymbol{a}^s(A) &= \boldsymbol{a}^s(P) + \partial_t^2\mathbf{R}\mathbf{R}^\top(A - P)
\end{aligned} \tag{4.1}$$

whilst the constant terms in (4.1) are equal to (cf. (2.14)), respectively

$$\begin{aligned}
&\boldsymbol{u}^m(P) = \boldsymbol{w} = \mathbf{R}\boldsymbol{w}^s \qquad \boldsymbol{u}^s(P) = \boldsymbol{w}^s = \mathbf{R}^\top\boldsymbol{w}\\
&\boldsymbol{v}^m(P) = \partial_t\boldsymbol{w} = \partial_t\mathbf{R}\boldsymbol{w}^s + \mathbf{R}\partial_t\boldsymbol{w}^s\\
&\boldsymbol{v}^s(P) = \mathbf{R}\partial_t\boldsymbol{w}^s = \partial_t\boldsymbol{w} - \partial_t\mathbf{R}\mathbf{R}^\top\boldsymbol{w}\\
&\boldsymbol{a}^m(P) = \partial_t^2\boldsymbol{w} = \mathbf{R}\partial_t^2\boldsymbol{w}^s + 2\partial_t\mathbf{R}\partial_t\boldsymbol{w}^s + \partial_t^2\mathbf{R}\boldsymbol{w}^s\\
&\boldsymbol{a}^s(P) = \mathbf{R}\partial_t^2\boldsymbol{w}^s + 2\partial_t\mathbf{R}\partial_t\boldsymbol{w}^s = \partial_t^2\boldsymbol{w} - \partial_t^2\mathbf{R}\mathbf{R}^\top\boldsymbol{w}
\end{aligned} \tag{4.2}$$

Appropriate representations for the tensors involved in (4.1) are derived in the sequel.

4.2. Rotation-related tensors

The most general rotation

If φ is an angle of rotation, and $\boldsymbol{i}$ is a unit vector parallel to the axis of rotation, then the rotation tensor admits the following representation

$$\begin{aligned}
&\mathbf{R} = \exp(\boldsymbol{\Phi}) \qquad \boldsymbol{\Phi} = -\mathbf{E}\varphi \qquad \varphi = -\frac{1}{2}\operatorname{tr}_{(2,4)(3,5)}(\mathbf{E} \otimes \boldsymbol{\Phi})\\
&\boldsymbol{\varphi} = \varphi\boldsymbol{i} \qquad \boldsymbol{i} \bullet \boldsymbol{i} = 1
\end{aligned} \tag{4.3}$$

where $\mathbf{E}$ means an alternating third-order tensor (the Ricci tensor), $\operatorname{tr}_{(2,4)(3,5)}$ indicates the appropriate double contraction operation. Thus, $\boldsymbol{\Phi}$ is a skew tensor the axial vector of which, named the rotation vector, is $\boldsymbol{\varphi}$. In a result

$$\operatorname{sym}\mathbf{R} = \mathbf{1} + (\cos\varphi - 1)(\mathbf{1} - \boldsymbol{i} \otimes \boldsymbol{i}) \qquad \frac{1}{2}\operatorname{tr}_{(2,4)(3,5)}(\mathbf{E} \otimes \mathbf{R}) = -\sin\varphi\boldsymbol{i} \tag{4.4}$$

The vector presented by $(4.4)_2$ is the axial vector corresponding to the skew part of the rotation tensor. Once the representation of $\mathbf{R}$ is established, the related tensors can be put in the analogous formulas. However, before doing that we disclose some useful identities. Namely, constancy of the magnitude of the $\boldsymbol{i}$ implies the following conditions

$$\boldsymbol{i} \bullet \partial_t \boldsymbol{i} = 0 \qquad \boldsymbol{i} \bullet \partial_t^2 \boldsymbol{i} + \partial_t \boldsymbol{i} \bullet \partial_t \boldsymbol{i} = 0 \tag{4.5}$$

which allow for

$$\begin{aligned} &(\mathbf{E}\boldsymbol{i})(\mathbf{E}\boldsymbol{i}) = \boldsymbol{i} \otimes \boldsymbol{i} - \mathbf{1} \qquad (\mathbf{E}\partial_t \boldsymbol{i})(\mathbf{E}\boldsymbol{i}) = \boldsymbol{i} \otimes \partial_t \boldsymbol{i} \\ &(\mathbf{E}\partial_t^2 \boldsymbol{i})(\mathbf{E}\boldsymbol{i}) = \boldsymbol{i} \otimes \partial_t^2 \boldsymbol{i} + (\partial_t \boldsymbol{i} \bullet \partial_t \boldsymbol{i})\mathbf{1} \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} &\operatorname{skw}\big((\mathbf{E}\partial_t \boldsymbol{i})\boldsymbol{i} \otimes \boldsymbol{i}\big) = \frac{1}{2}\mathbf{E}\partial_t \boldsymbol{i} \qquad \operatorname{skw}\big((\mathbf{E}\boldsymbol{i})\partial_t \boldsymbol{i} \otimes \partial_t \boldsymbol{i}\big) = \frac{1}{2}(\partial_t \boldsymbol{i} \bullet \partial_t \boldsymbol{i})\mathbf{E}\boldsymbol{i} \\ &\operatorname{skw}\big((\mathbf{E}\partial_t^2 \boldsymbol{i})\boldsymbol{i} \otimes \boldsymbol{i}\big) = \frac{1}{2}(\mathbf{E}\partial_t^2 \boldsymbol{i} + (\partial_t \boldsymbol{i} \bullet \partial_t \boldsymbol{i})\mathbf{E}\boldsymbol{i}) \end{aligned} \tag{4.7}$$

Successive differentiation of (4.4) yields

$$\begin{aligned} &\operatorname{sym} \partial_t \mathbf{R} = \partial_t \varphi \sin\varphi(\boldsymbol{i} \otimes \boldsymbol{i} - \mathbf{1}) + 2(1-\cos\varphi)\operatorname{sym}(\boldsymbol{i} \otimes \partial_t \boldsymbol{i}) \\ &\frac{1}{2}\operatorname{tr}_{(2,4)(3,5)}(\mathbf{E} \otimes \operatorname{skw} \partial_t \mathbf{R}) = -\partial_t \varphi \cos\varphi \boldsymbol{i} - \sin\varphi \partial_t \boldsymbol{i} \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} \operatorname{sym} \partial_t^2 \mathbf{R} &= (\partial_t^2 \varphi \sin\varphi + (\partial_t \varphi)^2 \cos\varphi)(\boldsymbol{i} \otimes \boldsymbol{i} - \mathbf{1}) + \\ &+ 2(1-\cos\varphi)\operatorname{sym}(\partial_t \boldsymbol{i} \otimes \partial_t \boldsymbol{i}) + \\ &+ 4\partial_t \varphi \sin\varphi \operatorname{sym}(\boldsymbol{i} \otimes \partial_t \boldsymbol{i}) + 2(1-\cos\varphi)\operatorname{sym}(\boldsymbol{i} \otimes \partial_t^2 \boldsymbol{i}) \\ \frac{1}{2}\operatorname{tr}_{(2,4)(3,5)}&(\mathbf{E} \otimes \partial_t^2 \mathbf{R}) = \\ &= (-\partial_t^2 \varphi \cos\varphi + (\partial_t \varphi)^2 \sin\varphi)\boldsymbol{i} - 2\partial_t \varphi \cos\varphi \partial_t \boldsymbol{i} - \sin\varphi \partial_t^2 \boldsymbol{i} \end{aligned} \tag{4.9}$$

The axial vector $\boldsymbol{\omega}$, called the angular velocity, corresponding to the skew tensor $(\partial_t \mathbf{R})\mathbf{R}^\top$ is equal to

$$\boldsymbol{\omega} = -\frac{1}{2}\operatorname{tr}_{(2,4)(3,5)}(\mathbf{E} \otimes ((\partial_t \mathbf{R})\mathbf{R}^\top)) = \partial_t \varphi \boldsymbol{i} + \sin\varphi \partial_t \boldsymbol{i} + (1-\cos\varphi)\boldsymbol{i} \times \partial_t \boldsymbol{i} \tag{4.10}$$

The symmetrical part, and the axial vector $\boldsymbol{\gamma}$, called the angular acceleration, corresponding to the skew part of the tensor $\partial_t^2\mathbf{R}\mathbf{R}^\top$ take the form

$$\begin{aligned}
&\operatorname{sym}(\partial_t^2\mathbf{R}\mathbf{R}^\top) = (\mathbf{E}\boldsymbol{\omega})(\mathbf{E}\boldsymbol{\omega}) = \boldsymbol{\omega}\otimes\boldsymbol{\omega} - (\boldsymbol{\omega}\bullet\boldsymbol{\omega})\mathbf{1} = \\
&\quad = (\boldsymbol{i}\otimes\boldsymbol{i} - \mathbf{1})\Big((\partial_t\varphi)^2 + (\partial_t\boldsymbol{i}\bullet\partial_t\boldsymbol{i})\sin^2\varphi\Big) + 2(\partial_t\varphi)\sin\varphi\operatorname{sym}(\boldsymbol{i}\otimes\partial_t\boldsymbol{i}) + \\
&\quad + 2(1-\cos\varphi)\Big(\operatorname{sym}(\boldsymbol{i}\times\partial_t\boldsymbol{i})\otimes(\partial_t\varphi\boldsymbol{i} + \sin\varphi\partial_t\boldsymbol{i}) + \cos\varphi(\partial_t\boldsymbol{i}\otimes\partial_t\boldsymbol{i}) + \\
&\quad - (\partial_t\boldsymbol{i}\bullet\partial_t\boldsymbol{i})\boldsymbol{i}\otimes\boldsymbol{i}\Big) \\
&\boldsymbol{\gamma} = \partial_t\boldsymbol{\omega} = -\frac{1}{2}\operatorname{tr}_{(2,4)(3,5)}(\mathbf{E}\otimes((\partial_t^2\mathbf{R})\mathbf{R}^\top)) = \partial_t^2\varphi\boldsymbol{i} + \\
&\quad + \partial_t\varphi(1+\cos\varphi)\partial_t\boldsymbol{i} + \sin\varphi\partial_t^2\boldsymbol{i} + \partial_t\varphi\sin\varphi\boldsymbol{i}\times\partial_t\boldsymbol{i} + (1-\cos\varphi)\boldsymbol{i}\times\partial_t^2\boldsymbol{i}
\end{aligned}\tag{4.11}$$

The advantage of the spatial description consists in the skewity of the velocity gradient as well as in the dependence of the symmetrical part of the acceleration gradient on the skew tensor, i.e., $\mathbf{E}\boldsymbol{\omega}$ (cf. $(4.11)_1$). Thereby, every non-constant component of both the velocity and acceleration representations $(4.1)_{4,6}$ can be expressed entirely with the use of the vector product of two vectors instead of the tensor product of a tensor and a vector. Explicitly

$$\begin{aligned}
(\partial_t\mathbf{R})\mathbf{R}^\top(A-P) &= \boldsymbol{\omega}\times(A-P) \\
(\partial_t^2\mathbf{R})\mathbf{R}^\top(A-P) &= \boldsymbol{\omega}\times(\boldsymbol{\omega}\times(A-P)) + \boldsymbol{\gamma}\times(A-P)
\end{aligned}\tag{4.12}$$

Rotation about nearly fixed axis

Let $(\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3)$ be a fixed orthonormal basis in $\mathcal{V}$. Assume that

$$\boldsymbol{i} = \boldsymbol{i}_3 + \boldsymbol{\psi} \qquad\qquad \boldsymbol{\psi} = \psi_1\boldsymbol{i}_1 + \psi_2\boldsymbol{i}_2 \tag{4.13}$$

where $\boldsymbol{\psi}$ is a time-dependent small vector. Hence

$$\partial_t\boldsymbol{i} = \partial_t\boldsymbol{\psi} \qquad\qquad \partial_t^2\boldsymbol{i} = \partial_t^2\boldsymbol{\psi} \tag{4.14}$$

Now that, $(4.3)_5$ and (4.5) change to

$$\begin{aligned}
&\boldsymbol{\psi}\bullet\boldsymbol{\psi} = 0 \qquad\qquad \boldsymbol{\psi}\bullet\partial_t\boldsymbol{\psi} = 0 \\
&\boldsymbol{\psi}\bullet\partial_t^2\boldsymbol{\psi} + \partial_t\boldsymbol{\psi}\bullet\partial_t\boldsymbol{\psi} = 0
\end{aligned}\tag{4.15}$$

Thus, retaining $\boldsymbol{\psi}$ and its time derivatives in the corresponding equations, we make the errors of the order of $O(\psi^2)$, $O(\psi\partial_t\psi)$, and $O(\psi\partial_t^2\psi + \partial_t\psi\partial_t\psi)$, where $\psi = \|\boldsymbol{\psi}\|$, respectively. By means of (4.13), the assertions (4.4) become

$$
\begin{aligned}
&\operatorname{sym}\mathbf{R} = \mathbf{1} + (\cos\varphi - 1)\Big(\mathbf{1} - \boldsymbol{i}_3 \otimes \boldsymbol{i}_3 - 2\operatorname{sym}(\boldsymbol{\psi}\otimes\boldsymbol{i}_3) + O(\psi^2)\Big)\\
&\frac{1}{2}\operatorname{tr}_{(2,4)(3,5)}(\mathbf{E}\otimes\mathbf{R}) = -\sin\varphi(\boldsymbol{i}_3 + \boldsymbol{\psi})
\end{aligned}
\tag{4.16}
$$

Carrying out time differentiation twice over, we obtain

$$
\begin{aligned}
&\operatorname{sym}\partial_t\mathbf{R} = \partial_t\varphi\sin\varphi\Big(\boldsymbol{i}_3\otimes\boldsymbol{i}_3 - \mathbf{1} + 2\operatorname{sym}(\boldsymbol{\psi}\otimes\boldsymbol{i}_3) + O(\psi^2)\Big) +\\
&\qquad + 2(1-\cos\varphi)\Big(\operatorname{sym}(\boldsymbol{i}_3\otimes\partial_t\boldsymbol{\psi}) + O(\psi\partial_t\psi)\Big)\\
&\frac{1}{2}\operatorname{tr}_{(2,4)(3,5)}(\mathbf{E}\otimes\partial_t\mathbf{R}) = \partial_t\varphi\cos\varphi(\boldsymbol{i}_3+\boldsymbol{\psi}) + \sin\varphi\partial_t\boldsymbol{\psi}
\end{aligned}
\tag{4.17}
$$

and next

$$
\begin{aligned}
&\operatorname{sym}\partial_t^2\mathbf{R} = \Big(\partial_t^2\varphi\sin\varphi + (\partial_t\varphi)^2\cos\varphi\Big)\Big(\boldsymbol{i}_3\otimes\boldsymbol{i}_3 - \mathbf{1} + \operatorname{sym}(\boldsymbol{\psi}\otimes\boldsymbol{i}_3) + O(\psi^2)\Big) +\\
&\qquad + 4\partial_t\varphi\sin\varphi\Big(\operatorname{sym}(\partial_t\boldsymbol{\psi}\otimes\boldsymbol{i}_3) + O(\psi\partial_t\psi)\Big) +\\
&\qquad + 2(1-\cos\varphi)\Big(\operatorname{sym}(\partial_t^2\boldsymbol{\psi}\otimes\boldsymbol{i}_3) + O(\psi\partial_t^2\psi + \partial_t\psi\partial_t\psi)\Big)\\
&\frac{1}{2}\operatorname{tr}_{(2,4)(3,5)}(\mathbf{E}\otimes\partial_t^2\mathbf{R}) = \Big(\partial_t^2\varphi\cos\varphi - (\partial_t\varphi)^2\sin\varphi\Big)(\boldsymbol{i}_3+\boldsymbol{\psi}) +\\
&\qquad + 2\partial_t\varphi\cos\varphi\partial_t\boldsymbol{\psi} + \sin\varphi\partial_t^2\boldsymbol{\psi}
\end{aligned}
\tag{4.18}
$$

The angular velocity has the form

$$
\boldsymbol{\omega} = \partial_t\varphi(\boldsymbol{i}_3+\boldsymbol{\psi}) + \sin\varphi\partial_t\boldsymbol{\psi} + (1-\cos\varphi)\Big((\boldsymbol{i}_3\times\partial_t\boldsymbol{\psi}) + O(\psi\partial_t\psi)\Big)
\tag{4.19}
$$

At last, using (4.19), we find

$$
\begin{aligned}
&(\mathbf{E}\boldsymbol{\omega})(\mathbf{E}\boldsymbol{\omega}) = \Big(\boldsymbol{i}_3\otimes\boldsymbol{i}_3 - \mathbf{1} + 2\operatorname{sym}(\boldsymbol{\psi}\otimes\boldsymbol{i}_3) + O(\psi^2)\Big)(\partial_t\varphi)^2 +\\
&\qquad + 2(\partial_t\varphi)\sin\varphi\operatorname{sym}(\partial_t\boldsymbol{\psi}\otimes\boldsymbol{i}_3) +\\
&\qquad + 2(1-\cos\varphi)(\partial_t\varphi)\operatorname{sym}\Big((\boldsymbol{i}_3\times\partial_t\boldsymbol{\psi})\otimes\boldsymbol{i}_3\Big) + O(\partial_t\psi\partial_t\psi) + (\partial_t\varphi)O(\psi\partial_t\psi)\\
&\boldsymbol{\gamma} = \partial_t^2\varphi(\boldsymbol{i}_3+\boldsymbol{\psi}) + \partial_t\varphi(1+\cos\varphi)\partial_t\boldsymbol{\psi} + \sin\varphi\partial_t^2\boldsymbol{\psi} +\\
&\qquad + \partial_t\varphi\sin\varphi\Big(\boldsymbol{i}_3\times\partial_t\boldsymbol{\psi} + O(\psi\partial_t\psi)\Big) + (1-\cos\varphi)\Big(\boldsymbol{i}_3\times\partial_t^2\boldsymbol{\psi} + O(\psi\partial_t^2\psi)\Big)
\end{aligned}
\tag{4.20}
$$

Rotation about fixed axis

Setting $\boldsymbol{\psi} = \mathbf{0}$ as well as $\partial_t \boldsymbol{\psi} = \mathbf{0}$ and $\partial_t^2 \boldsymbol{\psi} = \mathbf{0}$ in the appropriate equations inferred previously, in other words, assuming

$$\boldsymbol{i} = \boldsymbol{i}_3 \tag{4.21}$$

we can examine the case in which the axis of rotation is time-independent. Particularly, now the rotation tensor is represented by

$$\begin{aligned} &\operatorname{sym} \mathsf{R} = cos\varphi\mathbf{1} + (1 - \cos\varphi)\boldsymbol{i}_3 \otimes \boldsymbol{i}_3 \\ &\frac{1}{2} \operatorname{tr}_{(2,4)(3,5)}(\mathsf{E} \otimes \mathsf{R}) = \sin\varphi \boldsymbol{i}_3 \end{aligned} \tag{4.22}$$

Relations (4.17) simplify to

$$\begin{aligned} &\operatorname{sym} \partial_t \mathsf{R} = \partial_t \varphi \sin\varphi (\boldsymbol{i}_3 \otimes \boldsymbol{i}_3 - \mathbf{1}) \\ &\frac{1}{2} \operatorname{tr}_{(2,4)(3,5)}(\mathsf{E} \otimes \partial_t \mathsf{R}) = \partial_t \varphi \cos\varphi \boldsymbol{i}_3 \end{aligned} \tag{4.23}$$

and (4.18) change to

$$\begin{aligned} &\operatorname{sym} \partial_t^2 \mathsf{R} = (\partial_t^2 \varphi \sin\varphi + (\partial_t \varphi)^2 \cos\varphi)(\boldsymbol{i}_3 \otimes \boldsymbol{i}_3 - \mathbf{1}) \\ &\frac{1}{2} \operatorname{tr}_{(2,4)(3,5)}(\mathsf{E} \otimes \partial_t^2 \mathsf{R}) = (\partial_t^2 \varphi \cos\varphi - (\partial_t \varphi)^2 \sin\varphi) \boldsymbol{i}_3 \end{aligned} \tag{4.24}$$

The angular velocity attains a simple form

$$\boldsymbol{\omega} = \partial_t \boldsymbol{\varphi} = \partial_t \varphi \boldsymbol{i}_3 \tag{4.25}$$

Similarly, (4.20) can be written down as

$$(\mathsf{E}\boldsymbol{\omega})(\mathsf{E}\boldsymbol{\omega}) = (\boldsymbol{i}_3 \otimes \boldsymbol{i}_3 - \mathbf{1})(\partial_t \boldsymbol{\varphi})^2 \qquad \boldsymbol{\gamma} = \partial_t^2 \boldsymbol{\varphi} = \partial_t^2 \varphi \boldsymbol{i}_3 \tag{4.26}$$

5. Simplifications due to zero-angle of rotation

When the angle of rotation φ at a certain time attains zero then the rotation tensor is equal to the identity tensor at this time (cf. (4.4)), i.e.,

$$\varphi = 0 \qquad \mathsf{R} = \mathbf{1} \tag{5.1}$$

Thus, the deformation χ is purely a translation. Time rates of the deformation tensor reduce to

$$\partial_t \mathbf{R} = \mathbf{E}\boldsymbol{\omega} \qquad \partial_t^2 \mathbf{R} = (\partial_t \mathbf{R})(\partial_t \mathbf{R}) + \mathbf{E}\boldsymbol{\gamma} \tag{5.2}$$

where

$$\boldsymbol{\omega} = \partial_t \varphi \boldsymbol{i} \qquad \boldsymbol{\gamma} = \partial_t^2 \varphi \boldsymbol{i} + 2\partial_t \varphi \partial_t \boldsymbol{i} \tag{5.3}$$

In particular, assumptions (5.1) are satisfied when the current configuration is taken to be the reference configuration, i.e., the deformation is the identity transformation (cf. Wang, 1979).

An obvious consequence of (5.1) is that both the displacement fields are constant and coincide at every point (cf. (2.14) and (2.15))

$$\boldsymbol{u}^s = \boldsymbol{u}^m \equiv \boldsymbol{u} \qquad \operatorname{grad} \boldsymbol{u} = \mathbf{0} \tag{5.4}$$

The appropriate equations regarding the material description do not undergo any change. However, the equations relating the spatial description of the displacement, velocity and acceleration simplify to

$$\begin{aligned} &\boldsymbol{v}^s = \partial_t \boldsymbol{u}^s \qquad \partial_t \boldsymbol{v}^s = \partial_t^2 \boldsymbol{u}^s + (\partial_t \mathbf{R})\boldsymbol{v}^s \\ &\boldsymbol{a}^s = \partial_t \boldsymbol{v}^s + (\partial_t \mathbf{R})\boldsymbol{v}^s = \partial_t^2 \boldsymbol{u}^s + 2(\partial_t \mathbf{R})\boldsymbol{v}^s \end{aligned} \tag{5.5}$$

Subtracting the material and spatial description of the velocity and acceleration, we find

$$\boldsymbol{v}^m - \boldsymbol{v}^s = \partial_t \boldsymbol{u}^m - \partial_t \boldsymbol{u}^s = (\partial_t \mathbf{R})\boldsymbol{u} \qquad \boldsymbol{a}^m - \boldsymbol{a}^s = (\partial_t^2 \mathbf{R})\boldsymbol{u} \tag{5.6}$$

whereas

$$\begin{aligned} &\partial_t \boldsymbol{v}^m - \partial_t \boldsymbol{v}^s = (\partial_t^2 \mathbf{R})\boldsymbol{u} + (\partial_t \mathbf{R})\boldsymbol{v}^s \\ &\partial_t^2 \boldsymbol{u}^m - \partial_t^2 \boldsymbol{u}^s = (\partial_t^2 \mathbf{R})\boldsymbol{u} + 2(\partial_t \mathbf{R})\boldsymbol{v}^s \end{aligned} \tag{5.7}$$

Taking the gradient of (5.6), it follows

$$\operatorname{grad} \boldsymbol{v}^m = \operatorname{grad} \boldsymbol{v}^s = \operatorname{grad} \partial_t \boldsymbol{u}^m = \operatorname{grad} \partial_t \boldsymbol{u}^s \qquad \operatorname{grad} \boldsymbol{a}^m = \operatorname{grad} \boldsymbol{a}^s \tag{5.8}$$

while, in turn, (5.7) leads to

$$\begin{aligned} &\operatorname{grad} \partial_t \boldsymbol{v}^m - \operatorname{grad} \partial_t \boldsymbol{v}^s = (\partial_t \mathbf{R})(\partial_t \mathbf{R}) \\ &\operatorname{grad} \partial_t^2 \boldsymbol{u}^m - \operatorname{grad} \partial_t^2 \boldsymbol{u}^s = 2(\partial_t \mathbf{R})(\partial_t \mathbf{R}) \end{aligned} \tag{5.9}$$

6. Motion assuming small rotation

Now assume that φ be small, i.e., turning back to deformation (2.3), ϑ_P be small. Then R, being equal to $\operatorname{grad}\vartheta_P$ (in fact, independent of P), reduces as follows (cf. (4.4), (4.8), (4.9))

$$\mathsf{R} = \mathbf{1} + \mathbf{\Phi} + O(\varphi^2) = \mathbf{1} + O(\varphi) \tag{6.1}$$

where $\mathbf{\Phi}$ is a small skew tensor. Making use of (6.1), we find

$$\begin{aligned}
&\mathsf{R} - \mathbf{1} = \mathbf{\Phi}(1 + O(\varphi)) && \mathbf{1} - \mathsf{R}^\top = \mathbf{\Phi}(1 + O(\varphi))\\
&\partial_t \mathsf{R} = \partial_t \mathbf{\Phi}(1 + O(\varphi)) && \partial_t \mathsf{R}\mathsf{R}^\top = \partial_t \mathbf{\Phi}(1 + O(\varphi))\\
&\partial_t^2 \mathsf{R} = \partial_t^2 \mathbf{\Phi}(1 + O(\varphi)) && \partial_t^2 \mathsf{R}\mathsf{R}^\top = \partial_t^2 \boldsymbol{\phi}(1 + O(\varphi))
\end{aligned} \tag{6.2}$$

Similarly

$$\boldsymbol{\omega} = \partial_t \boldsymbol{\varphi}(1 + O(\varphi)) \qquad \boldsymbol{\gamma} = \partial_t \boldsymbol{\omega}(1 + O(\varphi)) \tag{6.3}$$

With the aid of (6.1) and (6.2), relations (2.14) and (3.13) change to

$$\begin{aligned}
&\boldsymbol{u}^s = \boldsymbol{u}^m(1 + O(\varphi))\\
&\boldsymbol{v}^m - \boldsymbol{v}^s = \partial_t \mathbf{\Phi}(1 + O(\varphi))\boldsymbol{u}^s = \partial_t \mathbf{\Phi}(1 + O(\varphi))\boldsymbol{u}^m\\
&\boldsymbol{a}^m - \boldsymbol{a}^s = \partial_t^2 \mathbf{\Phi}(1 + O(\varphi))\boldsymbol{u}^s = \partial_t^2 \mathbf{\Phi}(1 + O(\varphi))\boldsymbol{u}^m
\end{aligned} \tag{6.4}$$

It is seen that, to within an error of $O(\varphi)$ as $\varphi \to 0$, the displacement field as well as the displacement, velocity, and acceleration gradients in the material and spatial description coincide (cf. $(6.4)_1$ and (6.2)). Moreover, all the above-mentioned gradients are skew. However, in general, this is not the case as far as the velocity and acceleration fields are concerned (cf. $(6.4)_{2,3}$).

7. Relative motion of a particle

Let a C^2-class function $\xi : \mathcal{T} \to \mathcal{E}$ be motion of a particle relative to the physical space (or a smaller body) assuming the physical space is in rest. Anyway, deformation of the physical space changes the position of the particle. Thus, the resultant position of the particle is given by a function $\eta : \mathcal{T} \to \mathcal{E}$ defined by

$$\eta(t) = \chi(\xi(t), t) \tag{7.1}$$

The velocity of the particle is the time rate of (6.1). The chain rule leads to

$$\partial_t \eta = \boldsymbol{v}^b + \boldsymbol{v}^p \tag{7.2}$$

where

$$\begin{aligned} &\boldsymbol{v}^b(t) = \partial_t \chi(\xi(t), t) = \boldsymbol{v}^m(\xi(t), t) = \boldsymbol{v}^s(\eta(t), t) \\ &\boldsymbol{v}^p(t) = \mathbf{R}(t)\partial_t \xi(t) \end{aligned} \tag{7.3}$$

The symbol ∂_t in $(7.3)_1$ denotes partial time differentiation holding the point $\xi(t)$ fixed. The superscripts b and p are the first letters of the words: *body* and *particle*. The acceleration of the particle is the time rate of (7.2). In view of

$$\partial_t \boldsymbol{v}^b = \boldsymbol{a}^b + (\partial_t \mathbf{R})\partial_t \xi \qquad \partial_t \boldsymbol{v}^p = \boldsymbol{a}^p + (\partial_t \mathbf{R})\partial_t \xi \tag{7.4}$$

where

$$\begin{aligned} &\boldsymbol{a}^b(t) = \partial_t^2 \chi(\xi(t), t) = \boldsymbol{a}^m(\xi(t), t) = \boldsymbol{a}^s(\eta(t), t) \\ &\boldsymbol{a}^p(t) = \mathbf{R}(t)\partial_t^2 \xi(t) \end{aligned} \tag{7.5}$$

we arrive at

$$\partial_t^2 \eta = \boldsymbol{a}^b + \boldsymbol{a}^p + \boldsymbol{a}^C \tag{7.6}$$

where the last term in (7.6) (named after Coriolis) is defined by

$$\boldsymbol{a}^C = 2(\partial_t \mathbf{R})\partial_t \xi = 2\boldsymbol{\omega} \times \boldsymbol{v}^p \tag{7.7}$$

8. Concluding remarks

As regards the comparison of kinematical fields in the material and spatial description, it is obvious that both forms of the displacement at any point take either zero or non-zero values (cf. (2.14)). Thus, if a given point at any time is a fixed point of the deformation mapping, then the displacements at this point at this time vanish, and otherwise. However, in general, this is not the case as far as the velocity and acceleration fields are concerned. Nevertheless, a more stringent condition can be formulated. Namely, if either form of the velocity (acceleration), i.e., the material or spatial form, is zero, and, in addition, the displacement is zero, then the other form of the velocity (acceleration) vanishes as well (cf. (3.13)). On the other hand, if a given point is a fixed point of the deformation mapping in any interval of time, then the velocities and accelerations at this point vanish together with the displacements during that interval of time.

Representation formulas (4.1) are composed of constant and variable parts. Obviously, relations of four types are possible, i.e., "material-material", "spatial-spatial", "material-spatial" and "spatial-material". The variable parts are affected by rotation and unaffected by translation, whereas the constant parts are influenced by translation, and, interestingly, most of them by rotation too. Precisely, the rotation does not affect only the "material-material" constant parts, which are expressed entirely in terms of $\boldsymbol{w}$ (cf. (4.2)).

The material and spatial forms of the displacement, velocity and acceleration coincide at every point of the physical space if and only if the rotation tensor is equal to the identity tensor. Moreover, all the above-mentioned vector-valued fields are then constant.

Assuming that the rotation is small, the difference between the material and spatial description proves to be immaterial so far as the displacement field as well as the displacement, velocity and acceleration gradients are concerned. In order to discard this difference in the case of the velocity and acceleration the additional requirement about smallness of the translation as a part of the deformation (cf. (2.3)) is necessary. For instance, small translations are arising from motion in a small interval of time.

At last, both the material and spatial descriptions are necessary to define the notion of both centrodes, i.e., the fixed and moving ones, in an elegant manner.

Despite our consideration is carried out under the assertion that the deformation transforms the physical space into itself, the results treated in a proper way are in fact valid for any non-coplanar set of points (cf. e.g. Brinkman and Klotz, 1971).

Unlike in many books on the subject, the presented paper does not use the indicial notation excluding some representation formulas provided in Section 3. In our hope, such an approach is better than that of the others as far as some general concepts are concerned and could reduce the difficulties that arise while studying mechanics (cf. Evans *et al.*, 2004). Remembering that all real materials are deformable, there is every reason to extend the presented treatment on less stringent constrained materials.

Acknowledgement

A prelude to some early results communicated in this paper was an investigation conducted under the Grant nr 503G10834400 of Warsaw University of Technology in the Year 2000.

References

1. Brinkman H.W., Klotz E.A., 1971, *Linear Algebra and Analytic Geometry*, Addison-Wesley, Reading
2. Easthope C.E., 1964, *Three Dimensional Dynamics, A Vectorial Treatment*, Buttenworths, London
3. Evans D.L., Gray G.L., Constanzo F., Cornwell P., Self B.P., 2004, Rigid body dynamics: student misconceptions and their diagnosis, *21st ICTAM*, Warsaw
4. Gurtin M.E., 1981, *An Introduction to Continuum Mechanics*, Academic Press, New York
5. Gurtin M.E., Williams W.O., 1976, A note on rigid fields, *J. of Elasticity*, **6**, 3, 331-333
6. Nwokah O.D.I, Hurmuzlu Y., 2002, *The Mechanical Systems Design Handbook, Modeling, Measurement, and Control*, CRC Press, Boca Raton
7. Papastavridis J.G., 1998, *Tensor Calculus and Analytical Dynamics*, Library of Engineering Mathematics, CRC Press, Boca Raton
8. Sławianowski J.J., 2004, Affine symmetry in mechanics of discrete and continues systems, *21st ICTAM*, Warsaw
9. Truesdell C., Toupin R.A., 1960, The Classical Field Theories, in: *Encyclopedia of Physics*, vol. III/1, *Principles of Classical Mechanics and Field Theory*, (ed. S. Flügge), Springer Verlag, Berlin
10. Uicker J.J. (Jr.), Pennock G.R., Shigley J.E., 2003, *Theory of Machines and Mechanisms*, Oxford Univ. Press, New York
11. Wang C.C., 1979, *Mathematical Principles of Mechanics and Electromagnetism, Part A: Analytical and Continuum Mechanics*, Plenum Press, New York

O materialnym i przestrzennym opisie ciał sztywnych

Streszczenie

Rozważono kinematykę ciała sztywnego w opisie materialnym, przestrzennym i mieszanym bez wprowadzania układu współrzędnych. Przedstawiono reprezentacje tensorowe gradientów pól przemieszczenia, prędkości i przyspieszenia. Wyodrębniono przypadki szczególne obrotu wokół stałej oraz prawie stałej osi. Zbadano ruch przy założeniu małego obrotu. Naszkicowano zagadnienie ruchu względnego cząstki.

Manuscript received January 5, 2005; accepted for print February 10, 2005